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Pareto-efficient situations in infinite and finite pure-strategy staircase-function games

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Abstract

A computationally tractable method is suggested for solving N-person games in which players' pure strategies are staircase functions. The solution is meant to be Pareto-efficient. Owing to the payoff subinterval-wise summing, the N-person staircase-function game is considered as a succession of subinterval N-person games in which strategies are constants. In the case of a finite staircase-function game, each constant-strategy game is an N-dimensional-matrix game whose size is relatively far smaller to solve it in a reasonable time. It is proved that any staircase-function game has a single Pareto-efficient situation if every constant-strategy game has a single Pareto-efficient situation, and vice versa. Besides, it is proved that, whichever the staircase-function game continuity is, any Pareto-efficient situation of staircase function-strategies is a stack of successive Pareto-efficient situations in the constant-strategy games. If a staircase-function game has multiple Pareto-efficient situations, the best efficient situation is one which is the farthest from the most unprofitable payoffs. In terms of 0-1-standardization, the best efficient situation is the farthest from the zero payoffs.

Keywords: game theory, payoff functional, Pareto efficiency, staircase-function strategy, N-dimensional-matrix game

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1 Introduction

In noncooperative game theory, an N-person game is used to model interactions and struggles for rationalizing the distribution of limited resources among N persons (sides, players). Term "resource" herein is meant in a wide sense implying real-world and abstract funds, facilities, tools, custodial penalties, access, energy, etc. [12, 16, 17, 22, 30]. The resource utility is assessed as the player's payoff [16, 23, 24]. To receive closely the best possible payoff under conditions of uncertainty generated by actions of the other players, the player uses pure and mixed strategies.

The strategy can be as a simple (point) action, as well as a process consisting of an order of simple actions [3, 8, 9, 18]. In the simplest case, the player's pure strategy is a short action whose duration is negligible and thus is represented as just a time point. In a more complicated case, the player's pure strategy is a function of time [19, 21, 25, 27], so the player's action is a complex process [1, 7, 17, 22]. Such strategies often feature multi-stage processes with and without adaptive decision-making based on multi-stage corrective action under influence of uncertainties and other competitive factors [1, 8, 9]. They are used in planning and controlling sequences [3, 14, 31], multi-stage optimization [13, 19], scheduling [23], etc. [15, 16, 22, 25].

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In mathematical terms, the interaction-and-struggle process in an N-person game is usually a selection-and-payoff event or a series of such events, without any differentiation or integration, but the interpretation of the eventual result sometimes appears uncertain enough. First, the optimality or the best decision (solution) has multiple types. This is so because the optimality requires equilibrium, efficiency (profitability), and fairness [16, 24, 25]. These types are often contradictory in a 2-person game [11], and they may be far more contradictory in a game of three and more players. For instance, an equilibrium situation may be efficient for one or a few players while it is not profitable for the remaining players. Second, an N-person game may have multiple equilibria along with multiple Pareto-efficient situations [6, 11]. This induces the solution uncertainty. The unfairness of the players' payoffs worsens the solution selection. Furthermore, even a finite N-person game may have a continuum of equilibria [16, 24], wherein the best decision selection is far more difficult.

Obviously, infinite (even when a set of pure strategies is countable) and, moreover, continuous N-person games are far more complicated than finite N-person games. Whereas the finite game has at least an equilibrium (generally speaking, in mixed strategies), an infinite game may not have an equilibrium, or it may be indeterminable, or it may be impracticable [1, 18, 28]. As the pure strategy structure becomes more complex, the practicability of a game solution is further sophisticated. This becomes more ungainly when there is a mixed strategy to be used in a game. This is explained by the simplicity of the pure strategy solution (not requiring repetitions of a game) and the sophistication of the mixed strategy solution (requiring repetitions of a game for a proper implementation of the solution). The greater number of players, the more likely game practical intractability is. The best option is to model the interaction-and-struggle process with a finite N-person game always rendered to an N-dimensional-matrix game [20, 25], and selecting a pure strategy solution.

The most trivially structured pure strategy is a decision corresponding to a one-stage event whose duration through time is (negligibly) short. As it is mentioned above, a strategy can be also a multi-stage process like a staircase-function defined on a time interval. In a pure strategy situation, a set of N such staircase-function strategies (from the players in an N-person game) is mapped into a real value [21, 29]. Obviously, when each of the players possesses a finite set of such function-strategies, the staircase-function game is finite. It is easily rendered down to an N-dimensional-matrix game, wherein a pure strategy is a conditional point (just like it is in ordinary finite games), which in reality is a staircase function (if the conditional point is "disclosed").

2 Motivation to the finite staircase game and efficient solution

Noncooperative game theory considers finite, infinite, and continuous games, depending on the players' sets of pure strategies. Nevertheless, the theory should sooner or later comply with practical events, phenomena, processes, development, evolution, etc. Thus, in real-world practice, the continuity of a process (regardless of its duration) is an ill-posed assumption due to natural constraints imposed by corpuscular nature of the matter. The latter implies that any activity is discontinuous. This is why any process through a definite time interval is a finite set of elementary actions. As the elementary action of a player is naturally constrained, the pure strategy is a staircase function defined on a time interval. Consequently, during an elementary action, the staircase function must be considered constant [5, 13, 14, 26].

To make a staircase-function game finite, the set of possible values of the player's pure strategy should be finite. In such a staircase-function N-dimensional-matrix game the player's selection of a pure strategy means using a staircase function on a time interval whereon every pure strategy is defined. The total number of the player's pure strategies in the staircase-function N-dimensional-matrix game is determined by the number of "stair" subintervals and the number of possible values of the player's pure strategy (staircase function). If M is the number of the elementary actions could be made by a player, then it is the number of "stair" subintervals at which the player's pure strategy is constant. The minimum of this number is 2, so $M \in \mathbb{N} \setminus \{1\}$. The number of "stair" subintervals must not be confused with the number of possible values of the player's pure strategy. The latter is determined by the "resolution" of the pure strategy (along ordinate axis). So, if there are just 2 possible values of the pure strategy (the minimal "resolution") at every player, the total number of situations in the finite staircase-function game is $\prod_{n=1}^{N} 2^M$. The minimum-sized game is obtained by $N = 2$ (two players) and $M = 2$ (two "stair" subintervals), for which the total number of situations is just 16. This, however, is the most trivial case rarely happened to be practically relevant. In a more practically relevant example, even with two players, the 8-valued pure strategy "resolution" at every player and $M = 4$ result in $8^4 \cdot 8^4 = 16777216$ situations in the respective bimatrix 4096×4096 game (to which this finite 2-person game is rendered). If a third player is added into this game, the respective trimatrix $4096 \times 4096 \times 4096$ game has already 68719476736 (more than 68.7 billion) pure strategy situations. Adding a fourth player results in an immensely gigantic

$4096 \times 4096 \times 4096 \times 4096$ game having 281474976710656 (more than 281 trillion) pure strategy situations. It is quite obvious that a solution even in a 4096×4096 game cannot be found in a reasonable amount of computational time by using a reasonably expensive hardware, let alone $4096 \times 4096 \times 4096$ or $4096 \times 4096 \times 4096 \times 4096$ game (it is worth remembering that a pure strategy situation in an N-person game played with staircase-function strategies is a set of N staircase functions rather than a set of N real numbers, so processing such a staircase situation takes far more computational time than processing a situation consisting of simple-point-action strategies). This means that straightforwardly solving staircase-function N-dimensional-matrix games is quite impracticable, whichever practical purposes are [1, 10, 24, 26].

Another question is what the solution should be. It is well-known that the equilibrium in noncooperative games often appears to be unprofitable for at least one of the players [16, 22, 24, 25]. Thus, although the property of solution stability (which is theoretically ensured by the equilibrium) is considered important, sometimes it is more rational to find a solution in a Pareto-efficient situation. For example, in a $2 \times 3 \times 3$ game with payoff matrices (the left, middle, and right submatrices correspond to the first, second, and third pure strategies of the third player)

$$\mathbf{F} = \left[\begin{bmatrix} -2 & 1 & -2 \\ 2 & 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 8 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & -1 & -5 \\ 2 & 1 & -2 \end{bmatrix} \right] \tag{2.1}$$

and

$$\mathbf{G} = \left[\begin{bmatrix} 4 & -2 & 4 \\ -3 & 3 & 0 \end{bmatrix} \begin{bmatrix} -4 & 4 & 2 \\ 5 & 5 & -1 \end{bmatrix} \begin{bmatrix} -2 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix} \right] \tag{2.2}$$

and

$$\mathbf{H} = \left[\begin{bmatrix} 5 & -1 & 4 \\ 2 & -3 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 5 & 1 \\ 0 & 0 & 3 \end{bmatrix} \right] \tag{2.3}$$

of the first, second, and third players, respectively, there are two pure strategy equilibria with payoffs

$$\{1,\ 1,\ 0\} \tag{2.4}$$

and

$$\{-2,\ 1,\ 3\}\,. \tag{2.5}$$

These equilibria are not Pareto-efficient. There are seven Pareto-efficient situations with payoffs

$$\begin{gathered} \{-2,\ 4,\ 5\}\,, \\ \{1,\ 4,\ 2\}\,, \\ \{-1,\ 2,\ 5\}\,, \\ \{8,\ 2,\ 3\}\,, \\ \{1,\ 5,\ 1\}\,, \\ \{3,\ 3,\ -3\}\,, \\ \{2,\ 5,\ -1\}\,, \end{gathered} \tag{2.6}$$

where the most profitable are payoffs (2.6). Although payoffs (2.6) are received in a non-equilibrium situation, payoffs (2.4) and (2.5) are absolutely unprofitable for the first and second players, and only the third player does not lose in the situation with payoffs (2.5), losing nevertheless in the situation with payoffs (2.4). Meanwhile, the equilibrium situation with payoffs (2.4) is more profitable for the first player compared to (2.5). On the contrary, the equilibrium situation with payoffs (2.5) is more profitable for the third player. This discrepancy therein, paradoxically, will definitely induce an instable behavior of the first and third players eventually shattering the formal equilibrium. In its turn, a stability of the Pareto-efficient situation with payoffs (2.6) may be eventually induced. This is a quite illustrative example of that equilibria in an N-person game may eventually become non-equilibrium being shattered by the discrepancy in payoffs for some players. Pareto-efficient situations then may attract the players instead. So, the Pareto efficiency is first to be checked. Although a Pareto-efficient situation is not formally stable, its stability will likely be induced in the way described above. Moreover, sometimes Pareto-efficient situations happen to be equilibrium situations additionally motivating to search for the efficiency.

3 Objective and tasks to be fulfilled

A situation in a noncooperative game may be constituted by players' strategies of a complex form. Such a situation has a duration period which cannot be neglected. The matter is that the situation is a multi-stage event, during which every player must "go through one's path" (staircase function). Whereas it is absolutely impracticable to straightforwardly solve finite noncooperative games played with staircase-function strategies, the objective is to develop a tractable method of solving such games. The solution is meant to be Pareto-efficient. The Nash equilibrium requirement is not raised. To meet the objective, the following eight tasks are to be fulfilled:

1. To formalize an infinite noncooperative game, in which the players' strategies are functions. Commonly, the function-strategy depends on time. Such time-dependent function-strategies are presumed to be bounded and Lebesgue-integrable.

2. To formalize an infinite noncooperative game, in which the players' strategies are time-dependent staircase functions, whereas the time is discrete. In such a game, the set of the player's pure strategies is a continuum of staircase functions.

3. To consider the case of when a (finite or infinite) staircase-function game has a single Pareto-efficient situation (constituted by staircase-function strategies of the players).

4. To study the structure of a Pareto-efficient situation in a (finite or infinite) staircase-function game.

5. To suggest a method of solving finite noncooperative staircase-function games by using the Pareto-efficiency criterion.

6. To give an illustrative example of how the suggested method can be practically used.

7. To discuss practical applicability and scientific significance of the method along with its weaknesses. It should be emphasized why this method must be important for the game theory and decision making development.

8. To state a final recapitulation on the suggested method.

4 A noncooperative game played with function-strategies

In a noncooperative game of N players, $N \in \mathbb{N} \backslash \{1\}$, in which the player's pure strategy is a function, let each of the players use strategies defined almost everywhere on (time) interval $[t_1;\ t_2]$ by $t_2 > t_1$. Denote a strategy of the n-th player by $x_n(t)$ for $n = \overline{1,\ N}$. Surely, function $x_n(t)$ is presumed to be bounded, i. e.

$$x_n^{(\min)} \leqslant x_n(t) \leqslant x_n^{(\max)} \ \text{by} \ x_n^{(\min)} < x_n^{(\max)} \ \ \forall\, n = \overline{1,\ N}. \tag{4.1}$$

Besides, the square of the function-strategy is presumed to be Lebesgue-integrable. Thus, pure strategies of the player belong to a rectangular functional space (of time functions):

$$\mathcal{X}_n = \left\{ x_n(t)\,,\ t \in [t_1;\ t_2]\,,\ t_1 < t_2 :\ x_n^{(\min)} \leqslant x_n(t) \leqslant x_n^{(\max)} \ \text{by}\ x_n^{(\min)} < x_n^{(\max)} \right\} \subset \mathbb{L}_2[t_1;\ t_2] \tag{4.2}$$

is the set of the n-th player's pure strategies, $n = \overline{1,\ N}$. Obviously, set (4.2) is infinite. In fact, it consists of a continuum of functions, so set (4.2) might be called a continuous set. However, it contains as continuous functions, as well as functions with discontinuities (although still Lebesgue-integrable) like staircase functions mentioned above.

The player's payoff in situation

$$\{x_n(t)\}_{n=1}^{N} \tag{4.3}$$

is presumed to be an integral functional [4, 21]. Thus, the n-th player's payoff in situation (4.3) is

$$K_n\left(\{x_i(t)\}_{i=1}^{N}\right) = \int\limits_{[t_1;\ t_2]} f_n\left(\{x_i(t)\}_{i=1}^{N}\,,\ t\right) d\mu(t) \tag{4.4}$$

by a function

$$f_n\left(\{x_i(t)\}_{i=1}^{N}\,,\ t\right) \tag{4.5}$$

of time functions in (4.3) explicitly including time t. Therefore, the infinite noncooperative game

$$\left\langle \{\mathcal{X}_n\}_{n=1}^{N}\,,\ \left\{K_n\left(\{x_i(t)\}_{i=1}^{N}\right)\right\}_{n=1}^{N} \right\rangle \tag{4.6}$$

is played with function-strategies from respective rectangular functional spaces (4.2). Game (4.6) might be called continuous also due to every set (4.2) is of a continuum of functions. Why time t is explicitly included into (4.5) will be explained below.

5 A noncooperative staircase-function game

A staircase-function game is formed in a natural way. A function-strategy becomes staircase because it is defined so by the (physical, economical, biological, social, etc.) laws of a system modeled by the game. The number of subintervals at which the player's pure strategy is constant must be the same for every player. Thus, in practical reality, noncooperative game (4.6) with strategies as functions is presumed to be played discretely through time interval $[t_1;\ t_2]$.

The player's pure staircase-function strategy may have at most M different values. If $\left\{\tau^{(l)}\right\}_{l=1}^{M-1}$ are time points at which the staircase-function strategy changes or may change its value, where

$$t_1 = \tau^{(0)} < \tau^{(1)} < \tau^{(2)} < \ldots < \tau^{(M-1)} < \tau^{(M)} = t_2, \tag{5.1}$$

then

$$\left\{x_n\left(\tau^{(l)}\right)\right\}_{l=0}^{M} \tag{5.2}$$

are the values of the n-th player's strategy in a play-off of game (4.6), $n = \overline{1,\ N}$. The time interval breaking by (4.2) is the same for every player, which is naturally defined by the laws of the system. Obviously, points $\left\{\tau^{(l)}\right\}_{l=0}^{M}$ are not necessarily to be equidistant.

The staircase-function strategies are right-continuous [4]:

$$\lim_{\substack{\varepsilon>0 \\ \varepsilon\to 0}} x_n\left(\tau^{(l)} + \varepsilon\right) = x_n\left(\tau^{(l)}\right) \quad \text{for} \quad l = \overline{1,\ M-1} \quad \text{by} \quad n = \overline{1,\ N}, \tag{5.3}$$

whereas

$$\lim_{\substack{\varepsilon>0 \\ \varepsilon\to 0}} x_n\left(\tau^{(l)} - \varepsilon\right) \neq x_n\left(\tau^{(l)}\right) \quad \text{for} \quad l = \overline{1,\ M-1} \quad \text{by} \quad n = \overline{1,\ N}. \tag{5.4}$$

It is easy to see that a strategy value on subinterval $\left[\tau^{(M-1)};\ \tau^{(M)}\right]$ should not change, i. e.

$$\lim_{\substack{\varepsilon>0 \\ \varepsilon\to 0}} x_n\left(\tau^{(M)} - \varepsilon\right) = x_n\left(\tau^{(M)}\right), \tag{5.5}$$

so

$$x_n\left(\tau^{(M-1)}\right) = x_n\left(\tau^{(M)}\right) \quad \forall\, n = \overline{1,\ N}. \tag{5.6}$$

Then constant values (5.2) by (5.1) mean that game (4.6) is an infinite staircase-function game

$$\left\langle \left\{\mathcal{X}_n^{(M)}\right\}_{n=1}^{N},\ \left\{K_n\left(\{x_i(t)\}_{i=1}^{N}\right)\right\}_{n=1}^{N} \right\rangle \tag{5.7}$$

where

$$\mathcal{X}_n^{(M)} = \Big\{x_n(t):\ x_n(t) = \alpha_{nl} \text{ by } \alpha_{nl} \in \left[x_n^{(\min)};\ x_n^{(\max)}\right]\ \forall\, t \in \left[\tau^{(l-1)};\ \tau^{(l)}\right) \\ \text{for } l = \overline{1,\ M-1} \text{ and } \forall\, t \in \left[\tau^{(M-1)};\ \tau^{(M)}\right]\Big\} \subset \mathcal{X}_n \tag{5.8}$$

is the n-th player's rectangular functional space of staircase-function strategies by (5.1) — (5.6). The staircase-function game can be thought of as it is a succession of M ordinary continuous noncooperative games

$$\left\langle \left\{\left[x_n^{(\min)};\ x_n^{(\max)}\right]\right\}_{n=1}^{N},\ \left\{K_n\left(\{\alpha_{il}\}_{i=1}^{N}\right)\right\}_{n=1}^{N} \right\rangle \tag{5.9}$$

each defined on hyperparallelepiped

$$\bigtimes_{n=1}^{N} \left[x_n^{(\min)};\ x_n^{(\max)}\right] \tag{5.10}$$

by

$$\alpha_{nl} = x_n(t) \in \left[x_n^{(\min)};\ x_n^{(\max)}\right] \text{ by } n = \overline{1,\ N}$$
$$\forall\, t \in \left[\tau^{(l-1)};\ \tau^{(l)}\right) \text{ for } l = \overline{1,\ M-1} \text{ and } \forall\, t \in \left[\tau^{(M-1)};\ \tau^{(M)}\right], \tag{5.11}$$

where the factual players' payoffs in situation $\{\alpha_{il}\}_{i=1}^{N}$ are

$$K_n\left(\{\alpha_{il}\}_{i=1}^{N}\right) = \int\limits_{\left[\tau^{(l-1)};\ \tau^{(l)}\right)} f_n\left(\{\alpha_{il}\}_{i=1}^{N},\ t\right) d\mu(t) \ \ \forall\, l = \overline{1,\ M-1} \tag{5.12}$$

by

$$K_n\left(\{\alpha_{iM}\}_{i=1}^{N}\right) = \int\limits_{\left[\tau^{(M-1)};\ \tau^{(M)}\right]} f_n\left(\{\alpha_{iM}\}_{i=1}^{N},\ t\right) d\mu(t) \tag{5.13}$$

for $n = \overline{1,\ N}$. A pure-strategy situation in staircase-function game (5.7) is a succession of M situations

$$\left\{\{\alpha_{il}\}_{i=1}^{N}\right\}_{l=1}^{M} \tag{5.14}$$

in games (5.9), where each situation corresponds to its subinterval. The payoff in situation $\{\alpha_{il}\}_{i=1}^{N}$ can be thought of as it is the payoff on a "stair" subinterval l, which is $\left[\tau^{(l-1)};\ \tau^{(l)}\right)$ for $l = \overline{1,\ M-1}$ and $\left[\tau^{(M-1)};\ \tau^{(M)}\right]$ (when $l = M$). The stack of successive situations (5.14) is a (staircase) situation in the respective staircase-function game (5.7). The succession allows considering players' payoffs in situation (4.3) of staircase functions in a simpler form.

Theorem 5.1. In a pure-strategy situation of the staircase-function game (5.7), represented as a succession of M continuous games (5.9) by (5.11)—(5.13), functional (4.4) is re-written as subinterval-wise sum

$$\begin{aligned} K_n\left(\{x_i(t)\}_{i=1}^{N}\right) &= \sum_{l=1}^{M} K_n\left(\{\alpha_{il}\}_{i=1}^{N}\right) \\ &= \sum_{l=1}^{M-1} \int\limits_{\left[\tau^{(l-1)};\ \tau^{(l)}\right)} f_n\left(\{\alpha_{il}\}_{i=1}^{N},\ t\right) d\mu(t) + \int\limits_{\left[\tau^{(M-1)};\ \tau^{(M)}\right]} f_n\left(\{\alpha_{iM}\}_{i=1}^{N},\ t\right) d\mu(t). \end{aligned} \tag{5.15}$$

Proof . Situation $\{\alpha_{il}\}_{i=1}^{N}$ is tied to half-interval $\left[\tau^{(l-1)};\ \tau^{(l)}\right)$ by $l = \overline{1,\ M-1}$ and to interval $\left[\tau^{(M-1)};\ \tau^{(M)}\right]$ by $l = M$. Function (4.5) in this situation is some function of time t. Denote this function by $\psi_{nl}(t)$. For situation $\{\alpha_{il}\}_{i=1}^{N}$ function

$$\psi_{nl}(t) = 0 \ \ \forall\, t \notin \left[\tau^{(l-1)};\ \tau^{(l)}\right), \tag{5.16}$$

and for situation $\{\alpha_{iM}\}_{i=1}^{N}$ function

$$\psi_{nM}(t) = 0 \ \ \forall\, t \notin \left[\tau^{(M-1)};\ \tau^{(M)}\right]. \tag{5.17}$$

Therefore,

$$f_n\left(\{x_i(t)\}_{i=1}^{N},\ t\right) = \sum_{l=1}^{M} \psi_{nl}(t) \tag{5.18}$$

in a pure-strategy situation $\{x_i(t)\}_{i=1}^{N}$ of staircase game (5.7), by using (5.16) and (5.17). Consequently,

$$\begin{aligned}
K_n\left(\{x_i(t)\}_{i=1}^{N}\right) &= \int\limits_{[t_1;\, t_2]} f_n\left(\{x_i(t)\}_{i=1}^{N},\ t\right) d\mu(t) \\
&= \sum_{l=1}^{M-1} \int\limits_{\left[\tau^{(l-1)};\, \tau^{(l)}\right)} \psi_{nl}(t)\, d\mu(t) + \int\limits_{\left[\tau^{(M-1)};\, \tau^{(M)}\right]} \psi_{nM}(t)\, d\mu(t) \\
&= \sum_{l=1}^{M-1} \int\limits_{\left[\tau^{(l-1)};\, \tau^{(l)}\right)} f_n\left(\{\alpha_{il}\}_{i=1}^{N},\ t\right) d\mu(t) + \int\limits_{\left[\tau^{(M-1)};\, \tau^{(M)}\right]} f_n\left(\{\alpha_{iM}\}_{i=1}^{N},\ t\right) d\mu(t) \\
&= \sum_{l=1}^{M} K_n\left(\{\alpha_{il}\}_{i=1}^{N}\right)
\end{aligned} \tag{5.19}$$

in a pure-strategy situation $\{x_i(t)\}_{i=1}^{N}$ of staircase game (5.7). □

So, now it is clear why time t is explicitly included into function (4.5). If time t is not explicitly included into function (4.5) under integral in (4.4), then the payoff value would depend only on the subinterval length (in the same situation on different subintervals). If the subinterval length does not change, there are M identical (ordinary) continuous games (5.9) by (5.11) — (5.13). The triviality of the equal-length-subinterval case is explained by a standstill of the players' strategies. Time variable t therefore is explicitly included into (5.12) and (5.13) to make the system change (and make the players modify their actions) as time goes by.

When the staircase-function game (5.7) is studied, Theorem 5.1 allows considering each game (5.9) separately by using the subinterval-wise summing in (5.15). Although Theorem 5.1 does not provide a method of solving the N-person staircase-function game, it provides a fundamental decomposition of the game. By this decomposition each subinterval game (5.9) can be solved separately, whereupon the subinterval games solutions are stacked (stitched) together owing to the subinterval-wise summing in (5.15).

6 When a Pareto-efficient stack is single

The occurrence when every subinterval N-person game has a single Pareto-efficient situation is rare. The likelihood of such an occurrence even for finite staircase-function games is roughly less than 1 %. Nevertheless, there is an interesting assertion addressed to this case.

Theorem 6.1. If each of M games (5.9) by (5.1) — (5.6) and (5.11) — (5.13) has a single Pareto-efficient situation, then the respective N-person staircase-function game (5.7) has a single Pareto-efficient situation, which is the stack of successive Pareto-efficient situations in games (5.9).

Proof . Let

$$\{\alpha_{il}^{*}\}_{i=1}^{N} \tag{6.1}$$

be the single efficient situation in the game on "stair" subinterval l. This implies that a set of N simultaneous strict inequalities

$$K_n\left(\{\alpha_{il}\}_{i=1}^{N}\right) > K_n\left(\{\alpha_{il}^{*}\}_{i=1}^{N}\right) \text{ for } n = \overline{1,\ N} \tag{6.2}$$

is impossible for any

$$\alpha_{il} \in \left[x_i^{(\min)};\ x_i^{(\max)}\right] \text{ for } l = \overline{1,\ M} \text{ by } i = \overline{1,\ N} \tag{6.3}$$

and none of N strict inequalities in (6.2) is possible taken separately, but $\exists\, n_0 \in \left\{\overline{1,\ N}\right\}$ such that

$$K_{n_0}\left(\{\alpha_{il}\}_{i=1}^{N}\right) = K_{n_0}\left(\{\alpha_{il}^{*}\}_{i=1}^{N}\right) \tag{6.4}$$

is possible by $\alpha_{il} \neq \alpha_{il}^{*}$. Then a set of N simultaneous strict inequalities

$$\sum_{l=1}^{M} K_n\left(\{\alpha_{il}\}_{i=1}^{N}\right) > \sum_{l=1}^{M} K_n\left(\{\alpha_{il}^{*}\}_{i=1}^{N}\right) \text{ for } n = \overline{1,\ N} \tag{6.5}$$

is impossible for any (6.3), but $\exists\, n_0 \in \left\{\overline{1,\ N}\right\}$ such that

$$\sum_{l=1}^{M} K_{n_0}\left(\{\alpha_{il}\}_{i=1}^{N}\right) = \sum_{l=1}^{M} K_{n_0}\left(\{\alpha_{il}^{*}\}_{i=1}^{N}\right) \tag{6.6}$$

is possible by $\alpha_{il} \neq \alpha_{il}^{*}$. By the efficiency definition, owing to Theorem 5.1, this implies that stack

$$\left\{\{\alpha_{il}^{*}\}_{i=1}^{N}\right\}_{l=1}^{M} \tag{6.7}$$

is a Pareto-efficient situation in the respective N-person staircase-function game (5.7). Suppose that there is another stack which is also Pareto-efficient. Consider the case when $M = 2$ (there are 2 “stair” subintervals, i. e. a player can make 2 elementary actions, and the staircase-function strategy may change just once). Let stack

$$\left\{\left\{\{\alpha_{i1}^{*}\}_{i=1}^{N} \setminus \{\alpha_{k1}^{*}\}_{k\in I}\right\} \cup \left\{\alpha_{k1}^{(0)}\right\}_{k\in I},\ \{\alpha_{i2}^{*}\}_{i=1}^{N}\right\} \tag{6.8}$$

be a Pareto-efficient situation by $\alpha_{k1}^{(0)} \neq \alpha_{k1}^{*}\ \forall\, k \in I \subset \left\{\overline{1,\ N}\right\}$. This implies that a set of N simultaneous strict inequalities

$$K_n\left(\{\alpha_{i1}\}_{i=1}^{N}\right) + K_n\left(\{\alpha_{i2}\}_{i=1}^{N}\right) > K_n\left(\left\{\{\alpha_{i1}^{*}\}_{i=1}^{N} \setminus \{\alpha_{k1}^{*}\}_{k\in I}\right\} \cup \left\{\alpha_{k1}^{(0)}\right\}_{k\in I}\right) + K_n\left(\{\alpha_{i2}^{*}\}_{i=1}^{N}\right) \ \text{ for } \ n = \overline{1,\ N} \tag{6.9}$$

is impossible for any (6.3) by $M = 2$. Plugging $\alpha_{i2} = \alpha_{i2}^{*}\ \forall\, i = \overline{1,\ N}$ in the left sides of inequalities (6.9) gives a set of N simultaneous strict inequalities

$$K_n\left(\{\alpha_{i1}\}_{i=1}^{N}\right) > K_n\left(\left\{\{\alpha_{i1}^{*}\}_{i=1}^{N} \setminus \{\alpha_{k1}^{*}\}_{k\in I}\right\} \cup \left\{\alpha_{k1}^{(0)}\right\}_{k\in I}\right) \ \text{ for } \ n = \overline{1,\ N}. \tag{6.10}$$

If the set of simultaneous inequalities (6.10) is impossible then situation

$$\left\{\{\alpha_{i1}^{*}\}_{i=1}^{N} \setminus \{\alpha_{k1}^{*}\}_{k\in I}\right\} \cup \left\{\alpha_{k1}^{(0)}\right\}_{k\in I} \tag{6.11}$$

must be efficient. Therefore, the supposition about Pareto-efficiency of situation (6.8) is contradictory. The Pareto-efficiency impossibility of any other versions of M-subinterval stacks (when a player can make more than just 2 elementary actions) is proved similarly by ascending induction. $\square$

So, if each of the subinterval N-person games has a single Pareto-efficient solution, Theorem 6.1 allows finding the Pareto-efficient solution of the respective N-person staircase-function game in a very simple way, just by stacking the subinterval solutions. It is easy to see that the assertion of Theorem 6.1 is reversible.

Theorem 6.2. If an N-person staircase-function game (5.7) has a single Pareto-efficient situation, then each of the respective M games (5.9) by (5.1) — (5.6) and (5.11) — (5.13) has a single Pareto-efficient situation.

Proof . Let stack (6.7) be a single Pareto-efficient situation in an N-person staircase-function game (5.7). This implies that a set of simultaneous inequalities (6.5) is impossible for any (6.3). Plugging $\alpha_{il} = \alpha_{il}^{*}\ \forall\, i = \overline{1,\ N}$ and $\forall\, l = \overline{2,\ M}$ in the left sides of inequalities (6.5) gives a set of N simultaneous strict inequalities

$$K_n\left(\{\alpha_{i1}\}_{i=1}^{N}\right) > K_n\left(\{\alpha_{i1}^{*}\}_{i=1}^{N}\right) \ \text{ for } \ n = \overline{1,\ N}, \tag{6.12}$$

which is impossible as well. Hence, situation

$$\{\alpha_{i1}^{*}\}_{i=1}^{N} \tag{6.13}$$

in the game

$$\left\langle \left\{\left[x_n^{(\min)};\ x_n^{(\max)}\right]\right\}_{n=1}^{N},\ \left\{K_n\left(\{\alpha_{i1}\}_{i=1}^{N}\right)\right\}_{n=1}^{N} \right\rangle \tag{6.14}$$

on the first subinterval $\left[\tau^{(0)};\ \tau^{(1)}\right)$ is efficient. The efficiency of the remaining subinterval situations is proved in the same way.

Suppose that, along with efficient situation (6.13), situation

$$\left\{\alpha_{i1}^{(0)}\right\}_{i=1}^{N} \tag{6.15}$$

in game (6.14) is efficient also. Thus, a set of N simultaneous strict inequalities

$$K_n\left(\{\alpha_{i1}\}_{i=1}^{N}\right) > K_n\left(\left\{\alpha_{i1}^{(0)}\right\}_{i=1}^{N}\right) \quad \text{for} \quad n=\overline{1,\ N} \tag{6.16}$$

is impossible for any

$$\alpha_{i1} \in \left[x_i^{(\min)};\ x_i^{(\max)}\right] \quad \text{by} \quad i=\overline{1,\ N}. \tag{6.17}$$

Stack

$$\left\{\left\{\alpha_{i1}^{(0)}\right\}_{i=1}^{N},\ \left\{\{\alpha_{il}^{*}\}_{i=1}^{N}\right\}_{l=2}^{M}\right\} \tag{6.18}$$

must not be efficient. This implies that a set of N simultaneous nonstrict inequalities

$$K_n\left(\left\{\alpha_{i1}^{(0)}\right\}_{i=1}^{N}\right) + \sum_{l=2}^{M} K_n\left(\{\alpha_{il}\}_{i=1}^{N}\right) \leqslant K_n\left(\{\alpha_{i1}^{*}\}_{i=1}^{N}\right) + \sum_{l=2}^{M} K_n\left(\{\alpha_{il}^{*}\}_{i=1}^{N}\right) \quad \text{for} \quad n=\overline{1,\ N} \tag{6.19}$$

holds and $\exists\, n_0 \in \left\{\overline{1,\ N}\right\}$ such that

$$K_{n_0}\left(\left\{\alpha_{i1}^{(0)}\right\}_{i=1}^{N}\right) + \sum_{l=2}^{M} K_{n_0}\left(\{\alpha_{il}\}_{i=1}^{N}\right) < K_{n_0}\left(\{\alpha_{i1}^{*}\}_{i=1}^{N}\right) + \sum_{l=2}^{M} K_{n_0}\left(\{\alpha_{il}^{*}\}_{i=1}^{N}\right) \tag{6.20}$$

holds for any

$$\alpha_{il} \in \left[x_i^{(\min)};\ x_i^{(\max)}\right] \quad \text{for} \quad l=\overline{2,\ M} \quad \text{by} \quad i=\overline{1,\ N}. \tag{6.21}$$

Plugging $\alpha_{il} = \alpha_{il}^{*}\ \forall\, i=\overline{1,\ N}$ and $\forall\, l=\overline{2,\ M}$ in the left sides of inequalities (6.19) gives a set of N simultaneous nonstrict inequalities

$$K_n\left(\left\{\alpha_{i1}^{(0)}\right\}_{i=1}^{N}\right) \leqslant K_n\left(\{\alpha_{i1}^{*}\}_{i=1}^{N}\right) \quad \text{for} \quad n=\overline{1,\ N} \tag{6.22}$$

where $\exists\, n_0 \in \left\{\overline{1,\ N}\right\}$ such that

$$K_{n_0}\left(\left\{\alpha_{i1}^{(0)}\right\}_{i=1}^{N}\right) < K_{n_0}\left(\{\alpha_{i1}^{*}\}_{i=1}^{N}\right). \tag{6.23}$$

But (6.22) and (6.23) mean that situation (6.15) is not efficient. Such a contradiction is similarly proved for any other subinterval situation and any other combinations of subinterval situations. □

So, Theorem 6.2 asserts that when a Pareto-efficient stack is single, it does directly mean that every subinterval N-person game must have a single Pareto-efficient situation. The question about multiple Pareto-efficient stacks is cleared right below.

7 What a Pareto-efficient stack consists of

The player in a finite N-person staircase-function game (5.7) may have multiple Pareto-efficient strategies. For example, a 3-person game with 2-subinterval 3-staircased function-strategies at the first player, 2-subinterval 4-staircased function-strategies at the second player, and 2-subinterval 2-staircased function-strategies at the third player represented with respective three-dimensional matrices

$$\mathbf{F}_1 = \left[\begin{bmatrix} 3 & -1 & 3 & 3 \\ 5 & 2 & 4 & 1 \\ -1 & 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} 3 & -3 & 3 & 4 \\ 4 & 1 & 4 & -1 \\ 3 & -1 & 1 & 0 \end{bmatrix}\right],$$

$$\mathbf{F}_2 = \left[\begin{bmatrix} -1 & 5 & 1 & 3 \\ 2 & -2 & -2 & -1 \\ 1 & 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} -1 & 2 & 1 & 1 \\ 5 & 3 & 4 & 4 \\ -3 & 2 & 0 & 5 \end{bmatrix}\right] \tag{7.1}$$

and

$$\mathbf{G}_1 = \begin{bmatrix} \begin{bmatrix} 0 & 2 & 0 & -2 \\ 0 & 1 & 3 & 1 \\ 1 & 4 & 2 & -1 \end{bmatrix} & \begin{bmatrix} 3 & 1 & 3 & -3 \\ 1 & 3 & -1 & 1 \\ 1 & -1 & 3 & 5 \end{bmatrix} \end{bmatrix},$$

$$\mathbf{G}_2 = \begin{bmatrix} \begin{bmatrix} 4 & 4 & 2 & 6 \\ 1 & 2 & 1 & 0 \\ 4 & 0 & 1 & 0 \end{bmatrix} & \begin{bmatrix} 2 & 4 & 3 & 3 \\ 4 & 1 & 1 & -2 \\ 4 & -2 & 2 & 3 \end{bmatrix} \end{bmatrix} \tag{7.2}$$

and

$$\mathbf{H}_1 = \begin{bmatrix} \begin{bmatrix} 1 & 3 & -1 & 0 \\ 1 & 2 & -1 & -1 \\ 2 & 3 & 1 & 4 \end{bmatrix} & \begin{bmatrix} -3 & 1 & 2 & 1 \\ -2 & -2 & 3 & -2 \\ 2 & 2 & -1 & 2 \end{bmatrix} \end{bmatrix},$$

$$\mathbf{H}_2 = \begin{bmatrix} \begin{bmatrix} -2 & 6 & 1 & 1 \\ 2 & -4 & 2 & 1 \\ 3 & 5 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 2 & 2 & -3 & 1 \\ 4 & 4 & 2 & 2 \\ 1 & 2 & 2 & -1 \end{bmatrix} \end{bmatrix} \tag{7.3}$$

has 6 Pareto-efficient situations (each situation is a 2-subinterval stack). They are the stacks of two efficient situations with payoffs:

$$\{5,\ 0,\ 1\} \ \text{ and } \ \{5,\ 4,\ 6\},$$
$$\{4,\ 4,\ 3\} \ \text{ and } \ \{5,\ 4,\ 6\},$$
$$\{4,\ 4,\ 3\} \ \text{ and } \ \{3,\ 6,\ 1\},$$
$$\{2,\ -1,\ 4\} \ \text{ and } \ \{5,\ 4,\ 6\},$$
$$\{0,\ 5,\ 2\} \ \text{ and } \ \{5,\ 4,\ 6\},$$
$$\{0,\ 5,\ 2\} \ \text{ and } \ \{3,\ 6,\ 1\},$$

where the respective payoffs (in the finite 3-person staircase-function game) are

$$\begin{gathered} \{10,\ 4,\ 7\}, \\ \{9,\ 8,\ 9\}, \\ \{7,\ 10,\ 4\}, \\ \{7,\ 3,\ 10\}, \\ \{5,\ 9,\ 8\}, \\ \{3,\ 11,\ 3\}. \end{gathered} \tag{7.4}$$

By the way, the stack of efficient situations with payoffs $\{5,\ 0,\ 1\}$ and $\{3,\ 6,\ 1\}$ is not an efficient situation because its payoffs $\{8,\ 6,\ 2\}$ are less than payoffs (7.4). The stack of efficient situations with payoffs $\{2,\ -1,\ 4\}$ and $\{3,\ 6,\ 1\}$ is not efficient also because its payoffs $\{5,\ 5,\ 5\}$ are less than payoffs (7.4). Obviously, a continuous N-person staircase-function game may have multiple Pareto-efficient situations as well.

Theorem 7.1. Any Pareto-efficient situation in an N-person staircase-function game (5.7) is a stack of successive Pareto-efficient situations in games (5.9) by (5.1) — (5.6) and (5.11) — (5.13).

Proof . Let stack (6.7) be a Pareto-efficient situation in the respective N-person staircase-function game (5.7), where (6.1) is a Pareto-efficient situation in the game on "stair" subinterval i. Suppose that, on the first "stair" subinterval, situation (6.15) is not efficient in game (6.14), but stack (6.18) is an efficient situation in staircase-function game (5.7). Then at least one inequality of N inequalities

$$K_n\left(\left\{\alpha_{i1}^{(0)}\right\}_{i=1}^{N}\right) + \sum_{l=2}^{M} K_n\left(\{\alpha_{il}^*\}_{i=1}^{N}\right) \quad \geqslant \quad \geqslant K_n\left(\{\alpha_{i1}^*\}_{i=1}^{N}\right) + \sum_{l=2}^{M} K_n\left(\{\alpha_{il}^*\}_{i=1}^{N}\right) \ \text{ for } \ n = \overline{1,\ N} \tag{7.5}$$

must hold. Suppose that $\exists\, n_0 \in \left\{\overline{1,\ N}\right\}$ such that inequality

$$K_{n_0}\left(\left\{\alpha_{i1}^{(0)}\right\}_{i=1}^{N}\right) + \sum_{l=2}^{M} K_{n_0}\left(\left\{\alpha_{il}^{*}\right\}_{i=1}^{N}\right) \geqslant K_{n_0}\left(\left\{\alpha_{i1}^{*}\right\}_{i=1}^{N}\right) + \sum_{l=2}^{M} K_{n_0}\left(\left\{\alpha_{il}^{*}\right\}_{i=1}^{N}\right) \tag{7.6}$$

holds. Then inequality

$$K_{n_0}\left(\left\{\alpha_{i1}^{(0)}\right\}_{i=1}^{N}\right) \geqslant K_{n_0}\left(\left\{\alpha_{i1}^{*}\right\}_{i=1}^{N}\right) \tag{7.7}$$

holds. As in game (6.14) situation (6.15) is not efficient, inequality (7.7) is only possible as equality

$$K_{n_0}\left(\left\{\alpha_{i1}^{(0)}\right\}_{i=1}^{N}\right) = K_{n_0}\left(\left\{\alpha_{i1}^{*}\right\}_{i=1}^{N}\right). \tag{7.8}$$

This directly gives the set of N simultaneous inequalities

$$K_{n}\left(\left\{\alpha_{i1}^{(0)}\right\}_{i=1}^{N}\right) + \sum_{l=2}^{M} K_{n}\left(\left\{\alpha_{il}^{*}\right\}_{i=1}^{N}\right) \quad \leqslant \quad K_{n}\left(\left\{\alpha_{i1}^{*}\right\}_{i=1}^{N}\right) + \sum_{l=2}^{M} K_{n}\left(\left\{\alpha_{il}^{*}\right\}_{i=1}^{N}\right) \;\text{ for }\; n = \overline{1,\ N}. \tag{7.9}$$

The set of inequalities (7.9) means that stack (6.18) is not an efficient situation in staircase-function game (5.7). Such contradictions implying that stack (6.18) cannot be efficient are similarly proved for any other subinterval situation. Suppose now that situations (6.15) and

$$\left\{\alpha_{i2}^{(0)}\right\}_{i=1}^{N} \tag{7.10}$$

are not efficient in the first two subinterval games, but stack

$$\left\{\left\{\alpha_{i1}^{(0)}\right\}_{i=1}^{N},\ \left\{\alpha_{i2}^{(0)}\right\}_{i=1}^{N},\ \left\{\left\{\alpha_{il}^{*}\right\}_{i=1}^{N}\right\}_{l=3}^{M}\right\} \tag{7.11}$$

is an efficient situation in staircase-function game (5.7). Then at least one inequality of N inequalities

$$\begin{gathered} K_{n}\left(\left\{\alpha_{i1}^{(0)}\right\}_{i=1}^{N}\right) + K_{n}\left(\left\{\alpha_{i2}^{(0)}\right\}_{i=1}^{N}\right) + \sum_{l=3}^{M} K_{n}\left(\left\{\alpha_{il}^{*}\right\}_{i=1}^{N}\right) \\ \geqslant K_{n}\left(\left\{\alpha_{i1}^{*}\right\}_{i=1}^{N}\right) + K_{n}\left(\left\{\alpha_{i2}^{*}\right\}_{i=1}^{N}\right) + \sum_{l=3}^{M} K_{n}\left(\left\{\alpha_{il}^{*}\right\}_{i=1}^{N}\right) \;\text{ for }\; n = \overline{1,\ N} \end{gathered} \tag{7.12}$$

must hold. Suppose that $\exists\, n_0 \in \left\{\overline{1,\ N}\right\}$ such that inequality

$$\begin{gathered} K_{n_0}\left(\left\{\alpha_{i1}^{(0)}\right\}_{i=1}^{N}\right) + K_{n_0}\left(\left\{\alpha_{i2}^{(0)}\right\}_{i=1}^{N}\right) + \sum_{l=3}^{M} K_{n_0}\left(\left\{\alpha_{il}^{*}\right\}_{i=1}^{N}\right) \\ \geqslant K_{n_0}\left(\left\{\alpha_{i1}^{*}\right\}_{i=1}^{N}\right) + K_{n_0}\left(\left\{\alpha_{i2}^{*}\right\}_{i=1}^{N}\right) + \sum_{l=3}^{M} K_{n_0}\left(\left\{\alpha_{il}^{*}\right\}_{i=1}^{N}\right) \end{gathered} \tag{7.13}$$

holds. Then inequality

$$K_{n_0}\left(\left\{\alpha_{i1}^{(0)}\right\}_{i=1}^{N}\right) + K_{n_0}\left(\left\{\alpha_{i2}^{(0)}\right\}_{i=1}^{N}\right) \geqslant K_{n_0}\left(\left\{\alpha_{i1}^{*}\right\}_{i=1}^{N}\right) + K_{n_0}\left(\left\{\alpha_{i2}^{*}\right\}_{i=1}^{N}\right) \tag{7.14}$$

holds. As situations (6.15) and (7.10) both are not efficient, inequality (7.14) is only possible as equality

$$K_{n_0}\left(\left\{\alpha_{i1}^{(0)}\right\}_{i=1}^{N}\right) + K_{n_0}\left(\left\{\alpha_{i2}^{(0)}\right\}_{i=1}^{N}\right) = K_{n_0}\left(\left\{\alpha_{i1}^{*}\right\}_{i=1}^{N}\right) + K_{n_0}\left(\left\{\alpha_{i2}^{*}\right\}_{i=1}^{N}\right). \tag{7.15}$$

This directly gives the set of N simultaneous inequalities

$$\begin{gathered} K_{n}\left(\left\{\alpha_{i1}^{(0)}\right\}_{i=1}^{N}\right) + K_{n}\left(\left\{\alpha_{i2}^{(0)}\right\}_{i=1}^{N}\right) + \sum_{l=3}^{M} K_{n}\left(\left\{\alpha_{il}^{*}\right\}_{i=1}^{N}\right) \\ \leqslant K_{n}\left(\left\{\alpha_{i1}^{*}\right\}_{i=1}^{N}\right) + K_{n}\left(\left\{\alpha_{i2}^{*}\right\}_{i=1}^{N}\right) + \sum_{l=3}^{M} K_{n}\left(\left\{\alpha_{il}^{*}\right\}_{i=1}^{N}\right) \;\text{ for }\; n = \overline{1,\ N}. \end{gathered} \tag{7.16}$$

The set of inequalities (7.16) means that stack (7.11) is not an efficient situation in staircase-function game (5.7). Such contradictions implying that stack (7.11) cannot be efficient are similarly proved for any other two subinterval situations. Furthermore, by using the considered ascending induction, such contradictions implying that a stack including non-efficient subinterval situations cannot be efficient are similarly proved for any number of situations. □

So, Theorem 7.1 does answer the question of what a Pareto-efficient stack consists of. Every efficient situation in an N-person staircase-function game (5.7) is built out of Pareto-efficient situations in "stair" subinterval games. Theorem 7.1 does not mean that any stack of successive efficient situations will be efficient, which has been illustrated above by the example of the games with matrices (7.1) — (7.3). However, what Theorem 7.1 directly implies is that if every subinterval (finite or continuous) game has a finite number of Pareto-efficient situations, then the number of all the Pareto-efficient situations (stacks) in the respective N-person staircase-function game (5.7) is finite. These Pareto-efficient stacks can be determined by just running over all possible stacks (whose number is finite) and selecting such stacks (6.7) whose payoffs (determined as subinterval-wise summing owing to Theorem 5.1) are efficient.

8 Solving a finite N-person staircase-function game

In a finite N-person staircase-function game, players (forcedly or deliberately) act within a finite subset of possible values of their pure strategies. That is, these values are

$$x_n^{(\min)} = x_n^{(0)} < x_n^{(1)} < x_n^{(2)} < \ldots < x_n^{(Q_n-1)} < x_n^{(Q_n)} = x_n^{(\max)} \tag{8.1}$$

for the n-th player, $Q_n \in \mathbb{N}\ \forall\, n = \overline{1,\ N}$ (i. e., the player's function-strategy must have at least two different values). Then the pure strategy set of the n-th player in finite N-person staircase-function game (5.7) is

$$\mathcal{X}_n^{(M)}(Q_n) = \left\{ x_n(t) :\ x_n(t) \in \left\{ x_n^{(m_n-1)} \right\}_{m_n=1}^{Q_n+1} \forall\, t \in \left[\tau^{(l-1)};\ \tau^{(l)}\right) \right. \\ \left. \text{for } l = \overline{1,\ M-1} \text{ and } \forall\, t \in \left[\tau^{(M-1)};\ \tau^{(M)}\right] \right\} \subset \mathcal{X}_n^{(M)} \subset \mathcal{X}_n \ \text{ by } \ n = \overline{1,\ N}. \tag{8.2}$$

Subsequently, the succession of M continuous games (5.9) by (5.1) — (5.6) and (5.11) — (5.13) becomes a succession of M finite (N-dimensional-matrix) games

$$\left\langle \left\{ \left\{ x_i^{(m_i-1)} \right\}_{m_i=1}^{Q_i+1} \right\}_{i=1}^{N},\ \left\{ \mathbf{H}_{il} \right\}_{i=1}^{N} \right\rangle \tag{8.3}$$

with the n-th player's payoff matrix

$$\mathbf{H}_{nl} = \left[h_{nl\boldsymbol{\Omega}} \right]_{\mathscr{F}} \tag{8.4}$$

whose format is

$$\mathscr{F} = \bigtimes_{n=1}^{N} (Q_n + 1) \tag{8.5}$$

and elements are

$$h_{nl\boldsymbol{\Omega}} = \int\limits_{\left[\tau^{(l-1)};\ \tau^{(l)}\right)} f_n\left(\left\{ x_i^{(m_i-1)} \right\}_{i=1}^{N},\ t \right) d\mu(t) \ \text{ for } \ l = \overline{1,\ M-1} \tag{8.6}$$

and

$$h_{nM\boldsymbol{\Omega}} = \int\limits_{\left[\tau^{(M-1)};\ \tau^{(M)}\right]} f_n\left(\left\{ x_i^{(m_i-1)} \right\}_{i=1}^{N},\ t \right) d\mu(t) \tag{8.7}$$

by indexing

$$\boldsymbol{\Omega} = \left\{ \omega_k \right\}_{k=1}^{N},\ \ \omega_k \in \left\{ \overline{1,\ Q_k+1} \right\} \ \ \forall\, k = \overline{1,\ N}. \tag{8.8}$$

Let N-dimensional-matrix game (8.3) have J_l efficient situations, $J_l \in \mathbb{N}$. And let

$$\left\{ \alpha_{ilj_l}^{*} \right\}_{i=1}^{N} \tag{8.9}$$

be an efficient situation in this game, where $j_l \in \left\{\overline{1, J_l}\right\}$. It is unknown whether "participation" of situation (8.9) in a stack makes the stack efficient or not. There are altogether

$$A = \prod_{l=1}^{M} J_l \tag{8.10}$$

stacks of M subinterval Pareto-efficient situations. Let

$$\left\{\left\{\alpha^*_{ilj_l}\right\}_{i=1}^{N}\right\}_{l=1}^{M} \tag{8.11}$$

be a stack in an N-person staircase-function game, which is the succession of M N-dimensional-matrix games (8.3), where

$$\alpha^*_{ilj_l} \in \left\{x_i^{(m_i-1)}\right\}_{m_i=1}^{Q_i+1}.$$

Thus, stack (8.11) produces payoffs

$$\left\{h^*_{nu}\right\}_{n=1}^{N} = \left\{\sum_{l=1}^{M} K_n\left(\left\{\alpha^*_{ilj_l}\right\}_{i=1}^{N}\right)\right\}_{n=1}^{N} \quad \text{by} \quad u = \overline{1, A}. \tag{8.12}$$

Without losing generality, presume that, after sorting all the possible stacks just by separating the efficient from the non-efficient stacks, namely the first U payoffs in (8.12), where

$$U \in \left\{\overline{1, A}\right\},$$

are produced by the efficient stacks (for instance, this can be done). So, U is the number of efficient stacks. It is worth to remember that the case of when $U = 1$ is only possible if $J_l = 1 \; \forall \, l = \overline{1, M}$ (see Theorem 6.2). The best efficient stack can be found by a method suggested in [25]. Payoffs (8.12) are 0-1-standardized and a u_*-th stack is found at which the respective efficient payoffs

$$\left\{h^*_{nu_*}\right\}_{n=1}^{N}$$

are the farthest from the zero payoffs

$$\{0\}_{n=1}^{N} \tag{8.13}$$

(the most unprofitable payoffs, which constitute the origin in the N-dimensional space $\mathbb{R}^N$):

$$u_* \in \arg\max_{u=\overline{1, U}} \sqrt{\sum_{n=1}^{N}\left(\frac{h^*_{nu} - \min\limits_{k=\overline{1, U}} h^*_{nk}}{\max\limits_{k=\overline{1, U}} h^*_{nk} - \min\limits_{k=\overline{1, U}} h^*_{nk}}\right)^2}. \tag{8.14}$$

Thus, (8.14) provides the u_*-th stack to be the best.

9 A 3-subinterval example with 4 players

Consider a game example in which 4 players act using staircase-function strategies during $t \in [1.4\pi; \; 2\pi]$ by

$$\left\{\tau^{(l)}\right\}_{l=0}^{3} = \{1.4\pi, \; 1.6\pi, \; 1.8\pi, \; 2\pi\} \tag{9.1}$$

and their sets

$$\begin{aligned}\mathcal{X}_1^{(3)}(2) &= \left\{x_1(t): \; x_1(t) \in \{2, \; 2.5, \; 3\} \; \forall \, t \in \left[\tau^{(l-1)}; \; \tau^{(l)}\right) \text{ for } l = \overline{1, 2} \text{ and } \forall \, t \in \left[\tau^{(2)}; \; \tau^{(3)}\right]\right\} \\ &\subset \mathcal{X}_1^{(3)} \subset \mathcal{X}_1,\end{aligned} \tag{9.2}$$

$$\begin{aligned}\mathcal{X}_2^{(3)}(3) &= \; = \left\{x_2(t): \; x_2(t) \in \{0.7, \; 0.8, \; 0.9, \; 1\} \; \forall \, t \in \left[\tau^{(l-1)}; \; \tau^{(l)}\right) \text{ for } l = \overline{1, 2} \text{ and } \forall \, t \in \left[\tau^{(2)}; \; \tau^{(3)}\right]\right\} \\ &\subset \mathcal{X}_2^{(3)} \subset \mathcal{X}_2,\end{aligned} \tag{9.3}$$

$$\mathcal{X}_3^{(3)}(1) = \left\{x_3(t):\ x_3(t) \in \{0.5,\ 0.75\}\ \forall\, t \in \left[\tau^{(l-1)};\ \tau^{(l)}\right)\ \text{for}\ l = \overline{1,\ 2}\ \text{and}\ \forall\, t \in \left[\tau^{(2)};\ \tau^{(3)}\right]\right\} \subset \mathcal{X}_3^{(3)} \subset \mathcal{X}_3, \tag{9.4}$$

$$\mathcal{X}_4^{(3)}(4) = \left\{x_4(t):\ x_4(t) \in \{4,\ 5,\ 6,\ 7,\ 8\}\ \forall\, t \in \left[\tau^{(l-1)};\ \tau^{(l)}\right)\ \text{for}\ l = \overline{1,\ 2}\ \text{and}\ \forall\, t \in \left[\tau^{(2)};\ \tau^{(3)}\right]\right\} \subset \mathcal{X}_4^{(3)} \subset \mathcal{X}_4. \tag{9.5}$$

So, by (9.1), the player can change the value of one's pure strategy only at time points

$$\left\{\tau^{(l)}\right\}_{l=1}^{2} = \{1.6\pi,\ 1.8\pi\}. \tag{9.6}$$

The players' payoff functionals (4.4) are

$$K_1\big(x_1(t),\ x_2(t),\ x_3(t),\ x_4(t)\big) = \int\limits_{[1.4\pi;\ 2\pi]} \sin\left(0.8x_1x_2x_3t + \frac{\pi}{12}\right) d\mu(t), \tag{9.7}$$

$$K_2\big(x_1(t),\ x_2(t),\ x_3(t),\ x_4(t)\big) = \int\limits_{[1.4\pi;\ 2\pi]} \sin\left(0.7x_2x_3x_4t - \frac{\pi}{3}\right) d\mu(t), \tag{9.8}$$

$$K_3\big(x_1(t),\ x_2(t),\ x_3(t),\ x_4(t)\big) = \int\limits_{[1.4\pi;\ 2\pi]} \sin\left(2.9x_1x_3x_4t - \frac{7\pi}{8}\right) d\mu(t), \tag{9.9}$$

$$K_4\big(x_1(t),\ x_2(t),\ x_3(t),\ x_4(t)\big) = \int\limits_{[1.4\pi;\ 2\pi]} \sin\left(2.3x_1x_3x_4t + \frac{5\pi}{9}\right) d\mu(t). \tag{9.10}$$

Owing to (9.1) — (9.6), this game can be thought of as a succession of three finite $3 \times 4 \times 2 \times 5$ (quadmatrix) games (8.3):

$$\left\langle \left\{\{2,\ 2.5,\ 3\},\ \{0.7,\ 0.8,\ 0.9,\ 1\},\ \{0.5,\ 0.75\},\ \{4,\ 5,\ 6,\ 7,\ 8\}\right\},\ \{\mathbf{H}_{1l},\ \mathbf{H}_{2l},\ \mathbf{H}_{3l},\ \mathbf{H}_{4l}\}\right\rangle\ \text{by}\ l = \overline{1,\ 3} \tag{9.11}$$

with first player's payoff matrices

$$\left\{\mathbf{H}_{1l} = [h_{1l\omega_1\omega_2\omega_3\omega_4}]_{3\times4\times2\times5}\right\}_{l=1}^{3} \tag{9.12}$$

whose elements are

$$h_{1lm_1m_2m_3m_4} = \int\limits_{[1.4\pi+0.2\cdot(l-1)\pi;\ 1.4\pi+0.2l\pi)} f_1\left(x_1^{(m_1-1)},\ x_2^{(m_2-1)},\ x_3^{(m_3-1)},\ x_4^{(m_4-1)},\ t\right) d\mu(t)$$
$$= \int\limits_{[1.4\pi+0.2\cdot(l-1)\pi;\ 1.4\pi+0.2l\pi)} \sin\left(0.8\cdot(1.5+0.5m_1)(0.6+0.1m_2)(0.25+0.25m_3)t + \frac{\pi}{12}\right) d\mu(t)\ \ \text{for}\ \ l = \overline{1,\ 2} \tag{9.13}$$

and

$$h_{1,3m_1m_2m_3m_4} = \int\limits_{[1.8\pi;\ 2\pi]} \sin\left(0.8\cdot(1.5+0.5m_1)(0.6+0.1m_2)(0.25+0.25m_3)t + \frac{\pi}{12}\right) d\mu(t), \tag{9.14}$$

with second player's payoff matrices

$$\left\{\mathbf{H}_{2l} = [h_{2l\omega_1\omega_2\omega_3\omega_4}]_{3\times4\times2\times5}\right\}_{l=1}^{3} \tag{9.15}$$

whose elements are

$$h_{2lm_1m_2m_3m_4} = \int\limits_{[1.4\pi+0.2\cdot(l-1)\pi;\ 1.4\pi+0.2l\pi)} f_2\left(x_1^{(m_1-1)},\ x_2^{(m_2-1)},\ x_3^{(m_3-1)},\ x_4^{(m_4-1)},\ t\right) d\mu\left(t\right)$$
$$= \int\limits_{[1.4\pi+0.2\cdot(l-1)\pi;\ 1.4\pi+0.2l\pi)} \sin\left(0.7\cdot\left(0.6+0.1m_2\right)\left(0.25+0.25m_3\right)\left(3+m_4\right)t-\frac{\pi}{3}\right) d\mu\left(t\right) \text{ for } l=\overline{1,\ 2} \tag{9.16}$$

and

$$h_{2,3m_1m_2m_3m_4} = \int\limits_{[1.8\pi;\ 2\pi]} \sin\left(0.7\cdot\left(0.6+0.1m_2\right)\left(0.25+0.25m_3\right)\left(3+m_4\right)t-\frac{\pi}{3}\right) d\mu\left(t\right), \tag{9.17}$$

with third player's payoff matrices

$$\left\{\mathbf{H}_{3l} = \left[h_{3l\omega_1\omega_2\omega_3\omega_4}\right]_{3\times4\times2\times5}\right\}_{l=1}^{3} \tag{9.18}$$

whose elements are

$$h_{3lm_1m_2m_3m_4} = \int\limits_{[1.4\pi+0.2\cdot(l-1)\pi;\ 1.4\pi+0.2l\pi)} f_3\left(x_1^{(m_1-1)},\ x_2^{(m_2-1)},\ x_3^{(m_3-1)},\ x_4^{(m_4-1)},\ t\right) d\mu\left(t\right) \tag{9.19}$$
$$= \int\limits_{[1.4\pi+0.2\cdot(l-1)\pi;\ 1.4\pi+0.2l\pi)} \sin\left(2.9\cdot\left(1.5+0.5m_1\right)\left(0.25+0.25m_3\right)\left(3+m_4\right)t-\frac{7\pi}{8}\right) d\mu\left(t\right) \text{ for } l=\overline{1,\ 2} \tag{9.20}$$

and

$$h_{3,3m_1m_2m_3m_4} = \int\limits_{[1.8\pi;\ 2\pi]} \sin\left(2.9\cdot\left(1.5+0.5m_1\right)\left(0.25+0.25m_3\right)\left(3+m_4\right)t-\frac{7\pi}{8}\right) d\mu\left(t\right), \tag{9.21}$$

and with fourth player's payoff matrices

$$\left\{\mathbf{H}_{4l} = \left[h_{4l\omega_1\omega_2\omega_3\omega_4}\right]_{3\times4\times2\times5}\right\}_{l=1}^{3} \tag{9.22}$$

whose elements are

$$h_{4lm_1m_2m_3m_4} = \int\limits_{[1.4\pi+0.2\cdot(l-1)\pi;\ 1.4\pi+0.2l\pi)} f_4\left(x_1^{(m_1-1)},\ x_2^{(m_2-1)},\ x_3^{(m_3-1)},\ x_4^{(m_4-1)},\ t\right) d\mu\left(t\right)$$
$$= \int\limits_{[1.4\pi+0.2\cdot(l-1)\pi;\ 1.4\pi+0.2l\pi)} \sin\left(2.3\cdot\left(1.5+0.5m_1\right)\left(0.25+0.25m_3\right)\left(3+m_4\right)t+\frac{5\pi}{9}\right) d\mu\left(t\right) \text{ for } l=\overline{1,\ 2} \tag{9.23}$$

and

$$h_{4,3m_1m_2m_3m_4} = \int\limits_{[1.8\pi;\ 2\pi]} \sin\left(2.3\cdot\left(1.5+0.5m_1\right)\left(0.25+0.25m_3\right)\left(3+m_4\right)t+\frac{5\pi}{9}\right) d\mu\left(t\right). \tag{9.24}$$

It is worth noting that this finite 4-person game is rendered to a quadmatrix $27\times64\times8\times125$ game. Such a quadmatrix game cannot be solved in a reasonable amount of computational time because there are 1728000 pure-strategy situations (searching for efficient situations through this number of situations would take too long).

The three quadmatrix $3\times4\times2\times5$ games (9.11) with (9.12) — (9.24) have 10, 25, 12 Pareto-efficient situations, respectively. Therefore, there are

$$A = \prod_{l=1}^{3} J_l = 10\cdot25\cdot12 = 3000 \tag{9.25}$$

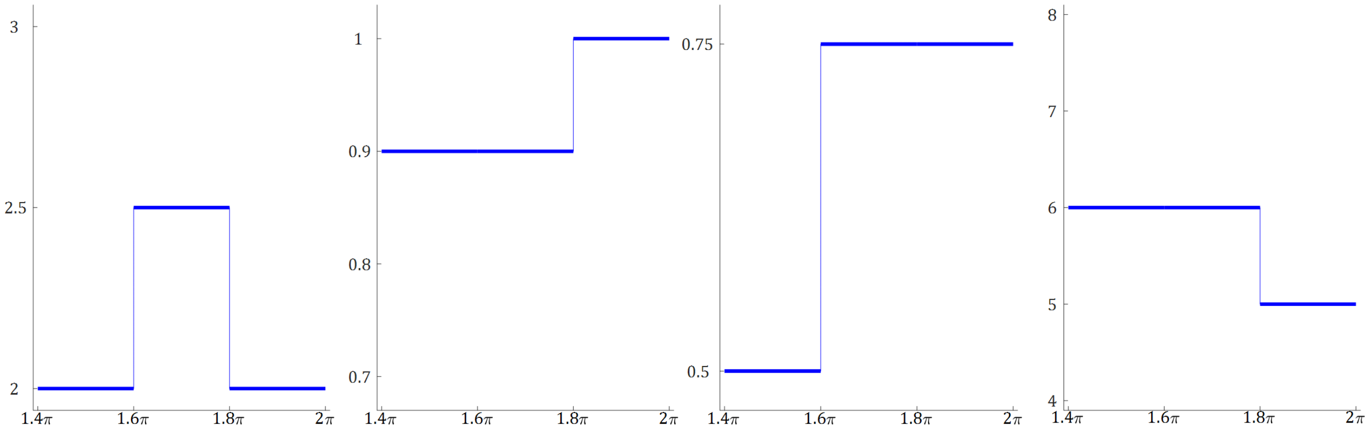

Figure 1: The best efficient staircase-function strategies producing payoffs (9.27) in the 4-person staircase-function game (5.7) by (9.1) — (9.10)

stacks of such situations. The respective 4-person staircase-function game by (9.1) — (9.10) has 474 Pareto-efficient stacks which are a subset of those 3000 ones due to Theorem 7.1. The single best efficient payoffs point calculated by (8.14) as

$$u_* \in \arg \max_{u=\overline{1,\,474}} \sqrt{\sum_{n=1}^{4} \left(\frac{h^*_{nu} - \min\limits_{k=\overline{1,\,474}} h^*_{nk}}{\max\limits_{k=\overline{1,\,474}} h^*_{nk} - \min\limits_{k=\overline{1,\,474}} h^*_{nk}} \right)^2} \tag{9.26}$$

corresponds to the best Pareto-efficient situation, whose players' strategies $\{x^*_n(t)\}_{n=1}^4$ are shown in Figure 1. The best efficient payoffs are

$$\left\{h^*_{nu_*}\right\}_{n=1}^4 = \{0.8177,\ 1.6351,\ 0.0585,\ 0.2466\}. \tag{9.27}$$

Note that payoffs (9.27) are not 0-1-standardized, so they may badly differ (in general, every player has own payoff measurement unit).

If to consider a time-shifted game differing from the game by (9.1) — (9.10) in that the players act during $t \in [1.6\pi;\ 2.2\pi]$ by

$$\left\{\tau^{(l)}\right\}_{l=0}^3 = \{1.6\pi,\ 1.8\pi,\ 2\pi,\ 2.2\pi\} \tag{9.28}$$

and the respective integration interval change in (9.7) — (9.10), (9.13), (9.14), (9.16), (9.17), (9.20), (9.21), (9.23), (9.24), then the respective three quadmatrix $3 \times 4 \times 2 \times 5$ games have 25, 12, 10 Pareto-efficient situations. In fact, there are (9.25) stacks of such situations, again. Now, however, the respective 4-person staircase-function game by (9.2) — (9.5) and (9.28) has 255 Pareto-efficient stacks (which are a subset of those 3000 ones due to Theorem 7.1). The single best efficient payoffs point calculated by (8.14) as

$$u_* \in \arg \max_{u=\overline{1,\,255}} \sqrt{\sum_{n=1}^{4} \left(\frac{h^*_{nu} - \min\limits_{k=\overline{1,\,255}} h^*_{nk}}{\max\limits_{k=\overline{1,\,255}} h^*_{nk} - \min\limits_{k=\overline{1,\,255}} h^*_{nk}} \right)^2} \tag{9.29}$$

corresponds to the best Pareto-efficient situation, whose players' strategies $\{x^*_n(t)\}_{n=1}^4$ are shown in Figure 2. The best efficient payoffs are

$$\left\{h^*_{nu_*}\right\}_{n=1}^4 = \{1.7050,\ 1.6175,\ 0.0769,\ 0.2836\}. \tag{9.30}$$

Compared to payoffs (9.27), the best efficient payoffs (9.30) in this time-shifted game change significantly for all the players except for the second player. The first player's payoff has increased more than twice. The third player's payoff has increased by 31.37 %. The fourth player's payoff has increased almost by 15 %.

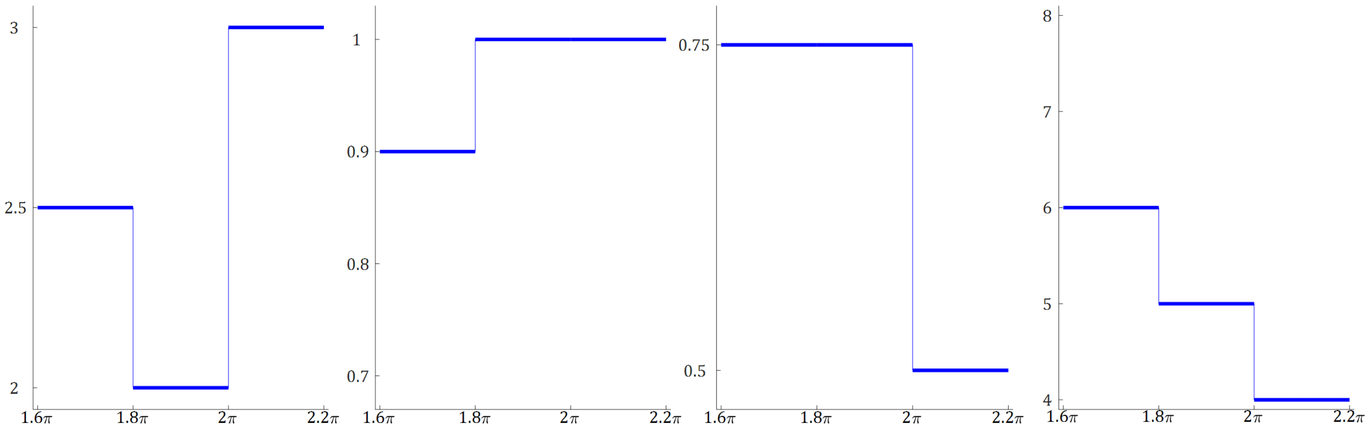

Figure 2: The best efficient staircase-function strategies producing payoffs (9.30) in the time-shifted 4-person staircase-function game (5.7) by (9.2) — (9.5) and (9.28)

An interesting fact is observed when comparing Figure 2 to Figure 1. The time-shifted game strategies in Figure 2 look like they are just shifted by 0.2π to the left. Meanwhile, the game whose best efficient payoffs are produced by strategies in Figure 2 is obtained by shifting the game by (9.1) — (9.10) by 0.2π. This property is regular, not just an occurrence. If a game is time-shifted, then its best efficient staircase-function strategies are shifted as well by the same time amount. This obviously follows from Theorem 5.1.

10 A 4-subinterval example with 4 players

Consider a 0.2π-extended game example which is obtained from the game by (9.1) — (9.10) by allowing the players to act using staircase-function strategies during $t \in [1.4\pi;\ 2.2\pi]$ by

$$\left\{\tau^{(l)}\right\}_{l=0}^{4} = \{1.4\pi,\ 1.6\pi,\ 1.8\pi,\ 2\pi,\ 2.2\pi\}\,. \tag{10.1}$$

As the respective integration interval is changed in (9.7) — (9.10), (9.13), (9.14), (9.16), (9.17), (9.20), (9.21), (9.23), (9.24) to $[1.4\pi;\ 2.2\pi]$, the respective four quadmatrix $3 \times 4 \times 2 \times 5$ games have 10, 25, 12, 10 Pareto-efficient situations. Therefore, there are

$$A = \prod_{l=1}^{4} J_l = 10 \cdot 25 \cdot 12 \cdot 10 = 30000 \tag{10.2}$$

stacks of such situations. The respective 4-person staircase-function game by

$$\begin{aligned}\mathcal{X}_1^{(4)}(2) &= \left\{x_1(t):\ x_1(t) \in \{2,\ 2.5,\ 3\}\ \forall\, t \in \left[\tau^{(l-1)};\ \tau^{(l)}\right)\ \text{for}\ l = \overline{1,\ 3}\ \text{and}\ \forall\, t \in \left[\tau^{(3)};\ \tau^{(4)}\right]\right\} \\ &\subset\ \mathcal{X}_1^{(4)} \subset \mathcal{X}_1,\end{aligned} \tag{10.3}$$

$$\begin{aligned}\mathcal{X}_2^{(4)}(3) &= \left\{x_2(t):\ x_2(t) \in \{0.7,\ 0.8,\ 0.9,\ 1\}\ \forall\, t \in \left[\tau^{(l-1)};\ \tau^{(l)}\right)\ \text{for}\ l = \overline{1,\ 3}\ \text{and}\ \forall\, t \in \left[\tau^{(3)};\ \tau^{(4)}\right]\right\} \\ &\subset\ \mathcal{X}_2^{(4)} \subset \mathcal{X}_2,\end{aligned} \tag{10.4}$$

$$\begin{aligned}\mathcal{X}_3^{(4)}(1) &= \left\{x_3(t):\ x_3(t) \in \{0.5,\ 0.75\}\ \forall\, t \in \left[\tau^{(l-1)};\ \tau^{(l)}\right)\ \text{for}\ l = \overline{1,\ 3}\ \text{and}\ \forall\, t \in \left[\tau^{(3)};\ \tau^{(4)}\right]\right\} \\ &\subset\ \mathcal{X}_3^{(4)} \subset \mathcal{X}_3,\end{aligned} \tag{10.5}$$

$$\begin{aligned}\mathcal{X}_4^{(4)}(4) &= \ = \left\{x_4(t):\ x_4(t) \in \{4,\ 5,\ 6,\ 7,\ 8\}\ \forall\, t \in \left[\tau^{(l-1)};\ \tau^{(l)}\right)\ \text{for}\ l = \overline{1,\ 3}\ \text{and}\ \forall\, t \in \left[\tau^{(3)};\ \tau^{(4)}\right]\right\} \\ &\subset\ \mathcal{X}_4^{(4)} \subset \mathcal{X}_4\end{aligned} \tag{10.6}$$

and (10.1) has 712 Pareto-efficient stacks (which are a subset of those 30000 ones due to Theorem 7.1). The single best efficient payoffs point calculated by (8.14) as

$$u_* \in \arg \max_{u=\overline{1,\, 712}} \sqrt{\sum_{n=1}^{4} \left(\frac{h^*_{nu} - \min\limits_{k=\overline{1,\, 712}} h^*_{nk}}{\max\limits_{k=\overline{1,\, 712}} h^*_{nk} - \min\limits_{k=\overline{1,\, 712}} h^*_{nk}} \right)^2} \tag{10.7}$$

corresponds to the best Pareto-efficient situation, whose players' strategies $\{x^*_n(t)\}_{n=1}^{4}$ are shown in Figure 3. The best efficient payoffs here are

$$\left\{h^*_{nu_*}\right\}_{n=1}^{4} = \{1.3992,\ 2.2096,\ 0.1313,\ 0.3805\}\,. \tag{10.8}$$

Obviously, payoffs (10.8) must be greater than payoffs (9.27), and so are they.

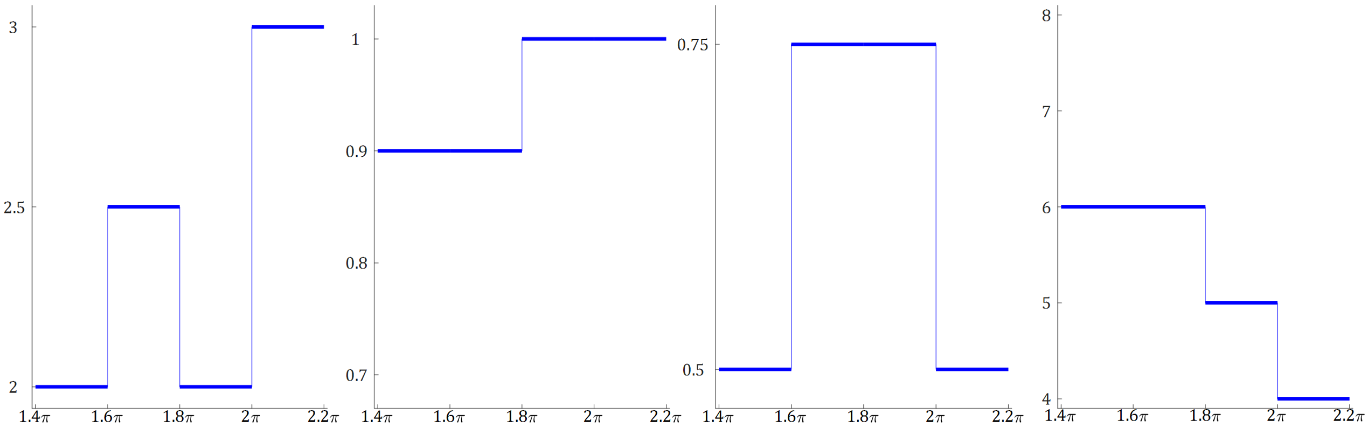

Figure 3: The best efficient staircase-function strategies producing payoffs (10.8) in the 0.2π-extended 4-person staircase-function game (5.7) by (10.3) — (10.6) and (10.1)

Another interesting fact is observed when comparing Figure 3 to the respective time-consistent overlap of Figure 2 and Figure 1. The 0.2π-extended game strategies in Figure 3 look like those in Figure 1 extended (continued) to Figure 2. Meanwhile, the game whose best efficient payoffs are produced by strategies in Figure 3 is obtained by extending time-forwardly the game by (9.1) — (9.10) by 0.2π. Following from Theorem 5.1, this property is regular similarly to the time-shifting considered above. If a game is time-extended (either forward or backward), then its best efficient staircase-function strategies are extended as well by the same time amount.

In a time-shifted game obtained from the 0.2π-extended game by allowing the players act during $t \in [1.6\pi;\ 2.4\pi]$ by

$$\left\{\tau^{(l)}\right\}_{l=0}^{4} = \{1.6\pi,\ 1.8\pi,\ 2\pi,\ 2.2\pi,\ 2.4\pi\} \tag{10.9}$$

and the respective integration interval change in (9.7) — (9.10), (9.13), (9.14), (9.16), (9.17), (9.20), (9.21), (9.23), (9.24), the respective four quadmatrix $3 \times 4 \times 2 \times 5$ games have 25, 12, 10, 27 Pareto-efficient situations. Now, there are

$$A = \prod_{l=1}^{4} J_l = 25 \cdot 12 \cdot 10 \cdot 27 = 81000 \tag{10.10}$$

stacks of such situations, where the respective 4-person staircase-function game by (9.2) — (9.5) and (10.9) has 1650 Pareto-efficient stacks (which are a subset of those 81000 ones due to Theorem 7.1). The single best efficient payoffs point calculated by (8.14) as

$$u_* \in \arg \max_{u=\overline{1,\, 1650}} \sqrt{\sum_{n=1}^{4} \left(\frac{h^*_{nu} - \min\limits_{k=\overline{1,\, 1650}} h^*_{nk}}{\max\limits_{k=\overline{1,\, 1650}} h^*_{nk} - \min\limits_{k=\overline{1,\, 1650}} h^*_{nk}} \right)^2} \tag{10.11}$$

corresponds to the best Pareto-efficient situation, whose players' strategies $\{x^*_n(t)\}_{n=1}^{4}$ are shown in Figure 4. The best efficient payoffs are

$$\left\{h^*_{nu_*}\right\}_{n=1}^{4} = \{2.1956,\ 2.1467,\ 0.0814,\ 0.3084\}\,. \tag{10.12}$$

Payoffs (10.12) are greater than payoffs (9.27) and (9.30). Compared to payoffs (10.8), the best efficient payoffs (10.12) in this time-shifted 0.2π-extended game change — most significantly for all the players except for the second player (just like in the 3-subinterval examples above). Meanwhile, only the first player's payoff has increased here (by 56.92 %). The third player has lost more than 38 % of the payoff.

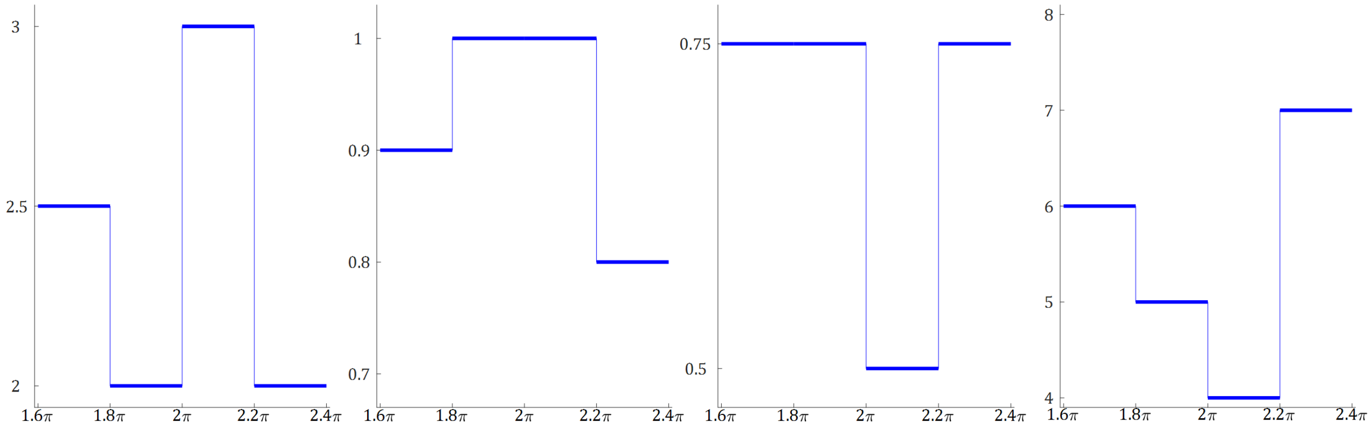

Figure 4: The best efficient staircase-function strategies producing payoffs (10.12) in the time-shifted 0.2π-extended game played by (10.9)

As in 3-subinterval examples above, the time-shifted 0.2π-extended game strategies in Figure 4 are just shifted by 0.2π to the left (compared to those ones in Figure 3). Followed by Theorem 5.1. if the 4-person staircase-function game is played for $t \in [1.4\pi;\ 2.4\pi]$, then its Pareto-efficient solution is possible to be obtained based on the solutions for $t \in [1.4\pi;\ 2.2\pi]$ and $t \in [1.6\pi;\ 2.4\pi]$ — just by overlapping them.

11 Discussion

The core of the method of solving N-person games played with staircase-function strategies consists in finding all Pareto-efficient situations in every subinterval N-person game. The computation time depends on the number of subintervals, i. e., on the "length" of the staircase-function game. In the examples considered by $t \in [1.4\pi;\ 2.2\pi]$ and $t \in [1.6\pi;\ 2.4\pi]$, the first player has 81 staircase-function strategies (one of them is shown in Figure 3 and Figure 4, for the respective interval of the staircase game). The second player has 256 staircase-function strategies, whereas the third player has only 16. Due to the greatest number of possible pure strategy values, the fourth player has the greatest number of such strategies (it is 625). Therefore, there are 207360000 pure-strategy situations (more than 207 million situations!). So, adding a subinterval has 120 times increased the "volume" of the staircase game. The examples show that, without "breaking" staircase game (5.7) into subinterval (classical) games, seeking for the efficiency in a finite N-person staircase-function game would be an extremely hard computational task. This task, however, is dramatically simplified by considering the respective succession of subinterval games. Another, concomitant, task (existing when the conditions of Theorem 6.1 do not hold, as it usually happens) is the selection of the best Pareto-efficient situation among U (staircase) Pareto-efficient situations.

If at least one subinterval game has a single efficient situation, this helps much in solving the staircase-function game. Then an efficient situation in the staircase-function game will definitely have the single efficient situation on the given subinterval (this is a direct corollary to Theorem 7.1).

The size of a finite subinterval game is defined by the sets of possible values of players' pure strategies. The size influences the computation time also. In particular cases, solving a continuous subinterval game may cause considerable delay or be just intractable itself. Then the continuous subinterval game must be approximated with a finite (i. e., N-dimensional-matrix) game using the known techniques [20].

Usually, an ordinary N-person game has multiple Pareto-efficient situations. The greater number N, the less likely a single Pareto-efficient situation is. However, this does not diminish the value of Theorem 6.1 whose proof directly follows from Theorem 7.1. The reversibility of Theorem 6.1 gives a definite practical impact. If it is known that a staircase-function game has a single Pareto-efficient situation then, according to Theorem 6.2, its search is organized by the principle of the early stop — once an efficient situation in a subinterval N-person game is found, the next subinterval game is solved.

It is clear that staircase-function N-dimensional-matrix games are solved easier. Moreover, there is no universal method to finding all Pareto-efficient situations in an infinite or continuous N-person game. The finite approximation may become a necessary intermediate in solving an infinite or continuous staircase-function game.

12 Conclusion

A finite N-person game whose players use staircase function-strategies is rendered to an N-dimensional-matrix game owing to the finiteness of the pure strategy sets. However, a finite N-person staircase-function game can hardly be solved as the N-dimensional-matrix game because of a gigantic number of pure-strategy situations. This number badly increases as either the number of players or the number of subintervals increases. Therefore, it is better to consider any N-person staircase-function game as a succession of N-person games in which the players' strategies are constants. This is possible owing to the payoff subinterval-wise summing by Theorem 5.1.

In the case of an infinite or continuous staircase-function game, where the player has a continuum (infinite, countable or uncountable set) of staircase function-strategies, each constant-strategy game is a classical infinite or continuous N-person game. In the case of a finite staircase-function game, each constant-strategy game is an N-dimensional-matrix game whose size is relatively far smaller to solve it in a reasonable time.

Theorem 6.1 ensures that the staircase-function game has a single Pareto-efficient situation if every constant-strategy game has a single Pareto-efficient situation. The inverse assertion by Theorem 6.2 is correct as well. Theorem 7.1 enlightens that, whichever the staircase-function game continuity is, any Pareto-efficient situation of staircase function-strategies is a stack of successive Pareto-efficient situations in the constant-strategy games. Therefore, if a staircase-function game has multiple Pareto-efficient situations (as it usually happens), the best efficient situation is one which is the farthest from the most unprofitable payoffs. In terms of 0-1-standardization, the best efficient situation is the farthest from the zero payoffs. However, this approach may raise computational difficulties in a case when there is a continuum of Pareto-efficient situations.

The suggested method of solving finite noncooperative games played with staircase-function strategies is a significant contribution to the mathematical N-person game theory and practice, where often a process is game-modeled so that every player must "go through one's path" (staircase function). As pure strategies are only considered, the method fits nonrepeatable games as well [2, 15]. It drastically simplifies any N-person game played with staircase-function strategies by (internally) "breaking" the staircase game into subinterval (classical) games, that allows to "deeinstellungize" the pure strategy structure complexity and solution search [21, 25]. The method is practically applicable owing to its tractability and simplicity, although the efficient situations search may be optimized for particular game classes. The question of stacking (overlapping) the best Pareto-efficient solutions of a staircase game played on disjoint (or partially overlapped) time intervals (just like in the examples with Figures 1 — 4) is a matter of an additional research. The future research will help to solve more efficiently too "long" staircase games by externally "breaking" them: a staircase game defined on a relatively long time interval will be disjointly "divided" into a few "shorter" staircase games whose best Pareto-efficient solutions are then stitched together.

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