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Sufficient conditions for the existence of solution for $(\omega - \sigma)$-higher order strongly variational inequality

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Abstract

In this paper, a new version of a higher-order strongly convex function is introduced which is named $(\omega-\sigma)$-higher-order strongly convex function. Sufficient conditions for the existence of minimum for $(\omega-\sigma)$-higher order strongly convex function is provided. The vector version of $(\omega-\sigma)$-higher order strongly convex function is given and by using KKM theory an existence results for a solution of it is proved. Moreover, the compactness of the solution set of the vector version of $(\omega-\sigma)$-higher order strongly convex function is investigated. The results of this article improve and extend the corresponding results presented in this area.

Keywords: $(\omega-\sigma)$-higher order convex function, Higher order variational inequalities, KKM theory
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1 Introduction

Convex functions have been extended and generalized in various directions in recent years [8, 11]. Mohsen et. al [6] introduced the concept of higher-order strongly convex functions and studied their properties. These results can be viewed as a significant refinement of the results presented by Lin and Fukushima in [4]. Higher-order strongly convex functions include the strongly convex functions, which were introduced and studied by Polyak [9]. Karmardian [3] used the strongly convex functions to discuss the unique existence of a solution for nonlinear complementarity problems. For the applications of strongly convex functions in optimization, variational inequalities and other branches of pure and applied sciences.

Let C be a nonempty, closed and convex subset of the Hilbert space H (throughout this paper H denotes a real Hilbert space). The variational inequality problem (VIP) is to find $x \in C$ such that

$$\langle F(x), y-x\rangle \geq 0, \quad \forall y \in C,$$

where $F : C \to H$ is a mapping and $< F(x), y-x >$ denotes the inner product between $F(x)$ and $y-x$. The variational inequality theory was introduced by Stampacchia [10] in the early 1960s to study some problems in partial differential equations with applications drawn from elasticity and potential theory. The first general theorem for the existence and uniqueness of solution for VIP was proved by Lions and Stampacchia [5] in 1967. Since then, the VIP has played fundamental and important roles in the study of a wide range of problems arising in optimization, mechanics, control

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theory, economics, operation research, management science, physics, elasticity, transportation and other branches of mathematical and engineering science[7]. In [6], the optimality conditions for the higher order strongly convex functions are characterized by a class of variational inequalities, which is called the higher order variational inequality is proved. Then the auxiliary technique to suggest an implicit method for solving higher-order strongly variational inequalities is used by Noor and et. al. Inspired by the research work going on in this field, a new class of the higher-order strongly convex functions and variational inequality problem is introduced. By using the KKM theory an existence result is established.

2 Preliminaries

Definition 2.1. [6] The function $F : C \to \mathbb{R}$ on the convex set C is said to be higher order strongly convex, if there exists a constant $\rho > 0$, such that for all $x, y \in C, \alpha \in [0, 1], k > 1$,

$$F(x + \alpha(y - x)) \leq (1 - \alpha)F(x) + \alpha F(y) - \rho(\alpha^k(1 - \alpha) + \alpha(1 - \alpha)^k)\|y - x\|^k.$$

We generalize Definition 2.1 as follows.

Definition 2.2. Let C be a nonempty and convex subset of $\mathbb{H}$. $F : C \to \mathbb{R}$ is $(\omega - \sigma)$-higher order strongly convex function if

$$F(x + \alpha(y - x)) \leq (1 - \alpha)F(x) + \alpha F(y) - \omega(\alpha)\|\sigma(x, y)\|^k,$$

where $\omega : [0, 1] \to \mathbb{R}^+$ and $\sigma : C \times C \to \mathbb{R}$.

It is clear that every $(\omega - \sigma)$-higher order strongly convex function is convex, but the following example shows that the converse may fail.

Example 2.3. We consider the Indicator function $i_{[0,1]} : \mathbb{R} \to (\infty, \infty]$ as follows,

$$i_{[0,1]}(x) = \begin{cases} 0 & x \in [0, 1] \\ \infty & x \notin [0, 1], \end{cases}$$

Since $[0, 1]$ is convex, it is obvious that the Indicator function is also convex. If we take $x = \frac{1}{3}$, $y = \frac{2}{3}$, $\alpha \in [0, 1]$ and define $\sigma(x, y) = y - x$,

$$\omega(\alpha) = \begin{cases} \tan(\frac{\pi}{2}\alpha) & \alpha \neq 1 \\ 0 & \alpha = 1, \end{cases}$$

where $\alpha \in [0, 1]$, then

$$0 = i_{[0,1]}(\alpha x + (1 - \alpha)y) \leq \alpha i_{[0,1]}(x) + (1 - \alpha)i_{[0,1]}(y) - \omega(\alpha)\|\frac{1}{3}\|^k,$$

which is impossible. This means that the indicator function, despite of being convex, is not $(\omega - \sigma)$-higher order strongly convex function.

The function F is said to be $(\omega - \sigma)$-higher order strongly concave, if and only if, $-F$ is $(\omega - \sigma)$-higher order strongly convex function.

Remark 2.4. Definition 2.2 reduces to the definition of the strongly convex function, introduced and studied by Polyak [9], by taking $\omega(\alpha) = \rho[\alpha(1 - \alpha)]$, $\sigma(x, y) = \|y - x\|$ and $k = 2$, where $\rho \geq 0$. For more details about the higher order strongly convex one can refer to [6] and the references therein.

The following definitions are needed in the sequel.

Definition 2.5. ([12]) A nonempty subset P of H is called a cone if $\alpha P \subset P$, for all $\alpha \geq 0$. Moreover, the cone P is said to be:

(i) convex if $\forall x_1, x_2 \in P : x_1 + x_2 \in P$;

(ii) pointed if $P \cap (-P) = \{0\}$.

Clearly, every convex pointed cone can induce a partial order relation $\preceq$ by

$$x \preceq y \Leftrightarrow y - x \in P.$$

Throughout this paper, H is ordered by a convex pointed cone P.

Definition 2.6. Let $P \subset H$ be a set. The positive polar cone P^* of P is defined as

$$P^* = \{x^* \in H^* : \langle x^*, p\rangle \geq 0, \forall p \in P\},$$

where H^* is the dual space of H, $\langle x^*, p\rangle$ denotes the value of the continuous linear functional x^* at the point p.

The next definition is vector version of Definition 2.2.

Definition 2.7. Let C be a nonempty and convex subset of H. $F : C \to H$ is generalized $(\omega - \sigma)$-higher order strongly convex function if and only if for all $x^* \in P^*$ the function $x^*oF : C \to \mathbb{R}$ satisfies in Definition 2.2, that is for all $\alpha \in [0, 1], k \geq 1$;

$$x^*oF(x + \alpha(y - x)) \leq (1 - \alpha)x^*oF(x) + \alpha x^*oF(y) - \omega(\alpha)\|\sigma(x, y)\|^k,$$

where $\omega : [0, 1] \to \mathbb{R}^+$ and $\sigma : C \times C \to \mathbb{R}$ is a bifunction.

Definition 2.8. [1] Recall that a mapping $F : X \to Y$ between Banach spaces, is Gateaux differentiable at $x \in X$ if, for each $v \in X$, the limit

$$F'(x, v) = F_x(v) = \lim_{\alpha\to 0} \frac{F(x + \alpha v) - F(x)}{\alpha}$$

exists, and F_x is continuous linear functional on X which is called Gateaux derivative of F at x.

Theorem 2.9. Let C be a nonempty and convex subset of H and $F : C \to \mathbb{R}$ be a Gateaux differentiable and $(\omega - \sigma)$-higher order strongly convex function. If $x \in C$ is the minimum of the function F and $\lim_{\alpha\to 0} \dfrac{\omega(\alpha)}{\alpha} = 1$, then

$$F(y) - F(x) \geq \|\delta(x, y)\|^K, \forall y \in C.$$

Proof . By the definition of the $(\omega - \sigma)$-higher order strongly convex for F we get

$$F(x + \alpha(y - x)) - F(x) \leq \alpha(F(y) - F(x)) - \omega(\alpha)\|\delta(x, y)\|^K,$$

which implies

$$\frac{F(x + \alpha(y - x)) - F(x)}{\alpha} \leq F(y) - F(x) - \frac{\omega(\alpha)}{\alpha}\|\delta(x, y)\|^K.$$

The Gateaux differentiability of F at x concludes that

$$F_x(y - x) \leq F(y) - F(x) - \|\delta(x, y)\|^K$$

thus

$$\|\delta(x, y)\|^K + F_x(y - x) \leq F(y) - F(x).$$

Now, the result follows from the last inequality and the assumption that x is the minimum of F.$\square$

Remark 2.10. It follows the last inequality of the proof of Theorem 2.9 that if $F : C \to \mathbb{R}$ is Gateaux differentiable, $(\omega - \sigma)$-higher order strongly convex function and it satisfies in the inequality

$$\langle F_x, y - x\rangle + \omega(\alpha)\|\delta(x, y)\|^K \geq 0, \forall y \in C,$$

then $x \in C$ is the minimum of the function F. In the other words, theses assumptions provide sufficient conditions which under them F attains its minimum.

If F is $(\omega-\sigma)$-higher order strongly convex function on H, we can not say F attains its minimum, therefore we need more assumptions to guarantee a convex function gets its minimum. In the following we give sufficient condition in order a convex function attains its minimum of the function composition.

There are examples that show that the $(\omega-\sigma)$-higher order strongly of the function F alone cannot guarantee the existence of a minimum for the function F. So we need more conditions. The following proposition provides sufficient conditions for this goal.

Proposition 2.11. Let C be a nonempty and convex subset of space H and $F : C \to H$ be a generalized $(\omega-\sigma)$-higher order strongly convex function. If

$$\limsup_{\alpha\to 0^+} \frac{x^*oF(x+\alpha(y-x)) - x^*oF(x)}{\alpha} \geq 0, \forall y \in C, x^* \in P^*, \tag{2.1}$$

and $\limsup_{\alpha\to 0^+} \frac{w(\alpha)}{\alpha} \geq 0$ hold, then for all $y \in C$

$$x^*oF(y) - x^*oF(x) \geq \|\sigma(x,y)\|^k,$$

and x is a minimum of F with respect to the ordering induced by P on H.

Proof . Due to the generalized $(\omega-\sigma)$-higher order strongly convex function of x^*oF, we get

$$x^*oF(x+\alpha(y-x)) \leq x^*oF(x) - \alpha(x^*oF(y) - x^*oF(x)) - \omega(\alpha)\|\sigma(x,y)\|^k.$$

Hence

$$x^*oF(x+\alpha(y-x)) - x^*oF(x) \leq \alpha(x^*oF(y) - x^*oF(x)) - \omega(\alpha)\|\sigma(x,y)\|^k.$$

This means that

$$\frac{x^*oF(x+\alpha(y-x)) - x^*oF(x)}{\alpha} \leq x^*oF(y) - x^*oF(x) - \frac{\omega(\alpha)}{\alpha}\|\sigma(x,y)\|^k$$

and by taking lim sup of both sides we get

$$x^*oF(y) - x^*oF(x) \geq \|\sigma(x,y)\|^k.$$

Now the Hahn-Banch theorem proves that x is a minimum of F with respect to the ordering induced by P on H and the proof is completed.□

Proposition 2.11 tells us that if x^*oF attains its minimum at x and moreover $x^*oF(x)$ is not the sharpest lower bound of the function x^*oF on C because the statement $x^*oF(x) + \|\sigma(x,y)\|^k$ is greater than $x^*oF(x)$. Hence, It seems why we defined the generalized $(\omega-\sigma)$-higher order strongly convex function.

Definition 2.12. Let C be a nonempty and convex subset of Hilbert space H. For a function $F : C \to H$, consider the problem of finding $x \in C$, such that

$$\langle F_x, y-x\rangle + g(x,y) \geq \omega(\alpha)\|\sigma(x,y)\|^k, \forall y \in C, k > 1$$

which is called the generalized $(\omega-\sigma)$-higher order strongly variational inequality, where $g : C \times C \to \mathbb{R}$ is a function.

Remark that, if $F = 0$, then the generalized $(\omega-\sigma)$-higher order strongly variational inequality reduces to the equilibrium problem and if $g = 0$, the generalized $(\omega-\sigma)$-higher order strongly variational inequality is the same as the variational inequality that introduced and studied by Stampacchia[10]. We need the following definitions and result in the next section.

Definition 2.13. Let $C \subset H$. A function $F : C \to H$ is called a KKM-map if

$$co(\{x_1, ..., x_n\}) \subseteq \cup_{i=1}^n F(x_i)$$

for any $x_1, ..., x_n \in C$.

Theorem 2.14. [2] Let C be a nonempty and convex subset of a Hausdorff topological vector space X. Suppose that $F : C \to X$ is a multivalued function and $F(x)$ is a closed subset of X such that is a KKM-map. If there is a nonempty compact convex set $B \subseteq C$, such that $cl_C(\bigcap_{x\in B} F(x))$ is compact (That cl is the note closure of set with respect to C), then $\bigcap_{x\in C} F(x) \neq \emptyset$.

Definition 2.15. A function $f : C \to \mathbb{R}$ is called upper semi-continuous at point $x_0 \in C$ if and only if

$$\limsup_{x\to x_0} f(x) \leq f(x_0).$$

3 Main results

In this section, by using Theorem 2.14 and suitable conditions an existence result of a solution for the generalized $(\omega - \sigma)$-higher order strongly variational inequality problem is established.

Theorem 3.1. Let C be a nonempty and convex of H and $F : C \to H$ be a function. Suppose that $g : C \times C \to \mathbb{R}$ is a bifunction such that

(i) $g(x, x) \geq 0$, for every $x \in C$,
(ii) $g(., y)$ is convex and upper semi-continuous and $\|\sigma(., y)\|^k$ is lower semicontinuous, for every $y \in C$,
(iii) the function $\psi : x \to \langle F(x), y - x\rangle$ is upper semi-continuous for each $y \in C$,
(iv) $\sigma(x, x) = 0 \ \ \forall x \in C$,
(v) $y \to \|\sigma(x, y)\|^k$ is concave.
(vi) there exist a nonempty compact subset A and a nonempty convex compact subset B of C such that, for each $x \in C \setminus A$, there exists $y \in B$ such that

$$\langle F(x), y - x\rangle + g(x, y) < \omega(\alpha)\|\sigma(x, y)\|^k, \ \ \forall\alpha \in [0, 1].$$

Then, the solution set of the generalized $(\omega - \sigma)$-higher order strongly variational inequality is nonempty and compact.

Proof . Define $S : C \rightrightarrows C$ by

$$S(y) := \{x \in C : \langle F(x), y - x\rangle + g(x, y) \geq \omega(\alpha)\|\sigma(x, y)\|^k, \exists\alpha \in [0, 1]\},$$

for every $y \in C$. It is easy to see that the set solution of the generalized $(\omega - \sigma)$-higher order strongly variational inequality equals to $\cap_{y\in C} S(y)$. Therefore it is enough to show that this intersection is nonempty. It is clear from our assumptions that $S(y) \neq \emptyset$. It follows from (ii) and (iii) that, for each $y \in C$, the set $S(y)$ is closed. We claim that S is a KKM mapping. Otherwise there exists $\{y_1, ..., y_n\} \subseteq C$ such that

$$co\{y_1, ..., y_n\} \nsubseteq \cup_{1\leq i\leq n} S(y_i).$$

Hence, there exists $w \in co\{y_1, ..., y_n\}$ such that $w \notin \cup_{1\leq i\leq n} S(y_i)$ where $w = \sum_{i=1}^n \alpha_i y_i$ such that $w = \sum_{i=1}^n \alpha_i = 1$. Thus

$$\langle F(w), y_i - w\rangle + g(w, y_i) < \omega(\alpha)\|\sigma(w, y_i)\|^k, \forall\alpha \in [0, 1].$$

Hence

$$\left\langle F(w), \sum_{i=1}^n \alpha_i y_i - w \right\rangle + \sum_{i=1}^n \alpha_i g(w, y_i) < \sum_{i=1}^n \alpha_i \omega(\alpha)\|\sigma(w, y_i)\|^k,$$

and

$$\sum_{i=1}^n \alpha_i g(w, y_i) < \omega(\alpha) \sum_{i=1}^n \alpha_i \|\sigma(w, y_i)\|^k,$$

for every $1 \leq i \leq n$. Thus, the hypotheses (i), (ii) and (v) imply that

$$0 \leq g(w, \sum_{i=1}^n \alpha_i y_i) < \omega(\alpha)\|\sigma(w, \sum_{i=1}^n \alpha_i y_i)\|^k,$$

that is

$$0 \leq g(w,w) < \omega(\alpha)\|\sigma(w,w)\|^k,$$

which is contradicted by (iv). Therefore $y \to S(y)$ is a KKM mapping. Condition (vi) implies

$$\cap_{y\in C} S(y) \subseteq \cap_{y\in B} S(y) \subseteq A,$$

which is closed and compact in C(this means the solution set of the generalized $(\omega-\sigma)$-higher order strongly variational inequality is compact). Hence, the set valued mapping S satisfies all the conditions of Theorem 2.14 which concludes

$$\cap_{y\in C} S(y) \neq \emptyset.$$

Then the solution set of the higher order strongly variational inequality problem is nonempty. The compactness of the solution set of the higher order strongly variational inequality problem directly follows from (vi) and closeness of the set $\cap_{y\in C} S(y)$. This completes the proof.□

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