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Multiplicity analysis of positive weak solutions in a quasi-linear Dirichlet problem inspired by Kirchhoff-type phenomena

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Abstract

The main focus of this paper lies in investigating the existence of infinitely many positive weak solutions for the following elliptic-Kirchhoff equation with Dirichlet boundary condition

$$\begin{cases} -\sum_{i=1}^{N} M_i\left(\int_\Omega \frac{1}{p_i(x)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)}dx\right)\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\right) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The methodology adopted revolves around the technical approach utilizing the direct variational method within the framework of anisotropic variable exponent Sobolev spaces.

Keywords: Nonlinear elliptic equations, Variational methods applied on PDEs, Positive solutions to PDEs
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1 Introduction

Over the past years, differential equations have been a focal point of research, owing to their extensive practical implications and widespread use in numerous fields.

Proposed by Kirchhoff [20], the Kirchhoff differential equations offer an extension to D'Alembert's wave equation, accommodating the effects of string length changes during vibrations

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.1}$$

in this context, L denotes the length of the chord, h represents the area of the cross-section, E stands for the Young's modulus of the material, denotes the density, and P_0 corresponds to the initial tension. The Kirchhoff equation (1.1)

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exhibits a unique characteristic in its inclusion of a non-local coefficient $\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx$, which is dependent on the average $\frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2$ of the kinetic energy $\frac{1}{2}\left|\frac{\partial u}{\partial x}\right|^2$ within the interval $[0, L]$. Thus, the equation loses its property of being a point-wise identity. See also [6, 19, 32] for related topics.

In recent times, numerous mathematicians, physicists, and engineers have shown a keen interest in anisotropic variable exponent Sobolev spaces. The motivation behind this stems from the crucial role these spaces play in modeling real-world phenomena, including electrorheological and image restoration, magnetorheological fluids, and elastic materials, (look at, for example [5, 8, 10, 30, 33, 34, 35, 36]).

In the present paper we study the existence of positive solutions of the nonhomogeneous anisotropic Kirchhoff problem

$$\begin{cases} -\sum_{i=1}^{N} M_i\left(\int_\Omega \frac{1}{p_i(x)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} dx\right)\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\right) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N, (N > 3)$ represents a bounded domain with a smooth boundary $\partial\Omega$, and p_i, $i = 1, \dots N$ are continuous functions. Additionally, for each $i = 1, \dots N$, M_i and f are continuous functions which satisfies some conditions detailed in Section 3.

The differential operator

$$\Delta_{\vec{p}(\cdot)} u = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\right), \tag{1.3}$$

involved in problem (1.2) is an anisotropic variable exponent $\vec{p}(\cdot)$-Laplace operator which represents an extension of the operator

$$\Delta_{p(\cdot)} u = \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right). \tag{1.4}$$

The $p(\cdot)$-Laplacian operator, obtained by setting each $p_i(x)$ to be equal to $p(x)$ for $i = 1, \dots N$, serves as a natural extension of the isotropic p-Laplacian operator

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right), \tag{1.5}$$

where $p > 1$ denotes a real constant. In the classical Sobolev spaces, F. J. S. A. Corrêa, R. G. Nascimento [12] have established the existence of solutions for problem (1.2) in this particular case p-Kirchhoff-type equation, for additional results, refer to [22, 28, 29, 31].

In the Sobolev variable exponent setting, G. Dai and D. Liu [13] has analyzed the problem (1.2) in the context of $p(x)$-Kirchhoff-type equation, see also [2, 9, 11, 14, 18] for related topics.

The investigation of problem (1.2) in anisotropic variable exponent Sobolev spaces has been previously addressed by other researchers (see [7, 16, 25]). However, our study stands apart due to the entirely distinct hypotheses adopted, which subsequently lead to different and novel findings.

The shift from a variable exponent to an anisotropic variable exponent inevitably introduces fresh complexities. To tackle these challenges, we adopt a combined approach, utilizing traditional methodologies alongside modern techniques specifically designed for handling problems of anisotropic nature with variable exponents. The organization of this paper is as follows: In Section 2, we provide an introduction to anisotropic variable exponent Sobolev spaces, laying the necessary groundwork for the subsequent analysis. Section 3 is dedicated to presenting the assumptions under which our problem yields positive solutions, accompanied by an illustrative example.

2 Preliminary

Let Ω denote a smooth bounded domain in $\mathbb{R}^N$, where we introduce the following definitions:

$$\mathcal{C}_+(\overline{\Omega}) = \left\{p \in \mathcal{C}(\overline{\Omega}) \quad \text{such that} \quad 1 < p^- \le p^+ < \infty\right\},$$

where

$$p^- = \operatorname{ess\,inf}\left\{p(x) : x \in \overline{\Omega}\right\} \quad \text{and} \quad p^+ = \operatorname{ess\,sup}\left\{p(x) : x \in \overline{\Omega}\right\}.$$

For any $p \in \mathcal{C}_+(\overline{\Omega})$, we introduce the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$, which encompasses all measurable functions $u : \Omega \longmapsto \mathbb{R}$ such that the convex modular

$$\rho_{p(\cdot)}(u) := \int_\Omega |u|^{p(x)} dx,$$

remains finite. Consequently, we define the norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \rho_{p(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1 \right\},$$

as the Luxemburg norm in $L^{p(\cdot)}(\Omega)$. As a separable Banach space, $(L^{p(\cdot)}(\Omega), |\cdot|_{p(\cdot)})$ exhibits desirable properties. Additionally, the space $L^{p(\cdot)}(\Omega)$ is uniformly convex, making it reflexive, and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, with $\dfrac{1}{p(\cdot)} + \dfrac{1}{p'(\cdot)} = 1$. Lastly, we arrive at the following Hölder-type inequality.

$$\left| \int_\Omega u(x)v(x)dx \right| \leq \Big(\frac{1}{p^-} + \frac{1}{(p^-)'}\Big) \|u\|_{p(\cdot)} \|v\|_{p'(\cdot)} \quad \text{for all } u \in L^{p(\cdot)}(\Omega) \text{ and } v \in L^{p'(\cdot)}(\Omega). \tag{2.1}$$

The modular $\rho_{p(\cdot)}$ of the space $L^{p(\cdot)}(\Omega)$ assumes a crucial role in handling the generalized Lebesgue spaces. The ensuing result is presented:

Proposition 2.1. (See [15]). Considering $u_n, u \in L^{p(\cdot)}(\Omega)$, with $p^+ < +\infty$, we observe the subsequent properties:

1. If $\|u\|_{p(\cdot)} > 1$, then $\|u\|_{p(\cdot)}^{p^-} < \rho_{p(\cdot)}(u) < \|u\|_{p(\cdot)}^{p^+}$.
2. For $\|u\|_{p(\cdot)} < 1$, we have $\|u\|_{p(\cdot)}^{p^+} < \rho_{p(\cdot)}(u) < \|u\|_{p(\cdot)}^{p^-}$.
3. The condition $\|u\|_{p(\cdot)} < 1$ (respectively $= 1; > 1$) is equivalent to $\rho_{p(\cdot)}(u) < 1$ (respectively $= 1; > 1$).
4. When $\|u_n\|_{p(\cdot)} \to 0$ (respectively $\to +\infty$), it implies $\rho_{p(\cdot)}(u_n) \to 0$ (respectively $\to +\infty$).
5. Lastly, we have $\rho_{p(\cdot)}\left(\dfrac{u}{\|u\|_{p(\cdot)}}\right) = 1$.

The definition of $W_0^{1,p(\cdot)}(\Omega)$ involves taking the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$, and

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N - p(x)} & \text{for } p(x) < N, \\ \infty & \text{for } p(x) \geq N. \end{cases}$$

Proposition 2.2. (See [15]).

(i) For $1 < p^- \leq p^+ < \infty$, both $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ are separable and reflexive Banach spaces.

(ii) If $q(x) \in C_+(\bar{\Omega})$ and $q(x) < p^*(x)$ holds true for each $x \in \Omega$, then the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous and compact.

We now introduce the anisotropic Sobolev space with variable exponent, which serves as the foundation for studying our main problem. Consider N variable exponents $p_1(\cdot), \ldots, p_N(\cdot)$ belonging to $\mathcal{C}_+(\overline{\Omega})$. We use the notation

$$\vec{p}(\cdot) = \Big\{p_1(\cdot), \ldots, p_N(\cdot)\Big\} \text{ and } D^i u = \frac{\partial u}{\partial x_i} \quad \text{for } i = 1, \ldots, N,$$

and we set it for all $x \in \overline{\Omega}$,

$$p_M(\cdot) = \max \Big\{p_1(\cdot), \ldots, p_N(\cdot)\Big\} \text{ and } p_m(\cdot) = \min \Big\{p_1(\cdot), ..., p_N(\cdot)\Big\}.$$

The following notations are introduced:

$$\underline{p} = \min\left\{p_1^-, p_2^-, \ldots, p_N^-\right\}, \quad \underline{p}^+ = \max\left\{p_1^-, p_2^-, \ldots, p_N^-\right\}, \quad \overline{p} = \max\left\{p_1^+, p_2^+, \ldots, p_N^+\right\}, \tag{2.2}$$

and

$$\underline{p}^* = \frac{N}{\sum_{i=1}^{N} \frac{1}{p_i^-} - 1}, \quad \underline{p}_{,\infty} = \max\left\{\underline{p}^*, \underline{p}^+\right\}. \tag{2.3}$$

In the context of this paper, we make the assumption that

$$\sum_{i=1}^{N} \frac{1}{p_i^-} > 1. \tag{2.4}$$

The definition of the anisotropic variable exponent Sobolev space $W^{1,\vec{p}(\cdot)}(\Omega)$ is as follows:

$$W^{1,\vec{p}(\cdot)}(\Omega) = \left\{D^i u \in L^{p_i(\cdot)}(\Omega), \quad i = 1, 2, \ldots, N\right\},$$

equipped with the norm

$$\|u\|_{W^{1,\vec{p}(\cdot)}(\Omega)} = \|u\|_{1,\vec{p}(\cdot)} = \sum_{i=1}^{N} \|D^i u\|_{L^{p_i(\cdot)}(\Omega)}, \tag{2.5}$$

Furthermore, $W_0^{1,\vec{p}(\cdot)}(\Omega)$ is defined as the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,\vec{p}(\cdot)}(\Omega)$ under the norm (2.5). The dual space of $W0^{1,\vec{p}(\cdot)}(\Omega)$ is denoted as $W^{-1,\vec{p}(\cdot)'}(\Omega)$, where $\vec{p'}(x) = p_0'(x), \ldots, p_N'(x)$, satisfying $\frac{1}{p_i'(x)} + \frac{1}{p_i(x)} = 1$ (see [26, 27]) for the constant exponent case). The reflexivity of the Banach space $\left(W_0^{1,\vec{p}(\cdot)}(\Omega), |u|_{1,\vec{p}(\cdot)}\right)$ has been established in [24]. For a more comprehensive treatment of anisotropic variable exponent Sobolev spaces, researchers may delve into [1, 3, 17, 21, 24].

Proposition 2.3. (See [4, 23]). The bounded domain $\Omega \subset \mathbb{R}^N$, with a smooth boundary and $N > 3$, satisfies relation (2.4).

1. Considering any $q \in \mathcal{C}+(\overline{\Omega})$ satisfying the condition $1 < q(x) < \underline{p}, \infty$, for all $x \in \overline{\Omega}$, then
$$W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega).$$
2. Assume that $\underline{p} > N$ then
$$W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow C^0(\overline{\Omega}).$$

3 Fundamental assumptions and main results

For the entirety of this paper, we make the assumption that the following set of conditions is satisfied:

Assume that $f : \Omega \times \mathbb{R} \longmapsto \mathbb{R}$ is Carathéodory functions satisfying the following condition

(H1) There exists a constant $\tau > 0$ such that $\sup_{t\in[0,\tau]} f(\cdot, t) \in L^\infty(\Omega)$.

(H2) Suppose that $(a_n)_n$ and $(b_n)_n$ be two positive sequences such that

$$0 < a_n < b_n, \ \lim_{n\to\infty} b_n = 0, \ \text{and} \ \int_0^{a_n} f(x,s)ds = \sup_{t\in[a_n,b_n]} \int_0^t f(x,s)ds \text{ for almost all } x \in \Omega \text{ and } n \in \mathbb{N}.$$

(H3) There is a sequence $(\vartheta_n)_n$, which is a subset of the interval $[0, b_n]$, such that

$$ess \inf_{\Omega} \int_0^{\vartheta_n} f(x,s)ds > 0.$$

For the function M_i, $i = 1, \dots N$, we set forth the subsequent assumptions.

(H4) M_i is a differentiable on $\mathbb{R}^+$ and there is positive constant m such that

$$M_i(t) \geq m \text{ for all } t \geq 0.$$

Functionals are defined for any $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ as follows:

$$\Phi(u) = \sum_{i=1}^{N} \widehat{M_i}\left(\int_\Omega \frac{1}{p_i(x)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} dx\right) - \int_\Omega F(x,u)dx, \tag{3.1}$$

where $F(x,t) = \displaystyle\int_0^t f(x,s)ds$ and $\widehat{M_i}(t) = \displaystyle\int_0^t M_i(s)ds$.

Definition 3.1. For any measurable function $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ to be considered a weak solution of the elliptic problem (1.2), it must satisfy the condition that, for all $v \in W_0^{1,\vec{p}(\cdot)}(\Omega)$,

$$\sum_{i=1}^{N} M_i\left(\int_\Omega \frac{1}{p_i(x)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} dx\right)\int_\Omega \sum_{i=1}^{N}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}dx = \int_\Omega f(x,u)v(x)dx. \tag{3.2}$$

It is easy to see that $\Phi \in \mathcal{C}^1(W_0^{1,\vec{p}(\cdot)}(\Omega), \mathbb{R})$ (see [7, 25]), and the function $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ is deemed a weak solution of (1.2) if and only if it corresponds to a critical point of the functional Φ.

Considering our assumptions on f, we can find positive constants k and τ such that $|f(\cdot,t)| \leq k$ for every $0 \leq \tau \leq t$ and almost every $x \in \Omega$. Without any loss of generality, we can suppose that $b_n \leq \tau$ for every $n \in \mathbb{N}$. Let's proceed by defining

$$g(\cdot,t) = \begin{cases} 0 & \text{if } t \leq 0, \\ f(\cdot,t) & \text{if } 0 < t \leq \tau, \\ f(\cdot,\tau) & \text{if } t > \tau. \end{cases} \tag{3.3}$$

Hence, we have

$$|g(\cdot,t)| \leq k, \tag{3.4}$$

for almost every $x \in \Omega$ and every $t \in \mathbb{R}$. Next, we take into account the following problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} M_i\left(\int_\Omega \frac{1}{p_i(x)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} dx\right)\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\right) = g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

We can identify the weak solutions of (3.5) as the critical points of the functional

$$\Psi(u) = \sum_{i=1}^{N} \widehat{M_i}\left(\int_\Omega \frac{1}{p_i(x)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} dx\right) - \int_\Omega G(x,u)dx, \tag{3.6}$$

where $G(x,t) = \displaystyle\int_0^t g(x,s)ds$. By (3.4), it is clear that Ψ is well defined and Gâteaux differentiable in $W_0^{1,\vec{p}(\cdot)}(\Omega)$ (see [7, 25]). For every fixed $n \in \mathbb{N}$, we define

$$K_n(u) = \left\{u \in W_0^{1,\vec{p}(\cdot)}(\Omega) : 0 < u(x) \leq b_n \text{ a.e. } \Omega\right\}. \tag{3.7}$$

Having established the necessary groundwork, we can now present the main findings of this paper.

Theorem 3.2. Assume assumptions **(H1)-(H4)** hold true and $f(\cdot,0) = 0$. Then, there exists a sequence $(u_n)_n \subset W_0^{1,\vec{p}(\cdot)}(\Omega)$ of positive, homoclinic weak solutions of (1.2) such that

$$\lim_{n\to+\infty} \Psi(u_n) = 0 \text{ and } \lim_{n\to+\infty} \|u_n\|_{1,\vec{p}(\cdot)} = 0. \tag{3.8}$$

Theorem 3.3. To enhance the organization and clarity, we divided the proof into three steps.

Step 1 :. Auxiliary lemmas.

Lemma 3.4. Assume assumptions **(H1)**, (3.4) and **(H4)** are satisfied. Then, the functionals Ψ is weakly lower semi-continuous.

Proof . For each $i = 0, \dots, N$ and any $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$, we can define the functionals J_i and $H : W_0^{1,\vec{p}(\cdot)}(\Omega) \longrightarrow \mathbb{R}$ as follows:

$$J_i = \int_\Omega \frac{1}{p_i(x)} \Big|\frac{\partial u}{\partial x_i}\Big|^{p_i(x)} dx, \quad \text{where} \quad \frac{\partial u}{\partial x_0} = u,$$

$$H(u) = -\int_\Omega G(x,u)dx.$$

Claim 1: Consider a sequence $(u_n)_n$ with the property that $u_n \rightharpoonup u$ in $W_0^{1,\vec{p}(\cdot)}(\Omega)$. As J_i is convex, for every n, we obtain

$$J_i(u) \le J_i(u_n) + \langle J_i'(u), u - u_n \rangle.$$

Taking the limit as $n \to \infty$ in the above inequality, we observe that J_i is sequentially weakly lower semi-continuous. As a result, we obtain:

$$\int_\Omega \sum_{i=1}^{N} \frac{1}{p_i(x)} \Big|\frac{\partial u}{\partial x_i}\Big|^{p_i(x)} dx \le \liminf_{n\to+\infty} \int_\Omega \sum_{i=0}^{N} \frac{1}{p_i(x)} \Big|\frac{\partial u_n}{\partial x_i}\Big|^{p_i(x)} dx. \tag{3.9}$$

By utilizing (3.9) and considering the continuity and monotonicity of $\widehat{M}_i$, we obtain

$$\begin{aligned} \liminf_{n\to+\infty} J(u_n) &= \liminf_{n\to+\infty} \sum_{i=0}^{N} \widehat{M}_i \left(\int_\Omega \frac{1}{p_i(x)} \Big|\frac{\partial u_n}{\partial x_i}\Big|^{p_i(x)} dx \right) \\ &\ge \sum_{i=0}^{N} \widehat{M}_i \left(\liminf_{n\to+\infty} \int_\Omega \frac{1}{p_i(x)} \Big|\frac{\partial u_n}{\partial x_i}\Big|^{p_i(x)} dx \right) \\ &\ge \sum_{i=0}^{N} \widehat{M}_i \left(\int_\Omega \frac{1}{p_i(x)} \Big|\frac{\partial u}{\partial x_i}\Big|^{p_i(x)} dx \right) \\ &\ge J(u). \end{aligned} \tag{3.10}$$

That is to say, J demonstrates sequential weak lower semi-continuity.

Claim 2: H is sequentially weakly continuous. Let $(u_n)_n$ be a sequence in $W_0^{1,\vec{p}(\cdot)}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,\vec{p}(\cdot)}(\Omega)$. So, by (3.4) and Proposition 2.3. Therefore, it is easy to show that $\lim_{n\to\infty} H(u_n) = H(u)$, and hence H is sequentially weakly lower semicontinuous. Similarly, just like we demonstrated for the mapping H, it is possible to establish the sequential weak lower semi-continuity of Φ. Since $\Psi = J - H$, we complete the proof. □

Lemma 3.5. On K_n, the functional Ψ is boundedly below, and the infimum m_n over K_n is attained at $u_n \in K_n$.

Proof . To begin with, considering any $u \in K_n$, we find that

$$\begin{aligned} \Psi(u) &= \sum_{i=1}^{N} \widehat{M}_i \left(\int_\Omega \frac{1}{p_i(x)} \Big|\frac{\partial u}{\partial x_i}\Big|^{p_i(x)} dx \right) - \int_\Omega G(x,u)dx \ge -\int_\Omega G(x,u)dx \\ &\ge -kb_n \text{meas}(\Omega). \end{aligned} \tag{3.11}$$

In conclusion, we deduce that Ψ is bounded from below on K_n. It is apparent that K_n possesses the properties of convexity and closedness, thus establishing its weak closedness within $W_0^{1,\vec{p}(\cdot)}(\Omega)$. Consider the sequence $(u_n)_n$ in K_n such that $\Psi(u_n)$ lies between m_n and $m_n + \frac{1}{n}$ for all $n \in \mathbb{N}$, where $m_n = \inf_{K_n} \Psi$. Next, if $\|u_n\|_{1,\vec{p}(\cdot)} \le 1$, our objective is achieved; otherwise, we proceed with the following steps

$$\begin{aligned}
\Psi(u) &= \sum_{i=1}^{N} \widehat{M}_i\left(\int_\Omega \frac{1}{p_i(x)}\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i(x)}dx\right) - \int_\Omega G(x,u)dx \\
&= \sum_{i=1}^{N}\int_0^{\left(\int_\Omega \frac{1}{p_i(x)}\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i(x)}dx\right)} M_i(s)ds - \int_\Omega G(x,u)dx \\
&\geq \frac{m}{\overline{p}}\sum_{i=1}^{N}\left(\Big\|\frac{\partial u}{\partial x_i}\Big\|_{p_i(\cdot)}^{\underline{p}} - 1\right) - \int_\Omega G(x,u)dx \\
&\geq \frac{m}{\overline{p}N^{\underline{p}-1}}\|u_n\|_{1,\vec{p}(\cdot)}^{\underline{p}} - \frac{mN}{\overline{p}N^{\underline{p}-1}} - kb_n(meas\Omega).
\end{aligned}$$

Which yields

$$\frac{m}{\overline{p}N^{\underline{p}-1}}\|u_n\|_{1,\vec{p}(\cdot)}^{\underline{p}} \leq m_n + 1 + \frac{mN}{\overline{p}N^{\underline{p}-1}} + kb_n|\Omega|, \tag{3.12}$$

for all $n \in \mathbb{N}$, thus $(u_n)_n$ is bounded in $W_0^{1,\vec{p}(\cdot)}(\Omega)$ which is a reflexive space. Therefore, by considering a sub-sequence denoted as $(u_n)_n$, we observe weak convergence towards a specific element $u_n \in K_n$. This leads us to the conclusion that $\Psi(u_n) = m_n$, utilizing the concept of weakly sequentially lower semi-continuity of Ψ. □

Step 2 :. A priori estimates.
We start this step by proving in the following result that the sequence $(u_n)_n$ is bounded almost everywhere.

Proposition 3.6. For all $n \in \mathbb{N}$, we have $0 \leq u_n(x) \leq a_n$ a.e. $x \in \Omega$.

Proof . Let $\Lambda_n = \{x \in \Omega : b_n \geq u_n(x) > a_n\}$ and suppose that $\text{meas}(\Lambda_n) > 0$. Define the function $\sigma_n(t) = \min\Big(\max(t,0), a_n\Big)$ and set $h_n = \sigma_n(u_n)$. It is clear that from the definition and the continuity of σ_n we get $h_n \in K_n$. As a consequence, we obtain that

$$h_n(x) = \begin{cases} u_n(x) & \text{if } x \in \Omega\backslash\Lambda_n, \\ a_n & \text{if } x \in \Lambda_n. \end{cases} \tag{3.13}$$

Then, we can write

$$\begin{aligned}
&\Psi(h_n) - \Psi(u_n) \\
&= \sum_{i=1}^{N}\widehat{M}_i\left(\int_\Omega \frac{1}{p_i(x)}\Big|\frac{\partial h_n}{\partial x_i}\Big|^{p_i(x)}dx\right) - \int_\Omega\int_0^{h_n} g(x,t)dtdx - \sum_{i=1}^{N}\widehat{M}_i\left(\int_\Omega \frac{1}{p_i(x)}\Big|\frac{\partial u_n}{\partial x_i}\Big|^{p_i(x)}dx\right) + \int_\Omega\int_0^{u_n} g(x,t)dtdx \\
&= \sum_{i=1}^{N}\widehat{M}_i\left(\int_{\Omega\backslash\Lambda_n} \frac{1}{p_i(x)}\Big|\frac{\partial h_n}{\partial x_i}\Big|^{p_i(x)}dx\right) + \sum_{i=1}^{N}\widehat{M}_i\left(\int_{\Lambda_n} \frac{1}{p_i(x)}\Big|\frac{\partial h_n}{\partial x_i}\Big|^{p_i(x)}dx\right) \\
&\quad - \int_{\Omega\backslash\Lambda_n}\int_0^{h_n} g(x,t)dtdx - \int_{\Lambda_n}\int_0^{h_n} g(x,t)dtdx \\
&\quad - \sum_{i=1}^{N}\widehat{M}_i\left(\int_{\Omega\backslash\Lambda_n} \frac{1}{p_i(x)}\Big|\frac{\partial u_n}{\partial x_i}\Big|^{p_i(x)}dx\right) - \sum_{i=1}^{N}\widehat{M}_i\left(\int_{\Lambda_n} \frac{1}{p_i(x)}\Big|\frac{\partial u_n}{\partial x_i}\Big|^{p_i(x)}dx\right) \\
&\quad + \int_{\Omega\backslash\Lambda_n}\int_0^{u_n} g(x,t)dtdx + \int_{\Lambda_n}\int_0^{u_n} g(x,t)dtdx \\
&= -\sum_{i=1}^{N}\widehat{M}_i\left(\int_{\Lambda_n} \frac{1}{p_i(x)}\Big|\frac{\partial u_n}{\partial x_i}\Big|^{p_i(x)}dx\right) - \int_{\Lambda_n}\int_0^{a_n} g(x,t)dtdx + \int_{\Lambda_n}\int_0^{u_n} g(x,t)dtdx \\
&= -\sum_{i=1}^{N}\widehat{M}_i\left(\int_{\Lambda_n} \frac{1}{p_i(x)}\Big|\frac{\partial u_n}{\partial x_i}\Big|^{p_i(x)}dx\right) - \int_{\Lambda_n}\int_{u_n}^{h_n} g(x,t)dtdx \\
&= -\sum_{i=1}^{N}\widehat{M}_i\left(\int_{\Lambda_n} \frac{1}{p_i(x)}\Big|\frac{\partial u_n}{\partial x_i}\Big|^{p_i(x)}dx\right) - \int_{\Lambda_n}\int_{u_n}^{a_n} g(x,t)dtdx \leq 0.
\end{aligned} \tag{3.14}$$

Because $\int_{\Lambda_n}\int_{u_n}^{a_n} g(x,t)dtdx \geq 0$. Hence, $\Psi(h_n) \geq \Psi(u_n) = \inf_{K_n}\Psi$, then every term should be zero. In particular,

$$\sum_{i=1}^{N}\int_{\Lambda_n}\frac{1}{p_i(x)}\Big|\frac{\partial u_n}{\partial x_i}\Big|^{p_i(x)}dx = \int_{\Lambda_n}\Big(G(x,a_n)-G(x,u_n)\Big)dx. \tag{3.15}$$

Therefore, $\text{meas}(\Lambda_n) = 0$, which means $0 \leq u_n(x) \leq a_n$ almost every where $x \in \Omega$. □ Next we show that the sequence $(u_n)_n$ formed of weak solutions of problem (3.5) as mentioned in the following result.

Proposition 3.7. The terms of $(u_n)_n$ are local minimum points of Ψ in $W_0^{1,\vec{p}(\cdot)}(\Omega)$.

Proof . Set $\Gamma_n = \{x \in \Omega : b_n \geq u(x) > a_n\}$. So, we have $\int_{\sigma_n(u)}^{u} g(x,t)dt = 0$ for any $x \in \Omega\backslash\Gamma_n$. In the other hand, if $x \in \Gamma_n$, then one has the following three cases.

(a) If $u(x) < 0$, then $\int_{\sigma_n(u)}^{u} g(x,t)dt = 0$.

(b) If $a_n < u(x) \leq b_n$, then by **(H2)**, $\int_{\sigma_n(u)}^{u} g(x,t)dt \leq 0$.

(c) If $b_n < u(x)$, then $\int_{\sigma_n(u)}^{u} g(x,t)dt = \int_{a_n}^{u} g(x,t)dt \leq \int_{a_n}^{u} kdt = k\Big(u(x)-a_n\Big)$, by (3.4).

Fix a real $\underline{p}_{,\infty}$ such that $\underline{p}_{,\infty} > q(x)+1 > \overline{p}$ for every $x \in \Omega$, then the following constant is finite

$$\lambda = \sup_{\mu \geq b_n}\frac{k(\mu - a_n)}{(\mu-a_n)^{q(x)+1}}.$$

Then, for almost every where $x \in \Omega$, we have $\int_{\sigma_n(u)}^{u} g(x,t)dt \leq \lambda|(u(x)-\sigma_n(u(x)))|^{q(x)+1}$. Then, since when $\underline{p} \leq N$, the space is $W^{1,\vec{p}(\cdot)}(\Omega)$ compactly embedded in $L^{q(\cdot)+1}(\Omega)$ and continuously embedded in $C^0(\overline{\Omega})$ elsewhere, there is a positive constant c such that

$$\int_{\Omega}\int_{\sigma_n(u)}^{u} g(x,t)dtdx \leq c^{q(x)+1}\lambda\|(u-\sigma_n(u))\|_{1,\vec{p}(\cdot)}^{q(x)+1}. \tag{3.16}$$

Therefore, we can write

$$\begin{aligned}
\Psi(u)-\Psi(\sigma_n(u)) &= \sum_{i=1}^{N}\widehat{M}_i\left(\int_{\Omega}\frac{1}{p_i(x)}\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i(x)}dx\right) - \int_{\Omega}\int_{0}^{u} g(x,t)dtdx \\
&\quad -\sum_{i=1}^{N}\widehat{M}_i\left(\int_{\Omega}\frac{1}{p_i(x)}\Big|\frac{\partial \sigma_n(u)}{\partial x_i}\Big|^{p_i(x)}dx\right) + \int_{\Omega}\int_{0}^{\sigma_n(u)} g(x,t)dtdx \\
&= \sum_{i=1}^{N}\widehat{M}_i\left(\int_{\Gamma_n}\frac{1}{p_i(x)}\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i(x)}dx\right) - \int_{\Gamma_n}\int_{\sigma_n(u)}^{u} g(x,t)dtdx \\
&= \sum_{i=1}^{N}\widehat{M}_i\left(\int_{\Omega}\frac{1}{p_i(x)}\Big|\frac{\partial u}{\partial x_i}-\frac{\partial \sigma_n(u)}{\partial x_i}\Big|^{p_i(x)}dx\right) - \int_{\Omega}\int_{\sigma_n(u)}^{u} g(x,t)dtdx \\
&\geq \sum_{i=1}^{N}\widehat{M}_i\left(\int_{\Omega}\frac{1}{p_i(x)}\Big|\frac{\partial u}{\partial x_i}-\frac{\partial \sigma_n(u)}{\partial x_i}\Big|^{p_i(x)}dx\right) - \lambda c^{q(x)+1}\|(u-\sigma_n(u))\|_{1,\vec{p}(\cdot)}^{q(x)+1} \quad \text{(by (3.16))} \\
&\geq \frac{m}{\underline{p}N^{\underline{p}-1}}\|(u-\sigma_n(u))\|_{1,\vec{p}(\cdot)}^{\overline{p}} - \lambda c^{q(x)+1}\|(u-\sigma_n(u))\|_{1,\vec{p}(\cdot)}^{q(x)+1},
\end{aligned} \tag{3.17}$$

Since $\sigma_n(u) \in K_n$, we have $\Psi(\sigma_n(u)) \geq \Psi(u_n)$ and preserving the generality of our analysis, let's assume that $\|(u-\sigma_n(u))\|_{1,\vec{p}(\cdot)} \leq 1$ cause we need small values of $\|(u-\sigma_n(u))\|_{1,\vec{p}(\cdot)}$. Then

$$\begin{aligned}\Psi(u) &\geq \Psi(u_n) + \frac{m}{\overline{p}(N)^{\underline{p}-1}}\|(u-\sigma_n(u))\|^{\overline{p}}_{1,\vec{p}(\cdot)} - \lambda c^{q(x)+1}\|(u-\sigma_n(u))\|^{q(x)+1}_{1,\vec{p}(\cdot)} \\ &\geq \Psi(u_n) + \left(\frac{m}{\overline{p}(N)^{\overline{p}-1}} - \lambda c^{q(x)+1}\|(u-\sigma_n(u))\|^{q(x)+1-\overline{p}}_{1,\vec{p}(\cdot)}\right)\|(u-\sigma_n(u))\|^{\overline{p}}_{1,\vec{p}(\cdot)}. \end{aligned} \tag{3.18}$$

The continuity of σ_n allows us to choose a positive value $\delta > 0$ such that, for any $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$, the condition

$$\|(u-\sigma_n(u))\|_{1,\vec{p}(\cdot)} < \delta, \quad \|(u-\sigma_n(u))\|^{q(x)+1-\overline{p}}_{1,\vec{p}(\cdot)} \leq \frac{m}{\overline{p}(N)^{\underline{p}-1}\lambda c^{q(x)+1}}, \tag{3.19}$$

this implies that u_n is a local minimum of Ψ. □

Proposition 3.8. The sequence $(m_n)_n$ is strictly negative and converges to zero.

Proof . In view of condition **(H3)**, we have $\vartheta_n \in K_n$

$$m_n \leq \Psi(\vartheta_n) = -\int_\Omega \int_0^{\vartheta_n} f(x,t)\,dt\,dx < 0. \tag{3.20}$$

To prove that $\lim\limits_{n\to+\infty} m_n = 0$ it is sufficient to observe that for every $n \in \mathbb{N}$ and $u \in K_n$, we have

$$0 > m_n = \Psi(u_n) \geq -kb_n|\Omega|. \tag{3.21}$$

Since $(b_n)_n$ converges to zero, we conclude the required result. □

Step 3 :. Proof of Theorem 3.2. Since the terms of $(u_n)_n$ are local minima of Ψ, they are weak solutions of (1.2). In virtue of Proposition 3.6, we have $0 \leq u_n(x) \leq a_n$ for almost every where $x \in \Omega$ and since $(a_n)_n$ converges to zero. An infinite number of distinct sequences $(u_n)_n$ can be found such that $\lim\limits_{n\to+\infty} \|u_n\|_{L^\infty(\Omega)} = 0$. Furthermore, we have

$$\begin{aligned} m_n = \Psi(u_n) &\geq \sum_{i=1}^N \widehat{M}_i\left(\int_\Omega \frac{1}{p_i(x)}\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)} dx\right) - \int_\Omega G(x,u_n)dx \\ &\geq \sum_{i=1}^N \widehat{M}_i\left(\int_\Omega \frac{1}{p_i(x)}\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)} dx\right) - kb_n|\Omega|. \end{aligned} \tag{3.22}$$

Thus, if $\|u_n\|_{1,\vec{p}(\cdot)} \leq 1$, we have

$$\frac{m}{\overline{p}N^{\underline{p}-1}}\|u_n\|^{\overline{p}}_{1,\vec{p}(\cdot)} \leq m_n + kb_n|\Omega| \longrightarrow 0. \tag{3.23}$$

Thus, $\lim_{n\to+\infty} \|u\|_{1,\vec{p}(\cdot)} = 0$, which completes our proof.

Now, we present an example to illustrate the main results.

Example 3.9. We define $M_i(t) = (1+t)^{\theta_i}$ for $i = 1, \dots N$, where $\theta_i > 0$. It is worth noting that $M_i(t) \geq 1$ for all $t \geq 0$, which directly verifies the condition stated in **(H4)**. Let

$$f(x,t) = \begin{cases} (1+|x|^2)(\underline{p}+2)t^{\underline{p}+1}\sin\left(\frac{1}{t^{\underline{p}}}\right) - \underline{p}(1+|x|^2)t\cos\left(\frac{1}{t^{\underline{p}}}\right) & \text{if } t > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.24}$$

It is easy to compute directly that

$$F(x,t) = \begin{cases} (1+|x|^2)t^{\underline{p}+2}\sin\left(\frac{1}{t^{\underline{p}}}\right) & \text{if } t > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.25}$$

We shall now consider the following nonlinear perturbed Kirchhoff problem

$$\begin{cases} -\sum_{i=1}^{N}\left[1+\int_{\Omega}\frac{1}{p_i(x)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)}dx\right]^{\theta_i}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\right) \\ = (1+|x|^2)(\underline{p}+2)t^{\underline{p}+1}\sin\left(\frac{1}{t^{\underline{p}}}\right) - \underline{p}(1+|x|^2)t\cos\left(\frac{1}{t^{\underline{p}}}\right) \quad \text{in } \Omega, \\ u=0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{3.26}$$

Let $(a_n)_n$, $(b_n)_n$, and $(\vartheta_n)_n$ be three positive sequences satisfying the conditions:

$$a_n = \left(\frac{1}{2n\pi+2\pi}\right)^{\frac{1}{\underline{p}}}, \quad b_n = \left(\frac{1}{2n\pi+\frac{3\pi}{2}}\right)^{\frac{1}{\underline{p}}} \quad \text{and} \quad \vartheta_n = \left(\frac{1}{4n\pi+\frac{\pi}{2}}\right)^{\frac{1}{\underline{p}}}, \tag{3.27}$$

for every $n \in \mathbb{N}$. Then one easily deduces

$$\int_0^{a_n} f(x,s)ds = \sup_{t\in[a_n,b_n]}\int_0^t f(x,s)ds,$$

and $F(x,\vartheta_n) > 0$. So conditions **(H2)** and **(H3)** have been verified. Having satisfied all the assumptions of Theorem 3.2, we can affirm the existence of a sequence of positive, homoclinic weak solutions $(u_n)_n$ in $W^{1,\vec{p}(x)}(\Omega)$ for the problem (3.26).

References

[1] A. Ahmed, H. Hjiaj, and A. Touzani, *Existence of infinitely many weak solutions for a Neumann elliptic equations involving the $\vec{p}(\cdot)$-Laplacian operator*, Rend. Circ. Mat. Palermo (2) **64** (2015), no. 3, 459–473.

[2] M. Allaoui and A. Ourraoui, *Existence results for a class of $p(x)$-Kirchhoff problem with a singular weight*, Mediterr. J. Math. **13** (2016), no. 2, 677–686.

[3] C.O. Alves and A. El Hamidi, *Existence of solution for an anisotropic equation with critical exponent*, Nonlinear Anal. TMA. **4** (2005), 611–624.

[4] A.E. Amrouss and A. El Mahraoui, *Infinitely many solutions for anisotropic elliptic equations with variable exponent*, Proyec. J. Math. **40** (2021), no. 5, 1071–1096.

[5] S.N. Antontsev and J.F. Rodrigues, *On stationary thermorheological viscous flows*, Ann. Univer. Ferrara. **52** (2006), no. 1, 19–36.

[6] A. Arosio and S. Panizzi, *On the well-posedness of the Kirchhoff string*, Trans. Amer. Math. Soc. **348** (1996), no. 1, 305–330.

[7] M. Avci, R.A. Mashiyev, and B. Cekic, *Solutions of an anisotropic nonlocal problem involving variable exponent*, Adv. Nonlinear Anal. **2** (2013), no. 3, 325–338.

[8] M.M. Boureanu, A. Matei, and M. Sofonea, *Non-linear problems with $p(\cdot)$-growth conditions and applications to anti-plane contact models*, Adv. Nonlinear Stud. **14** (2014), no. 2, 295–313.

[9] F. Cammaroto and L. Vilasi, *Multiple solutions for a Kirchhoff-type problem involving the $p(x)$-Laplacian operator*, Nonlinear Anal. *74* (2011), no. 5, 1841–1852.

[10] Y. Chen, S. Levine, and R. Rao, *Variable exponent, linear growth functionals in image restoration*, SIAM J. Appl. Math. **66** (2006), no. 4, 1383–1406.

[11] N.T. Chung, *Multiplicity results for a class of $p(x)$-Kirchhoff type equations with combined nonlinearities*, Electron. J. Qual. Theory Differ. Equ. **42** (2012), 1–13.

[12] F.J.S.A. Corrêa and R.G. Nascimento, *On a nonlocal elliptic system of p-Kirchhoff-type under Neumann boundary condition*, Math. Model. Comput. **49** (2009), no. 3-4, 598–604.

[13] G. Dai and D. Liu, *Infinitely many positive solutions for a $p(x)$-Kirchhoff-type equation*, J. Math. Anal. Appl. **359** (2009), no. 2, 704–710.

[14] G. Dai and R. Hao, *Existence of solutions for a $p(x)$-Kirchhoff-type equation*, J. Math. Anal. Appl. **359** (2009), no. 1, 275–284.

[15] L. Diening, P. Harjulehto, P. Hästö, and M. Růžička, *Lebesgue and Sobolev Spaces with Variable Exponents*, Lecture Notes in Mathematics, vol. 2017, Springer-Verlag, Heidelberg, 2011.

[16] G.C.G. dos Santos, J.R.S. Silva, S.C.Q. Arruda, and L.S. Tavares, *Existence and multiplicity results for critical anisotropic Kirchhoff-type problems with nonlocal nonlinearities*, Complex Var. Elliptic Equ. **67** (2022), no. 4, 822–842.

[17] X.L. Fan, *Anisotropic variable exponent Sobolev spaces and $\vec{p}(\cdot)$-Laplacian equations*, Complex Var. Elliptic Equ. **56** (2011), no. 7-9, 623–642.

[18] X.L. Fan, *On nonlocal $p(x)$-Laplacian Dirichlet problems*, Nonlinear Anal. **72** (2010), no. 7-8, 3314–3323.

[19] J.R. Graef, S. Heidarkhani, and L. Kong, *A variational approach to a Kirchhoff-type problem involving two parameters*, Results Math. **63** (2013), no. 3-4, 877–889.

[20] G. Kirchhoff, *Mechanik, Teubner, Leipzig*, Germany, 1883.

[21] B. Kone, S. Ouaro, and S. Traore, *Weak solutions for anisotropic nonlinear elliptic equations with variable exponents*, Electron. J. Differ. Equ. **144** (2009), 1–11.

[22] D. Liu, *On a p-Kirchhof equation via fountain theorem and dual fountain theorem,* Nonlinear Anal. **72** (2010), no. 1, 302-308.

[23] M. Mihăilescu and G. Morosanu, *Existence and multiplicity of solutions for an anisotropic elliptic problem involving variable exponent growth conditions*, Appl. Anal. **89** (2010), no. 2, 257–271.

[24] M. Mihăilescu, P. Pucci, and V. Rădulescu, *Eigenvalue problems for anisotropic quasilinear elliptic equations with variable exponent*, J. Math. Anal. Appl. **340** (2008), no. 1, 687–698.

[25] A. Ourraoui, *Multiplicity of solutions for $\vec{p}(\cdot)$-Laplacian elliptic Kirchhoff type equations*, Appl. Math. E-Notes. **20** (2020), 124–132.

[26] J. Rákosník, *Some remarks to anisotropic Sobolev spaces I*, Beiträge Anal. **13** (1979), 55-68.

[27] J. Rákosník, *Some remarks to anisotropic Sobolev spaces II*, Beiträge Anal. **15** (1981), 127–140.

[28] A. Razani and G. M. Figueiredo, *Degenerated and competing anisotropic (p,q)-Laplacians with weights*, Appl. Anal. **102** (2023), no. 16, 4471–4488.

[29] A. Razani and G.M. Figueiredo, *A positive solution for an anisotropic (p,q)-Laplacian*, Discrete Contin. Dyn. Syst. Ser. S **16** (2023), no. 6, 1629–1643.

[30] A. Razani and G.M. Figueiredo, *Existence of infinitely many solutions for an anisotropic equation using genus theory*, Math. Meth. Appl. Sci. **45** (2022), no. 12, 7591-7606

[31] A. Razani, *Nonstandard competing anisotropic (p,q)-Laplacians with convolution*, Bound. Value Probl. **2022** (2022), 87.

[32] B. Ricceri, *On an elliptic Kirchhoff-type problem depending on two parameters*, J. Glob. Optim. **46** (2010), no. 4, 543–549.

[33] M. Růžička, *Electrorheological Fluids: Modeling and Mathematical Theory*, Lecture Notes in Mathematics, Springer, Berlin. 1748, 2000.

[34] T. Soltani and A. Razani, *Solutions for an anisotropic elliptic problem involving nonlinear terms*, Q. Math. (2023), 1-20. https://doi.org/10.2989/16073606.2023.2190045

[35] R. Stanway, J.L. Sproston, and A.K. El-Wahed, *Applications of electro-rheological fluids in vibration control: A survey*, Smart Mater. Struct. **5** (1996), no. 4, 464-482.

[36] V.V. Zhikov, *Averaging of functionals in the calculus of variations and elasticity*, Math. USSR-Izvestiya **29** (1987), no. 1, 33–66.