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Edge-preserving smoothing of Perona-Malik nonlinear diffusion in two-dimensions

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Abstract

It has been thirty years since Perona and Malik (PM) introduced the nonlinear diffusion equation in image processing and analysis. The problem's complexity was to find a suitable and adaptive diffusion function that smooths away noise or textures while preserving sharp edges of a sufficiently smooth intensity. This paper provides a new two-dimensional analysis of the PM diffusion equation to examine its behavior during scales and an explicit formula to select the right diffusion function adequately. In this context, we study the PM equation at the zero crossings of the first and second directional derivatives of a sufficiently smooth function in the gradient direction.

Keywords: Nonlinear PDE, Diffusion Function, Scale-Space, Edge-Detection, Image Processing
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1 Introduction

At the lowest levels in the chain of information processing of a visual system, one of the approaches used for analyzing image data is the use of scale-space representation, which is characterized by a specific operator. The researchers' most frequently asked question was: What is the most appropriate operator, and how large it should be for the information extraction tasks?

It all began in *1984*, when Witkin [17] formulated a continuous approach to multi-scale description, namely the scale-space formulation. Subsequently, Koenderink indicated in his paper [9] that the structure of an image at different scales, in a scale-space formulation, are related explicitly by the heat equation:

$$\frac{\partial I}{\partial t}(x, y; t) - \Delta I(x, y; t) = 0. \tag{1.1}$$

Besides, particularly in image processing, this formulation can be viewed as a filtering process. Moreover, for many reasons [2, 8, 18], this equation is unique by the fact that it is equivalent to a Gaussian convolution, and consequently, it satisfies the *Causality criterion* [9]; that is, no spurious detail should be generated, passing from finer to coarser scales. In the mathematical imaging community, the equation (1.1) is recognized as a linear isotropic diffusion equation. However, it was observed that this linear formulation could not preserve the differential invariant features and their localizations through scales. For this reason, Perona and Malik devised in their seminal paper [13] a new concept for

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the scale-space representation. They made the diffusion locally adaptive to the structure of the image by introducing a diffusion function g in the equation (1.1), as shown in the following nonlinear partial differential equation (PDE):

$$\frac{\partial I}{\partial t}(x,y;t) = \nabla \cdot \left[g\left(\|\nabla I\|^2 \right) \nabla I(x,y;t) \right]. \tag{1.2}$$

Perona and Malik chose such a process to preserve significant features like edges while reducing irrelevant information such as noise in a homogeneous area. Since then, several scientific works develop new adaptive diffusion processes for edge detection or image denoising by considering an appropriate diffusion function g [3, 5, 11, 12, 14].

This paper will provide in Section 2 a definition of nonlinear scale-space that is deducted from the PM equation. After examining the causality requirement on the PM nonlinear diffusion in Section 3, Section 4 will answer one of the fundamental questions about examining and selecting the best diffusion function g relevant for accomplishing different tasks. We terminate this work with a conclusion in Section 5.

2 Nonlinear Scale-Space Representation

From a physical perspective, Koenderink [9] derived the notion of causality and proclaimed that while the scale increases, the number of the structures of an image should be reduced and that no new structure has to be generated. He showed that the relationship between structures at different scales leads to the diffusion equation using differential geometry. From this perspective, one can define the *Nonlinear Scale Space*:

$$I : \Omega \times (0,T) \to \mathbb{R}$$

which is considered infinitely differentiable function, as the family of derived images constructed by the following continuous PM nonlinear diffusion:

$$I_t = \nabla \cdot \left[g\left(\|\nabla I\|^2 \right) \nabla I \right] = 2g'\left(\|\nabla I\|^2 \right) \left(\nabla I \cdot \mathbf{H}_I \nabla I \right) + g\left(\|\nabla I\|^2 \right) \Delta I \tag{2.1}$$

where $\Omega := (0,a) \times (0,b)$ is a rectangular image domain of $\mathbb{R}^2$, I_t denotes the partial derivative with respect to t, $\mathbf{H}_I$ is the Hessian matrix of I, and g is a real-valued diffusion function. At a specific location $(x,y) \in \Omega$, It is quite clear that the nonlinear diffusion (2.1) depends on the characteristics of the function g. Hence, the question raises the issue of determining a suitable diffusion function g for a specific purpose. Therefore, it is of utmost importance that one examines and performs two-dimensional analysis on the connection between the smoothing-enhancing property of the equation (2.1) and the curve of the function g at specific points, where there are local extrema or authentic edges [6].

3 Non-Enhancement of Local Extrema

One of the essential properties of scale-space, as mentioned earlier, is that there are fewer details at higher levels of scales [9]. To illustrate this requirement, the critical points, as defined below, are considered as part of the image structure.

Definition 3.1. (*Local Extrema*)

At a certain scale level $\bar{t}$.

- A point $\left(\overline{x}, \overline{y}; \overline{t}\right) \in \Omega \times (0,T)$ is said to be a *critical point* for the real function I if

$$\nabla I\left(\overline{x}, \overline{y}; \overline{t}\right) = \mathbf{0}.$$

- At a critical point $\left(\overline{x}, \overline{y}; \overline{t}\right)$, $I\left(\overline{x}, \overline{y}; \overline{t}\right)$ is a *strict local maximum*(*minimum*) if the Hessian matrix of I is *negative* (*positive*) *definite*. In this case, $\left(\overline{x}, \overline{y}; \overline{t}\right)$ is said to be a *local maximum* (*minimum*) *point*.

Thus, one would expect that if for some scale level $\bar{t}$, a point $(\overline{x}, \overline{y})$ is a local maximum (minimum) for the nonlinear scale-space representation I. Then, at this point, I must not increase (decrease) while $\bar{t}$ increases. Within the more general context, Weickert demonstrated in his book [16] the following result:

Theorem 3.2. (*Non-enhancement of local extrema*) While the scale parameter t increases and at a critical point $(\overline{x},\overline{y};t)$ where $\nabla I=\mathbf{0}$.

$$\text{No new local extrema is generated} \Leftrightarrow \mathbf{sign}\left(I_t\right)=\mathbf{sign}\left(\Delta I\right) \Leftrightarrow g\left(0\right)>0. \tag{3.1}$$

Proof . At a critical point, we have $\nabla I=\mathbf{0}$. Then, by using the equation (2.1) we obtain,

$$I_t=g\left(0\right)\Delta I. \tag{3.2}$$

Besides, let us assume that $(\overline{x},\overline{y};t)$ is a local maximum where the Hessian matrix of I is negative definite and possesses negative eigenvalues α_+ and α_- as follows:

$$\alpha_{+/-}=\frac{1}{2}\left(\Delta I\pm\sqrt{\Delta I^2-4\left|\mathbf{H}_I\right|}\right)<0 \tag{3.3}$$

where $\left|\mathbf{H}_I\right|$ is the determinant of the Hessian matrix $\mathbf{H}_I$. We have

$$\alpha_+ + \alpha_- = \Delta I<0. \tag{3.4}$$

Then,

$$\begin{aligned}\text{No new local maximum is created at } (\overline{x},\overline{y};t) &\Longleftrightarrow (I_t<0) \text{ and } \left(\alpha_{+/-}<0\right)\\ &\overset{(3.4)}{\Longleftrightarrow} (I_t<0) \text{ and } (\Delta I<0)\,.\end{aligned}$$

Similarly, one proceeds and obtains:

$$\text{No new local minimum is created at} \left(\overline{x},\overline{y};t\right) \Longleftrightarrow \left(I_t>0\right) and \left(\Delta I>0\right).$$

Thus, in either case, we have:

$$\begin{aligned}\text{No new local extrema is created at} \left(\overline{x},\overline{y};t\right) &\Longleftrightarrow \mathbf{sign}\left(I_t\right)=\mathbf{sign}\left(\Delta I\right)\\ &\overset{(3.2)}{\Longleftrightarrow} g\left(0\right)>0.\end{aligned}$$

Which completes the proof. □

The requirement (3.1) prevents local extrema from being enhanced and thus avoids the creation of some false structures in scale-space representation.

4 Behavioral Analysis of the Two-Dimensional PM Nonlinear Diffusion

4.1 Smoothing-Enhancing of PM Diffusion

To analyze the PM equation's smoothing-enhancing property, it is significant that we examine the temporal variation of the gradient magnitude of I:

- In the one-dimensional case

Perona and Malik investigated the following diffusion process:

$$I_t=\left[g\left(I_x^2\right)I_x\right]_x. \tag{4.1}$$

If there exist a sufficiently smooth solution I it satisfies:

$$\left(I_x^2\right)_t=2I_x\left(I_x\right)_t=8I_x^4I_{xx}^2g''\left(I_x^2\right)+4I_x^2\left(3I_{xx}^2+I_xI_{xxx}\right)g'\left(I_x^2\right)+2I_xI_{xxx}g\left(I_x^2\right). \tag{4.2}$$

- In the N-dimensional case $(N\in\mathbb{N}^*)$

For a sufficiently smooth intensity $I:\Omega\times(0,T)\longrightarrow\mathbb{R}\left(\Omega\subset\mathbb{R}^N \text{ in this case}\right)$,we have the following extension:

Lemma 4.1. The temporal variation of the gradient magnitude of I is defined as follows:

$$\left(\|\nabla I\|^2\right)_t = 8g'' \left[\nabla I \cdot \mathbf{H}_I \nabla I\right]^2 + 4g' \left[2\left(\nabla I \cdot \mathbf{H}_I^2 \nabla I\right) + \mathbf{tr}(\mathbf{H}_I)\left(\nabla I \cdot \mathbf{H}_I \nabla I\right) + \left(\nabla I \cdot \mathbf{A}_I \nabla I\right)\right] + 2g\, \mathbf{tr}(\mathbf{A}_I) \tag{4.3}$$

where $g := g\left(\|\nabla I\|^2\right)$ and $\mathbf{A}_I = (a_{i,j})$ is real symmetric matrix of $\mathbb{R}^{N\times N}$ such that

$$a_{i,j} = \nabla I \cdot \nabla\left[\nabla\left(\nabla I \cdot e_i\right) \cdot e_j\right]$$

and $(e_1, e_2, ..., e_N)$ is the canonical basis of $\mathbb{R}^N$.

Proof . We have

$$\nabla g = \nabla g\left(\|\nabla I\|^2\right) = 2g' \mathbf{H}_I \nabla I, \tag{4.4}$$

and by using the PM equation (2.1) in $\mathbb{R}^N$, we get

$$\left(\|\nabla I\|^2\right)_t = 2\nabla I \cdot \nabla I_t = 2\nabla I \cdot \nabla\left[\nabla g \cdot \nabla I + g\Delta I\right].$$

Then, according to (4.4), we obtain

$$\begin{aligned}\left(\|\nabla I\|^2\right)_t =& 2\nabla I \cdot \nabla\left[2g'\left(\nabla I \cdot \mathbf{H}_I \nabla I\right) + g\Delta I\right] \\ =& 2\nabla I \cdot \left[2\left(\nabla I \cdot \mathbf{H}_I \nabla I\right)\nabla g' + 2g'\nabla\left(\nabla I \cdot \mathbf{H}_I \nabla I\right) + \Delta I \nabla g + g\nabla\left(\Delta I\right)\right],\end{aligned}$$

and knowing that

$$\begin{aligned}\nabla\left[\nabla I \cdot \mathbf{H}_I \nabla I\right] &= \mathbf{J}^T_{\mathbf{H}_I \nabla I} \nabla I + \mathbf{J}^T_{\nabla I} \mathbf{H}_I \nabla I \\ &= \mathbf{J}^T_{\mathbf{H}_I \nabla I} \nabla I + \mathbf{H}_I^2 \nabla I \\ &= 2\mathbf{H}_I^2 \nabla I + \mathbf{A}_I \nabla I\end{aligned} \tag{4.5}$$

where, $\mathbf{J}^T_{\mathbf{H}_I \nabla I}$ is the transpose of the Jacobian matrix of $\mathbf{H}_I \nabla I$. we get

$$\begin{aligned}\left(\|\nabla I\|^2\right)_t =& 2\nabla I \cdot \left[4g''\left(\nabla I \cdot \mathbf{H}_I \nabla I\right)\mathbf{H}_I \nabla I + 2g'\left(2\mathbf{H}_I^2 \nabla I + \mathbf{A}_I \nabla I\right) + 2g'\Delta I \mathbf{H}_I \nabla I + g\nabla\left(\Delta I\right)\right] \\ =& 8g''\left[\nabla I \cdot \mathbf{H}_I \nabla I\right]^2 + 4g'\Big[2\left(\nabla I \cdot \mathbf{H}_I^2 \nabla I\right) + \left(\nabla I \cdot \mathbf{A}_I \nabla I\right) + \Delta I\left(\nabla I \cdot \mathbf{H}_I \nabla I\right)\Big] \\ &+ 2g\nabla I \cdot \nabla\left(\Delta I\right).\end{aligned}$$

Which completes the proof. □

Corollary 4.2. In the two-dimensional case, The temporal variation of the gradient magnitude of I is defined as follows:

$$\begin{aligned}\left(\|\nabla I\|^2\right)_t =& 8g''\left[\nabla I \cdot \mathbf{H}_I \nabla I\right]^2 \\ &+ 4g'\Big[3\,\mathbf{tr}(\mathbf{H}_I)\left(\nabla I \cdot \mathbf{H}_I \nabla I\right) - 2\|\nabla I\|^2 |\mathbf{H}_I| + \left(\nabla I \cdot \mathbf{A}_I \nabla I\right)\Big] \\ &+ 2g\,\mathbf{tr}(\mathbf{A}_I)\end{aligned} \tag{4.6}$$

where

$$\mathbf{A}_I = \begin{pmatrix}\nabla I \cdot \nabla I_{xx} & \nabla I \cdot \nabla I_{xy} \\ \nabla I \cdot \nabla I_{xy} & \nabla I \cdot \nabla I_{yy}\end{pmatrix}. \tag{4.7}$$

Proof . According to ***Cayley–Hamilton theorem*** the matrix $\mathbf{H}_I$ is annihilated by its characteristic polynomial. Which means that

$$\mathbf{H}_I^2 - \mathbf{tr}\left(\mathbf{H}_I\right)\mathbf{H}_I + |\mathbf{H}_I|\,\mathbf{I}_2 = \mathbf{0}$$

where $\mathbf{I}_2$ is the identity matrix. Then,

$$\nabla I \cdot \mathbf{H}_I^2 \nabla I = \mathbf{tr}\left(\mathbf{H}_I\right)\left(\nabla I \cdot \mathbf{H}_I \nabla I\right) - \|\nabla I\|^2 |\mathbf{H}_I|. \tag{4.8}$$

Finally, by using the equation (4.8) in the Lemma 4.1, we find the equation (4.6). Which completes the proof. □

Besides, the quantity $\|\nabla I\|^2$ measures the maximum rate of change in the intensity I at the location (x,y) of Ω. Hence, by studying its variation in time (4.6) in every point (x,y;t) of the space $\Omega \times (0, T)$, one can obtain valuable insights into how the PM nonlinear diffusion behaves.

4.2 Smoothing-Enhancing of the Edge

It is well known that edges are one of the necessary image features that one should detect for the sake of many applications, especially those related to the segmentation and recognition paradigm. Therefore, we will restrict our analysis to the edges of objects in the image.

- In the one-dimensional case

Perona and Malik examined the behavior of their Model [13] at edge location where

$$I_{xx} = 0, \text{ and } I_x I_{xxx} < 0,$$

and stated:

$$\begin{aligned} \text{The edge is enhanced by (4.1)} &\iff \left(I_x^2\right)_t = 2I_x I_{xxx}\left[2I_x^2 g'\left(I_x^2\right) + g\left(I_x^2\right)\right] > 0 \\ &\iff 2I_x^2 g'\left(I_x^2\right) + g\left(I_x^2\right) < 0 \\ &\iff \phi'\left(I_x\right) < 0, \end{aligned} \tag{4.9}$$

$$\begin{aligned} \text{the edge is smoothed by (4.1)} &\iff \left(I_x^2\right)_t = 2I_x I_{xxx}\left[2I_x^2 g'\left(I_x^2\right) + g\left(I_x^2\right)\right] < 0 \\ &\iff 2I_x^2 g'\left(I_x^2\right) + g\left(I_x^2\right) > 0 \\ &\iff \phi'\left(I_x\right) > 0, \end{aligned} \tag{4.10}$$

where ϕ is the flux function defined as $\phi\left(I_x\right) = g\left(I_x^2\right) I_x$.

- In the two-dimensional case

We will extend the last result and determine the factors influencing the PM equation's diffusion behavior. Lindeberg [10] defined the edges as "the set of points for which the gradient magnitude assumes a maximum in the gradient direction," which is known as Haralick and Canny's edge detection method [7, 4].

By introducing a local orthonormal coordinate system $(\boldsymbol{u}, \boldsymbol{v})$, such that in each point, the $\boldsymbol{v}$-direction is parallel to the maximal change of the intensity (the gradient direction of I), and the other is perpendicular to it. One can define an edge, at any scale t, as the points on the zero-crossing curves of the second-derivative of I in the $\boldsymbol{v}$-direction ($I_{\boldsymbol{vv}}$) for which the third-derivative $I_{\boldsymbol{vvv}}$ is strictly negative, which leads to the following differential geometric definition of an edge [10]:

Definition 4.3. (*Edge location*)

- For a fixed time t, a point $(x, y; t)$ is regarded as a gradient maximum in the gradient direction of I if the following conditions are met:

$$I_{\boldsymbol{vv}} = \frac{I_x^2 I_{xx} + 2I_x I_y I_{xy} + I_y^2 I_{yy}}{I_x^2 + I_y^2} = 0, \tag{4.11}$$

$$I_{\boldsymbol{vvv}} = \frac{I_x^3 I_{xxx} + 3I_x^2 I_y I_{xxy} + 3I_x I_y^2 I_{xyy} + I_y^3 I_{yyy}}{\left(\sqrt{I_x^2 + I_y^2}\right)^3} < 0. \tag{4.12}$$

- We say that an edge point $(x, y; t)$ is smoothed by a PDE if in a neighborhood of $(x, y; t)$, $\left(\left\|\nabla I\left(x, y; t\right)\right\|^2\right)_t < 0$, i.e. if $\left\|\nabla I\left(x, y; t\right)\right\|^2$ decreases as t increases.
- We say that an edge point $(x, y; t)$ is enhanced by a PDE if in a neighborhood of $(x, y; t)$, $\left(\left\|\nabla I\left(x, y; t\right)\right\|^2\right)_t > 0$, i.e. if $\left\|\nabla I\left(x, y; t\right)\right\|^2$ increases as t increases.

Besides, Clark stated in his paper [6] that the points satisfying (4.11) and (4.12) are the authentic edges, and the points satisfying (4.11) with $I_{\boldsymbol{vvv}} > 0$ are the phantom edges, while the latter ones are not the edges at all. Furthermore, the two conditions (4.11) and (4.12) could be represented in a more simplified manner:

We know that

$$I_v = \nabla I \cdot \frac{1}{\|\nabla I\|} \nabla I = \|\nabla I\| . \tag{4.13}$$

Since,

$$I_{vv} = \nabla I_v \cdot \frac{1}{\|\nabla I\|} \nabla I = \nabla \left(\|\nabla I\| \right) \cdot \frac{1}{\|\nabla I\|} \nabla I \quad \text{and} \quad \nabla \left(\|\nabla I\| \right) = \frac{1}{\|\nabla I\|} \mathbf{H}_I \nabla I.$$

Therefore, we obtain the second directional derivative of I in the gradient direction that corresponds to (4.11):

$$I_{vv} = \frac{1}{\|\nabla I\|^2} \left[\nabla I \cdot \mathbf{H}_I \nabla I \right] = 0. \tag{4.14}$$

Lemma 4.4. The third directional derivative of I in the gradient direction can be defined as:

$$I_{vvv} = \frac{1}{\|\nabla I\|^3} \left[-2 \left(\frac{\nabla I \cdot \mathbf{H}_I \nabla I}{\|\nabla I\|} \right)^2 + 2\, \mathbf{tr}\left(\mathbf{H}_I \right) \left(\nabla I \cdot \mathbf{H}_I \nabla I \right) - 2 \|\nabla I\|^2 \left| \mathbf{H}_I \right| + \left(\nabla I \cdot \mathbf{A}_I \nabla I \right) \right] \tag{4.15}$$

where $\mathbf{A}_I$ is the real symmetric matrix defined in (4.7)

Proof . we have

$$\begin{aligned} I_{vvv} &= \nabla I_{vv} \cdot \frac{1}{\|\nabla I\|} \nabla I \\ &= \nabla \left[\frac{1}{\|\nabla I\|^2} \left(\nabla I \cdot \mathbf{H}_I \nabla I \right) \right] \cdot \frac{1}{\|\nabla I\|} \nabla I \\ &= \left[\left(\nabla I \cdot \mathbf{H}_I \nabla I \right) \nabla \left(\frac{1}{\|\nabla I\|^2} \right) + \frac{1}{\|\nabla I\|^2} \nabla \left(\nabla I \cdot \mathbf{H}_I \nabla I \right) \right] \cdot \frac{1}{\|\nabla I\|} \nabla I. \end{aligned} \tag{4.16}$$

Since,

$$\nabla \left(\frac{1}{\|\nabla I\|^2} \right) = \frac{-2}{\|\nabla I\|^4} \mathbf{H}_I \nabla I \tag{4.17}$$

and by using (4.5), we get

$$I_{vvv} = \frac{1}{\|\nabla I\|^3} \left[-2 \left(\frac{\nabla I \cdot \mathbf{H}_I \nabla I}{\|\nabla I\|} \right)^2 + 2 \left(\nabla I \cdot \mathbf{H}_I^2 \nabla I \right) + \left(\nabla I \cdot \mathbf{A}_I \nabla I \right) \right]. \tag{4.18}$$

Finally, by using (4.8) in (4.18) we obtain:

$$I_{vvv} = \frac{1}{\|\nabla I\|^3} \left[-2 \left(\frac{\nabla I \cdot \mathbf{H}_I \nabla I}{\|\nabla I\|} \right)^2 + 2\, \mathbf{tr}\left(\mathbf{H}_I \right) \left(\nabla I \cdot \mathbf{H}_I \nabla I \right) - 2 \|\nabla I\|^2 \left| \mathbf{H}_I \right| + \left(\nabla I \cdot \mathbf{A}_I \nabla I \right) \right].$$

Which completes the proof. □

Proposition 4.5. The condition(4.12) is equivalent to:

$$I_{vvv} = \frac{1}{\|\nabla I\|^3} \left[\nabla I \cdot \mathbf{A}_I \nabla I \right] < 0 \tag{4.19}$$

Proof . By considering the condition (4.14) and the fact that $\mathbf{H}_I$ is a real symmetric matrix. One can derive the following:

$$I_{vv} = \alpha_+ \cos^2(\varphi) + \alpha_- \sin^2(\varphi) = 0$$

where $\alpha_{+/-}$ are the eigenvalues of $\mathbf{H}_I$ and φ is the angle between $\boldsymbol{v}$ and an eigenvector of α_+. Which leads us to three cases [15]:

1. If $\alpha_+ = \alpha_- = 0$, then we have a *flat* shape, which is a case that has to be excluded.
2. If $\alpha_+ \times \alpha_- < 0$, then we obtain *saddle* shapes, that are not considered as edges; This case is also excluded.
3. If $\begin{cases} \alpha_+ = 0, \varphi \equiv 0 \mod (\pi) \\ \alpha_- = 0, \varphi \equiv \frac{\pi}{2} \mod (\pi) \end{cases}$ then we have *cylindrical* shapes.

Accordingly, the only case that corresponds to edge shapes is the third one. Which means that $|\mathbf{H}_I| = \alpha_+ \times \alpha_- = 0$. Thus, by considering this result with that of (4.14), we conclude from lemma 4.4 that $I_{vvv} = \frac{1}{\|\nabla I\|^3} [\nabla I \cdot \mathbf{A}_I \nabla I] < 0$. Which completes the proof. □

Since, the matrix $\mathbf{A}_I$ is negative definite (4.19). It has negative eigenvalues μ_+ and μ_-, and we have

$$\mu_+ + \mu_- = \mathbf{tr}(\mathbf{A}_I) < 0 \tag{4.20}$$

where $\mathbf{tr}(\mathbf{A}_I)$ is the trace of $\mathbf{A}_I$, and

$$\mu_{+/-} = \frac{1}{2}\left(\mathbf{tr}(\mathbf{A}_I) \pm \sqrt{\mathbf{tr}(\mathbf{A}_I)^2 - 4|\mathbf{A}_I|}\right) = \frac{\mathbf{tr}(\mathbf{A}_I)}{2}\beta_{+/-} \tag{4.21}$$

with

$$\beta_{+/-} = 1 \pm \sqrt{1 - 4\frac{|\mathbf{A}_I|}{\mathbf{tr}(\mathbf{A}_I)^2}} \tag{4.22}$$

and

$$\begin{cases} 1 \le \beta_+ < 2, \\ 0 < \beta_- \le 1. \end{cases} \tag{4.23}$$

On the other hand, since $\mathbf{A}_I$ is a real symmetric matrix. Then, there exists an orthogonal matrix $\mathbf{O}$ such that

$$\mathbf{A}_I = \frac{1}{2}\mathbf{tr}(\mathbf{A}_I)\,\mathbf{O}\Lambda\mathbf{O}^{-1} = \frac{1}{2}\mathbf{tr}(\mathbf{A}_I)\,\mathbf{O}\Lambda\mathbf{O}^T \tag{4.24}$$

where Λ is real diagonal matrix:

$$\Lambda = \begin{pmatrix} \beta_- & 0 \\ 0 & \beta_+ \end{pmatrix}. \tag{4.25}$$

While the eigenvectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are the corresponding columns of $\mathbf{O}$. In investigating the PM equation's behavior (2.1) at edge locations, we provide our main theorem that considers the relationship between the curve of the diffusion function g in the PDE (2.1) and the Smoothing-Enhancing property.

Theorem 4.6. (Smoothing-Enhancing Edges) At a maximum gradient point where the two conditions (4.14) and (4.19) are met, we have the following results:

$$\text{The edge is enhanced by (2.1)} \iff \|\nabla I\|^2 g'\left(\|\nabla I\|^2\right) Q(\boldsymbol{w}) + g\left(\|\nabla I\|^2\right) < 0, \tag{4.26}$$

$$\text{the edge is smoothed by (2.1)} \iff \|\nabla I\|^2 g'\left(\|\nabla I\|^2\right) Q(\boldsymbol{w}) + g\left(\|\nabla I\|^2\right) > 0, \tag{4.27}$$

where, Q denotes the quadratic form associated to Λ and $\boldsymbol{w}^T = (\boldsymbol{v}\cdot\boldsymbol{a}_1\ \ \boldsymbol{v}\cdot\boldsymbol{a}_2)^T$ is a unit vector such that:

$$\beta_- \le Q(\boldsymbol{w}) \le \beta_+. \tag{4.28}$$

Proof . Suppose that the conditions (4.14) and (4.19) are satisfied. We have $|\mathbf{H}_I| = 0$ and according to corollary 4.2, we get:

$$\left(\|\nabla I\|^2\right)_t = 4g'\left(\|\nabla I\|^2\right)(\nabla I \cdot \mathbf{A}_I \nabla I) + 2g\left(\|\nabla I\|^2\right)\mathbf{tr}(\mathbf{A}_I). \tag{4.29}$$

Substituting (4.24) into the equation (4.29), we find:

$$\begin{aligned}\left(\|\nabla I\|^2\right)_t =& 4g'\left(\|\nabla I\|^2\right)\nabla I \cdot \left(\frac{1}{2}\mathbf{tr}(\mathbf{A}_I)\,\mathbf{O}\Lambda\mathbf{O}^T\right)\nabla I + 2g\left(\|\nabla I\|^2\right)\mathbf{tr}(\mathbf{A}_I) \\ =& 2\,\mathbf{tr}(\mathbf{A}_I)\left[\|\nabla I\|^2 g'\left(\|\nabla I\|^2\right)\left(\boldsymbol{v}^T\mathbf{O}\Lambda\mathbf{O}^T\boldsymbol{v}\right) + g\left(\|\nabla I\|^2\right)\right].\end{aligned} \tag{4.30}$$

Then, by putting $\boldsymbol{w} = \mathbf{O}^T \boldsymbol{v}$ and $Q(\boldsymbol{w}) = \boldsymbol{w}^T \Lambda \boldsymbol{w}$. We obtain:

$$\left(\|\nabla I\|^2\right)_t = 2\,\mathbf{tr}\left(\mathbf{A}_I\right)\left[\|\nabla I\|^2 g'\left(\|\nabla I\|^2\right) Q(\boldsymbol{w}) + g\left(\|\nabla I\|^2\right)\right]. \tag{4.31}$$

As long as $\mathbf{tr}\left(\mathbf{A}_I\right) < 0$ (4.20), we conclude that

$$\mathbf{sign}\left[\left(\|\nabla I\|^2\right)_t\right] = -\,\mathbf{sign}\left[\|\nabla I\|^2 g'\left(\|\nabla I\|^2\right) Q(\boldsymbol{w}) + g\left(\|\nabla I\|^2\right)\right], \tag{4.32}$$

and since

$$\|\boldsymbol{w}\| = \sqrt{\left(\boldsymbol{v}\cdot\boldsymbol{a}_1\right)^2 + \left(\boldsymbol{v}\cdot\boldsymbol{a}_2\right)^2} = \sqrt{\cos^2(\theta) + \cos^2\left(\frac{\pi}{2} - \theta\right)} = 1 \tag{4.33}$$

with θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{a}_1$, we get

$$Q(\boldsymbol{w}) = \beta_- \cos^2(\theta) + \beta_+ \sin^2(\theta) \tag{4.34}$$

where $\beta_- \le Q(\boldsymbol{w}) \le \beta_+$. Which completes the proof. □

Remark 4.7. if $g' = 0$ and $g = \kappa \in \mathbb{R}$, then $\left(\|\nabla I\|^2\right)_t = 2\kappa\,\mathbf{tr}\left(\mathbf{A}_I\right)$ with $\mathbf{tr}\left(\mathbf{A}_I\right) < 0$. In that case, we realize the *Heat* equation and the edge is smoothed if $\kappa > 0$ and is enhanced if $\kappa < 0$.

Remark 4.8. For every increasing strictly positive (decreasing strictly negative) real function g, the edges are smoothed (enhanced) during the diffusion process (2.1).

Remark 4.9. Aubert and Kornprobost investigated in [1] the parabolicity of the equation (2.1) and provided the following result: The equation (2.1) is parabolic if, and only if,

$$b\left(\|\nabla I\|^2\right) = 2\|\nabla I\|^2 g'\left(\|\nabla I\|^2\right) + g\left(\|\nabla I\|^2\right) > 0. \tag{4.35}$$

Hence, we knew that $Q(\boldsymbol{w}) \le \beta_+ < 2$. Then by choosing $g > 0$ and $g' < 0$, we get

$$0 < b\left(\|\nabla I\|^2\right) < \|\nabla I\|^2 Q(\boldsymbol{w})\, g'\left(\|\nabla I\|^2\right) + g\left(\|\nabla I\|^2\right). \tag{4.36}$$

Thus, if the equation (2.1) is parabolic, then the edges are always smoothed during the diffusion process using a positive decreasing function.

Example 4.10. By considering the following diffusion function

$$g\left(s^2\right) = \frac{1}{1 + \left(\frac{s}{\lambda}\right)^2} \tag{4.37}$$

where $\lambda > 0$ is a threshold parameter. Perona and Malik [13] established a strategy for detecting a particular feature labeled as "the edge". It may appear at first glance, and as in the one-dimensional case [13], their model's behavior depends on the curve of the diffusion function g and the threshold parameter λ. We have the following result:

$$\|\nabla I\|^2 g'\left(\|\nabla I\|^2\right) Q(\boldsymbol{w}) + g\left(\|\nabla I\|^2\right) = \frac{1 + \left(1 - Q(\boldsymbol{w})\right)\left(\frac{\|\nabla I\|}{\lambda}\right)^2}{\left[1 + \left(\frac{\|\nabla I\|}{\lambda}\right)^2\right]^2}. \tag{4.38}$$

Then, from (4.22) and (4.34) one can derive:

$$\begin{aligned} 1 - Q(\boldsymbol{w}) &= 1 - \left(\beta_- \cos^2(\theta) + \beta_+ \sin^2(\theta)\right) \\ &= \left(1 - \beta_-\right)\cos^2(\theta) + \left(1 - \beta_+\right)\sin^2(\theta) \\ &= \left(1 - \beta_-\right)\left(\cos^2(\theta) - \sin^2(\theta)\right). \end{aligned}$$

Therefore,

- $Q(\boldsymbol{w}) = 1$ if and only if $\begin{cases} \beta_- = \beta_+ = 1 \\ \text{or} \\ \theta = \frac{\pi}{4} + \frac{k\pi}{2}, k \in \mathbb{Z} \end{cases}$ if and only if $\begin{cases} (\nabla I \cdot \nabla I_{xx} - \nabla I \cdot \nabla I_{yy})^2 + 4(\nabla I \cdot \nabla I_{xy})^2 = 0 \\ \text{or} \\ \theta = \frac{\pi}{4} + \frac{k\pi}{2}, k \in \mathbb{Z} \end{cases}$
  if and only if $\begin{cases} \mathbf{A}_I = (\nabla I \cdot \nabla I_{xx}) \mathbf{I}_2 = (\nabla I \cdot \nabla I_{yy}) \mathbf{I}_2 \\ \text{or} \\ \theta = \frac{\pi}{4} + \frac{k\pi}{2}, k \in \mathbb{Z}, \end{cases}$
- $Q(\boldsymbol{w}) > 1$ if and only if $\theta \in \cup_{k\in\mathbb{Z}} \left] \frac{\pi}{4} + k\pi, \frac{3\pi}{4} + k\pi \right[$,
- $Q(\boldsymbol{w}) < 1$ if and only if $\theta \in \cup_{k\in\mathbb{Z}} \left] -\frac{\pi}{4} + k\pi, \frac{\pi}{4} + k\pi \right[$.

Then, we conclude that:

- The edge is enhanced if, and only if,

$$\left(\|\nabla I\| > \frac{\lambda}{\sqrt{Q(\boldsymbol{w}) - 1}} \text{ and } \theta \in \cup_{k\in\mathbb{Z}} \left] \frac{\pi}{4} + k\pi, \frac{3\pi}{4} + k\pi \right[\right). \tag{4.39}$$

- The edge is smoothed if, and only if,

$$\left[\left(\|\nabla I\| < \frac{\lambda}{\sqrt{Q(\boldsymbol{w}) - 1}} \text{ and } \theta \in \cup_{k\in\mathbb{Z}} \left] \frac{\pi}{4} + k\pi, \frac{3\pi}{4} + k\pi \right[\right) \text{ or } \left(\mathbf{A}_I = \rho \mathbf{I}_2, \rho \in \mathbb{R}_-^* \right) \right. \\ \text{or} \\ \left. \left(\theta \in \cup_{k\in\mathbb{Z}} \left[-\frac{\pi}{4} + k\pi, \frac{\pi}{4} + k\pi \right] \right) \right]. \tag{4.40}$$

Formerly, at the edge locations and in contrast to the one-dimensional case of the equation (4.1) with the PM function (4.37), the diffusion behavior in the two-dimensional case depends not only on the threshold parameter λ but also on the direction of the gradient and a local structure expressed by the matrix $\mathbf{A}_I$.

Example 4.11. The second example concerns the Charbonnier diffusion function [5]:

$$g\left(s^2\right) = \frac{1}{\sqrt{1 + \left(\frac{s}{\lambda}\right)^2}}, \text{ where } \lambda > 0. \tag{4.41}$$

Then, by using this function in the result of theorem 4.6, we get:

$$\|\nabla I\|^2 g'\left(\|\nabla I\|^2\right) Q(\boldsymbol{w}) + g\left(\|\nabla I\|^2\right) = \frac{2 + (2 - Q(\boldsymbol{w}))\left(\frac{\|\nabla I\|}{\lambda}\right)^2}{2\left(\sqrt{1 + \left(\frac{\|\nabla I\|}{\lambda}\right)^2}\right)^3}. \tag{4.42}$$

Then, as long as $Q(\boldsymbol{w}) < 2$, the Charbonnier diffusion process smooths the edges constantly.

Under these circumstances, it is quite clear that the diffusion process's behavior is closely related to the curve of the diffusion function. Consequently, selecting the appropriate diffusion function using the Theorem 4.6 for a specified task would be interesting.

5 Conclusion

To sum up, the diffusion function curve has a direct impact on the PM equation's behavior. The present paper has provided an analysis at local extrema and edge locations to gain insight into how the two dimensional PM diffusion behaves. Meanwhile, in contrast to one dimension analysis, We have demonstrated that the diffusion function's characteristics, the magnitude and the direction of the gradient, and a local structure matrix control the nonlinear PM diffusion. Besides, by using the theorem, one can determine the adequate diffusion function for the task in hand.

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