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Hybrid direct fuzzy quantum control design for a class of quantum stochastic systems and its application in pairs trading strategy

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Abstract

In this paper, a new hybrid direct fuzzy quantum control is designed for a class of quantum stochastic systems where the dynamics of the state variable is prescribed via a Quantum Stochastic Differential Equation (QSDE) with respect to a quantum Brownian motion on a quantum probability space. The presented control is comprised of two parts, an adaptive fuzzy control part that performs the main control action and a quantum-fuzzy control part that is implemented when the existence and uniqueness of the solution is not established. Thereby, the adjusted laws of the control parameters and the quantum-fuzzy rules are designed via the Lyapunov-based technique such that stability of the system is guaranteed. One theorem for facilitating the Fuzzy controller design algorithm is presented and proved. The proposed control method enhances applicability of the quantum stochastic control theory for many practical control problems such as the portfolio management. Therefore, theoretical results are illustrated through simulating pairs trading problem. According to simulation results, performance of pairs trading strategy is improved as increasing return portfolio that is controlled by the proposed method.

Keywords: Quantum stochastic differential equation, Fuzzy controller, Quantum Brownian motion, Quantum probability space, Pairs trading strategy
2020 MSC: 93C42, 93C42, 91G50, 93E03

1 Introduction

The control systems are exposed to uncertain and imperfectly known disturbances that can be taken as random variables. The stochastic control context may be discrete or continuous time frameworks. In the discrete time stochastic control, for all of the time intervals, sum of expected values of a non-linear function is optimized [9]. In the continuous time framework, the state of the system is known for controlling in any time and the objective is that maximum of integral of a concave function of a state variable be obtained over a horizon from initial time to a terminal time T or maximum of a concave function of a state variable will be obtained at some future date T [14]. Also, in the stochastic optimal control theory, the time path of random variables controlled is designed by a desired control task such that cost function of the system be minimized [8]. There are various stochastic optimal problems with various type of the cost function. The first class of such problems is Linear Quadratic Regulator problems [3]. Additionally, the path

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integral methods can solve a class of non-linear and non-quadratic control problems [13]. Moreover, there are two well-known approaches , using a partial differential equation Homilton-Jacobi-Bellman(HJB) which is the basis of the dynamic programming method and the other is the Pontryagin Maximum Principle (PMP) which is based on the calculus of variations that yields a pair of ordinary differential equations [19]

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The state of a stochastic system is described by a Stochastic Differential Equation(SDE) on finite or infinite horizon, in the controlled diffusion framework [19]. Note that, there are different types SDEs such as the classical SDEs with respect to Brownian motion, Levy processes and Poisson point processes [22]. Also, it is possible that the parameters of a stochastic differential equation (SDE) be unknown. In such cases, the learning techniques on finite/infinite horizon [14], the partial observability problems [10, 23, 21], the joint inference and control problems [16] can be useful. Additionally, the expectation values of the utility function, in the dynamic programming approah, are computed. Thus, all states need to be observed and some tedious calculations must be done. The reinforcement learning approach encounters these intractabilities [28].

A quantum control system can be manipulated as a control system for achieving a desired state in a Hilbert space [27], based on quantum mechanics as a flexible and powerful framework for solving the control problems. Also, in addition to the path integral approach, for describing a stochastic control problem as a linear formalism, the quantum transformations can be used [15]. A time-symmetric stochastic control approach based on Nelson's probability theoretical approach was presented as a representative of quantum mechanics [15]. If the evolution of a quantum system is described by a Hudson-Parthasarathy stochastic differential equation in Fock space and the performance functional associated with the system is a quadratic function then a quantum stochastic optimal control problem of LQR type is obtained that can be solved [26].

Fuzzy control has a specific effect in the control theory, because the fuzzy controllers provides an efficient framework for incorporating linguistic information from human experts for many practical control problems. There are many fuzzy control approach for controlling nonlinear stochastic systems [27]. For this purpose, in some papers [1, 2] , the Takagi-Sugeno (T-S) Fuzzy systems with multiplicative noise term is structured to describe the nonlinear stochastic system. In determining optimal controls for nonlinear, bilinear, or even linear stochastic systems with control constraints, a HJB equation has to be solved. The fuzzy approaches are techniques that determine optimal controls without using the HJB equation [5]. Additionally, due to the importance of optimal control of Brownian motion process or stochastic differential equations and essential role of fuzziness in describing the real systems, as fuzzy counterpart of stochastic process and Brownian motion, fuzzy process and C process was introduced by Liu. Therefore, a fuzzy optimal control problem was presented based on the concept of the Liu fuzzy process [24].

A fuzzy controller must guarantee the common issues of any control problem such as stability and admissible performance of the underlying closed-loop system. The stability of a fuzzy control system can be guaranteed by two different approaches. In the first approach, the structure and parameters of the fuzzy controller are specified such that the closed-loop system with the fuzzy controller is stable [29]. In the second approach, the fuzzy controller is designed without any stability consideration, then another controller is appended to the fuzzy controller for considering the stability requirement [7]. Because of flexibility in designing the fuzzy controller via in the second approach, the resulting fuzzy control system is expected to show better performance, usually.

In this paper, main objective is to design a new hybrid direct fuzzy quantum control for a class of quantum stochastic systems in which the state is governed by a Quantum Stochastic Differential Equation(QSDE). Indeed, according to conditions of the existence and uniqueness of solutions to the QSDES, $\|\hat{H}|\phi_t\rangle\| < 1$ where $|\phi_t\rangle$ and $\hat{H}$ are the quantum variable and the Hamiltonian of the quantum corresponding system, respectively, the new hybrid control is comprised of two parts, an adaptive fuzzy control part that performs the main control action and a quantum-fuzzy control part that is implemented when the existence and uniqueness condition of solution of the QSDE is not true. Thereby, the adjusted laws of the control parameters and the quantum-fuzzy rule are designed via the Lyapunov-based technique such that stability of the system is guaranteed. One theorem for facilitating the fuzzy controller design algorithm, are proved. The proposed control method enhances applicability of the quantum stochastic control theory for many practical control problems such as the portfolio management. Therefore, theoretical results are illustrated through simulating pairs trading problem. According to simulation results, performance of pairs trading strategy is improved as increasing return of the portfolio that is controlled by the proposed method.

The paper is organized as follows; problem formulation in quantum probability space is presented in section 2. In section 3, the new hybrid direct fuzzy quantum control is designed for quantum controllable diffusion process. The pairs trading problem is simulated by proposed method in section 4. Finally, the conclusion is presented in section 5

2 Problem Formulation in Quantum Probability Space

Consider a system with possible interaction outcome set $\Gamma \subseteq N$. Assume that each element of Γ is corresponded to a complex number by function $\rho : \Gamma \to \mathbb{C}$. Also, consider the orthonormal set $\{|\nu_n\rangle\}_{n\in\Gamma}$ where $|\nu_n\rangle$ is a vector with components that are defined by the following function [11]:

$$\nu_n(k) = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases} \tag{2.1}$$

For example, if $\Gamma = \{-q, ..., q\}$ then we have:

$$\{|\nu_n\rangle\}_{n\in\Gamma} = \left\{ |\nu_{-q}\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, ..., |\nu_q\rangle = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right\}. \tag{2.2}$$

Consider the space H of all vectors $|\rho_t\rangle = \sum_{n\in\Gamma} \rho(n)|\nu_n\rangle$ where $\langle\nu_n|\rho\rangle = \rho(n)$, with Hermitian conjugate vector $\langle\rho|$ that is defined as:

$$\begin{cases} \langle\rho| : H \to \mathbb{C} \\ |\lambda\rangle \xrightarrow{\langle\rho|} \langle\rho|\lambda\rangle \end{cases} \tag{2.3}$$

Therefore, under the following inner product:

$$\langle\rho, \lambda\rangle = \sum_{n\in\Gamma} \overline{\rho\,(n)}\lambda\,(n) \tag{2.4}$$

the space H is a Hilbert space. Now, consider the following normalized function:

$$\phi : \Gamma \to \mathbb{C}\ ,\ \phi(n) = \frac{1}{\sqrt{\langle\rho|\rho\rangle}}\rho(n), \tag{2.5}$$

such that

$$\sum_{n\in\Gamma} |\phi(n)|^2 = 1. \tag{2.6}$$

The Hilbert space $\mathcal{H}$ of all normalized vectors $|\phi\rangle$ is the quantum space which is corresponded to the system with possible interaction outcome set Γ. Note that the normalized vector $|\phi\rangle$ satisfies the Schrödinger equation:

$$\begin{cases} i\frac{\partial|\phi\rangle}{\partial t} = \hat{H}|\phi\rangle \\ |\phi(0)\rangle = |\phi_0\rangle, \end{cases} \tag{2.7}$$

where, i is the imaginary unit and $\hat{H}$ is a linear Hamiltonian operator on the Hilbert space $\mathcal{H}$[6].

Lemma 2.1. Consider the quantum probability space $\mathcal{H}$ corresponded to the outcomes set Γ, Filtration $\mathcal{F}$ and Measure $\mathcal{M}$. In the corresponding quantum space $\mathcal{H}$ comprised of vectors $|\phi\rangle$, by considering the normalized function $\phi : \Gamma \to \mathbb{C}$ as $\mathcal{F}$-measurable random variable, the following function is a probability measure on the quantum probability space $\mathcal{H}$:

$$\begin{cases} \mathcal{M} : \mathcal{F} \longrightarrow [0,1] \\ \mathcal{M}(\phi(n)) = |\phi(n)|^2. \end{cases} \tag{2.8}$$

Proof . Refer to [25]. □

2.1 Brownian Motion in Corresponding Quantum Space

Brownian motion is an important stochastic process that has key roles in stochastic problems modeling and stochastic optimal control problems. The following Theorem shows, under what conditions, a quantum adapted process is an Brownian motion.

Theorem 2.2. Consider the random variable ϕ_t that is satisfied the Schrödinger differential (2.7). Every quantum adapted process, $\{\phi_t\}_{t\in[0,\infty]}$ is a Brownian motion.

Proof . Refer to [25]. □

2.2 Quantum Controlled Diffusion Processes

Consider a quantum control system model via the following quantum stochastic differential equation system (QSDES):

$$\begin{cases} dX_t = \Big(f(X_t,t) + g(X_t,t)u_t\Big)dt + u_t\sigma(X_t,t)d|\phi_t\rangle \\ d|\phi_t\rangle = -i\hat{H}|\phi_t\rangle dt \\ X(0) = X_0 \end{cases} \tag{2.9}$$

where, X_0 and $|\phi_t\rangle$ are initial value of the random variable X_t and the d-dimensional quantum Brownian motion, respectively, on a quantum filtered probability space $(\mathcal{H}, (\mathcal{F}_t)_{t>0}, \mathcal{M})$. The control $u = (u_t) \in \hat{A} \subseteq \mathbb{C}$ is a $\mathcal{F}$-measurable process. The coefficients $\Big(f(X_t,t) + g(X_t,t)u_t\Big)$ and $\sigma(X_t,t)$ are measurable functions on $\mathbb{C}^n \times \hat{A}$ into $\mathbb{C}^n$.

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where, X_0 and $|\phi_t\rangle$ are initial value of the random variable X_t and the d-dimensional quantum Brownian motion, respectively, on a quantum filtered probability space $(\mathcal{H}, (\mathcal{F}_t)_{t>0}, \mathcal{M})$. The control $u = (u_t) \in \hat{A} \subseteq \mathbb{C}$ is a $\mathcal{F}$-measurable process. The coefficients $\Big(f(X_t,t) + g(X_t,t)u_t\Big)$ and $\sigma(X_t,t)$ are measurable functions on $\mathbb{C}^n \times \hat{A}$ into $\mathbb{C}^n$ and $\mathbb{C}^{n\times d}$, respectively.and $\mathbb{C}^{n\times d}$, respectively. For simplicity discussion, the following augmented equation is considered:

$$\begin{cases} dX_t = \Big(f(X_t,t) + g(X_t,t)u_t\Big)dt - iu_t\sigma(X_t,t)\hat{H}|\phi_t\rangle dt \\ X(0) = X_0. \end{cases} \tag{2.11}$$

It is trivial that Equation (2.10) is equivalent to Equation(2.11). Now, the conditions of the existence and uniqueness of solutions to the quantum stochastic differential (2.11) are presented in the following theorem:

Theorem 2.3. For all $u_t \in \hat{\mathcal{A}}$ and any initial condition $(t, X_0) \in [0,T] \times \mathbb{C}^n$, where, $0 \le T \le \infty$, the augmented QSDE (2.11) satisfies the existence and uniqueness conditions of solutions to stochastic differential equations, if

1. $\|\hat{H}|\phi\rangle\| < 1$.
2. The following stochastic differential equation satisfies the existence and uniqueness conditions.

$$dX_t = \Big(f(X_t,t) + g(X_t,t)u_t\Big)dt + u_t\sigma(X_t,t)dW_t, \tag{2.12}$$

where, X_0 and W_t are initial value of random variable X_t and d-dimensional Brownian motion, respectively, on a filtered probability space $(\Gamma, (\mathcal{F}_t)_{t>0}, P)$. Additionally, the control $u_t = (u_t)$, $\Big(f(X_t,t) + g(X_t,t)u_t\Big)$ and $\sigma(X_t,t)$ are as (2.10).

Proof . Refer to [25]. □

The main problem that is solved by presented controller in this paper, is: if the first condition of the existence and uniqueness of the solution presented in theorem 2 is not established, how can find an optimal control signal. For this propose, in the next section, a hybrid direct fuzzy quantum control is designed.

3 Hybrid Direct Fuzzy Quantum Control

In this section, before persenting the hybrid direct fuzzy quantum control, the fuzzy logic system is introduced, briefly.

3.1 Fuzzy Systems

A fuzzy system includes a set of fuzzy rules in the following form:

$$R^i : IF\ x_1\ is\ A_1^i \text{ and } \cdots \text{and } x_n\ is\ A_n^i \text{ then } y^i\ is\ B^i. \tag{3.1}$$

Indeed, a fuzzy logic system introduces a nonlinear mapping from $U = U_1 \times U_2 \times \cdots \times U_n \subset \mathbb{R}^n$ to $\mathbb{R}$. By using of singleton fuzzification, product inference and center average defuzzification for m fuzzy rules, the output function of the fuzzy system is as follows: [7]:

$$y(\mathbf{x}) = \frac{\sum_{i=1}^{n} \gamma_i \prod_{j=1}^{n} \mu_{A_j^i}(x_i)}{\sum_{i=1}^{m} \Big[\prod_{j=1}^{n} \mu_{A_j^i}(x_i)\Big]}. \tag{3.2}$$

where the vector $\mathbf{x} = [x_1, x_2, \cdots, x_n]^T \in \mathbb{R}^n$ with membership function $\mu_{A_j^i}(x_j)$ and the variable $y \in \mathbb{R}$ denote the linguistic input and output variables of the fuzzy logic system, respectively. The Equation (3.2) can be rewritten as:

$$y(\mathbf{x}) = \theta^T W(\mathbf{x}), \tag{3.3}$$

where $\theta = [\gamma_1, \cdots, \gamma_m]^T$ and

$$W(\mathbf{x}) = \frac{\prod_{j=1}^{m} \mu_{A_j^i}(x_i)}{\sum_{i=1}^{n} \Big[\prod_{j=1}^{m} \mu_{A_j^i}(x_i)\Big]}. \tag{3.4}$$

In this system, m denotes the number of the fuzzy rules and n indicates the number of inputs for fuzzy logic system.

Now, consider a quantum stochastic controllable system that its dynamics is described by QSDE (2.11). Assume that the control u_t is designed as an adaptive fuzzy control system (3.3), i.e.:

$$u(t) = u_F(t) = \theta^T W(\mathbf{x}). \tag{3.5}$$

According to the theorem(2.3), a necessary condition for existence and uniqueness solution of the QSDE (2.11) is that the quantum Brownian motion $\{|\phi_t\rangle\}_{t\geq 0}$ satisfies inequality $\|\hat{H}|\phi_t\rangle\| < 1$. Thus, for controlling the system, a new hybrid control is designed that is comprised of two parts, an adaptive fuzzy control u_F and a quantum-fuzzy control u_{Fq}. If $\|\hat{H}|\phi_t\rangle\| < 1$ then u_F performs control action and if $\|\hat{H}|\phi_t\rangle\| \geq 1$ then the quantum-fuzzy control u_{Fq} collapses the Hamiltonian effects of the quantum corresponding system such that the QSDE (2.11) satisfies the necessary condition for existence and uniqueness solution. Therefore, consider the following hybrid control law:

$$u(t) = \alpha u_F(X(t)) + (1-\alpha)u_{Fq}(X(t)), \tag{3.6}$$

where,

$$\alpha = \begin{cases} 1, & \|\hat{H}|\phi_t\rangle\| < 1 \\ 0, & \|\hat{H}|\phi_t\rangle\| \geq 1 \end{cases} \tag{3.7}$$

In the next section, the stability of the presented quantum stochastic controllable system (2.11) is investigated.

3.2 Fuzzy- Quantum Control Design and Analysis of stability

Now, through the following theorem, the quantum control u_{Fq} is designed to guarantee $\|\hat{H}|\phi_t\rangle\| < 1$.

Theorem 3.1. Consider the quantum stochastic controllable system (2.11). The hybrid control (3.6) guarantees the stability and necessary condition of the existence and uniqueness solution of the QSDE(2.11) if two following conditions are satisfied, simultaneously.

1. For the fuzzy control u_F in (3.5) we have

$$\dot{\theta} = \eta X_t^T \Big(g(X_t) - i\sigma(X_t)\hat{H}|\phi_t\rangle\Big)\alpha W_t, \qquad \eta > 0. \tag{3.8}$$

2. Fuzzy quantum control law $u_{Fq}(t)$ is defined as follows:

$$u_{Fq}(t) = M(t)u_F(t), \tag{3.9}$$

where

$$M(t) = -ksgn(X_t^T \dot{X}_t), \tag{3.10}$$

such that positive constant k is satisfied the following inequality:

$$k \leq \frac{1}{\|\hat{H}|\phi_t\rangle\|}. \tag{3.11}$$

Proof . Consider the following Lyapunov function:

$$V(t) = \frac{1}{2}\|X_t\|^2 + \frac{1}{2\eta}\theta^T\theta. \tag{3.12}$$

According to the QSDE(2.11) and the hybride control (3.6), we have:

$$\begin{aligned} \dot{V}(t) &= X_t^T \dot{X}_t + \tfrac{1}{\eta}\theta^T\dot{\theta} \\ &= X_t^T\Big(f(X_t) + (g(X_t) - i\sigma(X_t)\hat{H}|\phi_t\rangle)u_t\Big) + \tfrac{1}{\eta}\theta^T\dot{\theta} \\ &= X_t^T\Big(f(X_t) + (g(X_t) - i\sigma(X_t)\hat{H}|\phi_t\rangle)(\alpha u_F + (1-\alpha)u_{Fq})\Big) + \tfrac{1}{\eta}\theta^T\dot{\theta}. \end{aligned} \tag{3.13}$$

Also, according to (3.9) and (3.5) it is yielded:

$$\begin{aligned} \dot{V}(t) &= X_t^T\Big(f(X_t) + (g(X_t) - i\sigma(X_t)\hat{H}|\phi_t\rangle)(\alpha + (1-\alpha)M)u_F)\Big) + \tfrac{1}{\eta}\theta^T\dot{\theta}. \\ &= X_t^T\Big(f(X_t) + (g(X_t) - i\sigma(X_t)\hat{H}|\phi_t\rangle)(\alpha + (1-\alpha)M)\theta^T W_t)\Big) + \tfrac{1}{\eta}\theta^T\dot{\theta}. \\ &= X_t^T\Big(f(X_t) + (g(X_t) - i\sigma(X_t)\hat{H}|\phi_t\rangle)(1-\alpha)M\theta^T W_t\Big) + \theta^T\Big(X_t^T(g(X_t) - i\sigma(X_t)\hat{H}|\phi_t\rangle)\alpha W_t - \tfrac{1}{\eta}\dot{\theta}\Big), \end{aligned} \tag{3.14}$$

by substituting respectively $\dot{\theta}$ and $M(t)$ form (3.8) and (3.10) in (3.14), one can conclude that

$$\dot{V} = -ksgn(X_t^T\dot{X}_t)X_t^T\dot{X}_t < 0 \tag{3.15}$$

Therefore, stability of the system is guaranteed. Note that if $\|\hat{H}|\phi_t\rangle\| < 1$ then the existence and uniqueness condition is satisfied and in this case, according to the hybrid control (3.6), $u_t = u_F$, else if $\|\hat{H}|\phi_t\rangle\| \geq 1$ then $u_t = u_{Fq}$, in this case the dynamics of the colsed-loop quantum stochastic control system is expressed as follows:

$$\dot{X} = \Big(f(X_t) + g(X_t)u_q\Big)dt - iu_F\sigma(X_t)\tilde{H}|\phi_t\rangle dt \tag{3.16}$$

where $\tilde{H} = ksgn(X_t^T \dot{X}_t)\hat{H}$. According to the inequality (3.11), we have:

$$\|\tilde{H}|\phi_t\rangle\| = \|ksgn(X_t^T \dot{X}_t)\hat{H}|\phi_t\rangle\| < \|\frac{1}{\|\hat{H}|\phi_t\rangle\|} sgn(X_t^T \dot{X}_t)\hat{H}|\phi_t\rangle\| \leq 1. \tag{3.17}$$

Therefore, the equation (3.17) has a unique solution. This completes the proof. □

Remark 3.2. According to the inequality (3.11) in the theorem(3.1), one can select $k = \frac{1}{\|\hat{H}\|}$ in simulating of the presented method. Because,

$$k = \frac{1}{\|\hat{H}\|} = \frac{1}{\sup_{\||\phi_i\rangle\|=1} \|\hat{H}|\phi_i\rangle\|} \leq \frac{1}{\|\hat{H}|\phi_i\rangle\|} \tag{3.18}$$

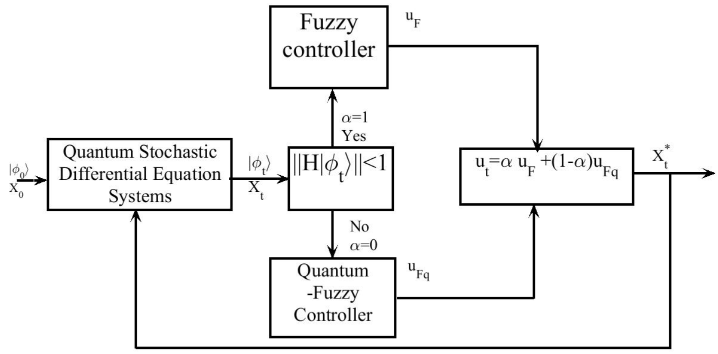

Figure 1: Schematic diagram of Hybrid Direct Fuzzy Quantum Control Method .

Figure 1 presents a schematic diagram of the proposed Hybrid Direct Fuzzy Quantum Controller. Also, for an initial condition $(u_0, X_0, |\phi_0\rangle)$ the method is implemented according to the following :

Hybrid Direct Fuzzy Quantum Control Algorithm:

Step 1. By using Equation (2.7), for initial condition $|\phi_0\rangle$, compute the quantum random process $\{|\phi_t\rangle\}_{t\in[0,\infty]}$ in the quantum space $(\mathcal{H}, \mathcal{F}_t, \mathcal{M})$ corresponded to the possible events of the stochastic system.

Step 2. According to value of function α by (3.7), for initial condition (u_0, X_0), compute $u(t)$ by Equations (3.6), (3.5) and (3.9). Also compute $X(t)$ by (2.11).

4 Pairs Trading by using Hybrid Direct Fuzzy Quantum Control

In this section, a numerical simulation of pairs trading problem is presented via the proposed Hybrid Direct Fuzzy Quantum Control (HDFQC) method to illustrate the advantages of the method.

An investment strategy for trading pairs of highly correlated stocks based on identifying them in a predictable relation to one another, is called pairs trading. Pars trading strategy trades in order to profit from modifying the predictable relation deviation [18]. Indeed, a trader reaches to profit from a return to the typical relationship. A pair of highly correlated stocks can be found in the same and homogenous industries, such as Coca-Cola and Pepsi. A pairs trading strategy is designed by the relationship between the pair of stocks regardless of the overall movement of their industry or market [18]. Some of the existing academic models can be found in [20].

In order to design an optimal pairs trading strategy, one can use a model [17] whereby the spread is an Ornstein-Uhlenbeck process. The spread is the difference between the log-price of a pair of stocks. The model formulates a dynamic portfolio optimization problem. By solving the optimal problem, an optimal pairs trading strategy is computed. According to the obtained optimal pairs trading strategy one can either buy and sell equal amounts of the stocks in the pair (trading based on the spread) or place money in a risk-free asset [17].

In this section, for designing pairs trading strategy based on proposed Hybrid Direct Fuzzy Quantum Control (HDFQC) method, the presented algorithm in section 3 is performed. According to the algorithm, spread is modeled as a quantum process that follows a Brownian motion in a quantum probability space of spread possible events. The fuzzy control system is designed based on fuzzy rule base that is constructed according to spread values. Also, the quantum dynamic portfolio based on main stock , paired and spread values, is formulated. Finally, the hybrid direct fuzzy quantum controller is applied for designing a portfolio.

4.1 Dynamics of Paired Stock Prices, Spread and Wealth value

Let $S_0(t)$ denotes the price of a risk-free asset with risk-free rate of r compounded continuously. Thus, $S_0(t)$ satisfies the following dynamics:

$$dS_0(t) = rS_0(t)dt. \tag{4.1}$$

Additionally, let $S_1(t)$ and $S_2(t)$ denote the prices of the pair of stocks 1 and 2 at time t, respectively. Assume that the price of stock 2 follows a quantum Brownian motion. So,

$$dS_2(t) = \mu S_2(t)dt + \sigma S_2(t)d\psi(t), \tag{4.2}$$

where μ , σ, $\psi(t)$ are the drift, the volatility and the quantum Brownian motion, respectively. The spread of two stocks at time t is denoted by $x(t)$ that is [17]:

$$x(t) = \ln(S_1(t)) - \ln(S_2(t)). \tag{4.3}$$

So, the dynamics of spread is described by the following quantum stochastic differential equation

$$dx(t) = K(\lambda - x(t))dt + \gamma d\phi_t, \tag{4.4}$$

where $K(\lambda - x(t))$ is the drift term that represents the expected instantaneous change in the spread at time t, and λ is the long-term spread equilibrium level. The rate of reversion is represented by the positive parameter K. The standard deviation parameter γ determines the volatility of the spread. Additionally, ϕ_t is a quantum process. Now, by using Equations from (4.2) and (4.4), the dynamics of $S_1(t)$ is obtained as follows:

$$\begin{aligned} \tfrac{dS_1}{S_1} &= dx + \tfrac{dS_2}{S_2} \\ &= K(\lambda - x)dt + \gamma d\phi_t(x) + \mu dt + \sigma d\psi \\ &= (K(\lambda - x) + \mu)dt + \gamma d\phi_t(x) + \sigma d\psi, \end{aligned} \tag{4.5}$$

Additionally, suppose that the value of a self-financing pairs-trading portfolio is denoted by $W(t)$. Also, let $u(t)$ and $\tilde{u}(t)$ denote the portfolio proportions for stocks 1 and 2 at time t, respectively. The portfolio is constrained to trading stocks 1 and 2 as a pair. So, a trader is allowed to sell (”go short”) one of them and buy (”go long”) the other in equal amount of his/her money. Therefore, the portfolio proportions are related as follows:

$$u(t) + \tilde{u}(t) = 0 \tag{4.6}$$

Note that the portfolio proportion on the risk-free asset is 1 [17]. Finally, consider the wealth dynamics of the portfolio value as follows:

$$dW(t) = W(t)\Big\{u(t)\tfrac{dS_1(t)}{S_1(t)} + \tilde{u}(t)\tfrac{dS_2(t)}{S_2(t)} + \tfrac{dS_0(t)}{S_0(t)}\Big\}. \tag{4.7}$$

By using Equations (4.1), (4.2), (4.5) and (4.6) , the Equation (4.7) can be rewritten as follows:

$$dW(t) = W(t)\Big\{[u(t)K(\lambda - x) + r]dt + u(t)\gamma d\phi_t(x)\Big\}, \tag{4.8}$$

where, ϕ_t satisfies $\frac{d}{dt}\phi_t = -i\hat{H}\phi_t$.

4.2 Fuzzy System Design based on Spread values

After identifying the pair of each stock, the trading rule is going to create a trading signal when the spread between main asset and it’s pair hits the boundary of the set $\{x : |x| < \lambda\}$ for a spread equilibrium level, λ [12]. Therefore, the fuzzy controller u_F is constructed from the following two fuzzy IF-THEN rules:

$$\begin{aligned} &\text{If } X_t > 0 \text{ then } u_t > 0 \\ &\text{If } X_t < 0 \text{ then } u_t = 0. \end{aligned} \tag{4.9}$$

where, $X_t = |x_t| - \lambda$. The fuzzy sets ”positive”, ”negative” and ”zero” are characterized by the following membership functions, respectively.

$$\mu_{\text{positive}}(X) = \frac{1}{1 + e^{-30X}} \tag{4.10}$$

$$\mu_{\text{negative}}(X) = \frac{1}{1+e^{30X}} \tag{4.11}$$

$$\mu_{\text{positive}}(u) = e^{-(u-5)^2} \tag{4.12}$$

$$\mu_{\text{zero}}(u) = e^{-u^2}. \tag{4.13}$$

Using the center average defuzzifier and the product inference engine, the fuzzy controller is obtained as follows:

$$u_F(X) = \frac{\gamma_1(\frac{1}{1+e^{-30X}}) + \gamma_2(\frac{1}{1+e^{30X}})}{(\frac{1}{1+e^{-30X}} + \frac{1}{1+e^{30X}})} \tag{4.14}$$

4.3 Simulation Results

In this section, according to the wealth dynamics of the portfolio value that is described by Equation (4.8), the hybrid direct fuzzy quantum control (3.6) is used to designing and implementing quantum pairs-trading strategy for a pair of stocks. According to the wave-particle property of the stocks in a stock market, the price process of stocks follow the quantum process that satisfies the following controlled Schrödinger wave equation [4]:

$$\dot{\phi}_t = -i\hat{H}\phi_t = -i(\beta\hat{R}\cos(\omega t))\phi_t, \tag{4.15}$$

where, β and ν are complex constants. Also, $\hat{R}$ is the return operator. Also, ϕ_t is a quantum variable in the quantum probability space $(\mathcal{H}, \mathcal{F}, \mathcal{M})$ corresponded to possible values for the rate of return of the stock. The rate of return in a trading day can not exceed a fixed limit $q\%$. So, in this example, the outcomes set is $\Gamma = \{-q, -q+1, ..., q-1, q\}$, for $q = 10$. It yields $\hat{R} = diag(-10, ..., 10)$ and also, $\omega = 1/5000$ and $\beta = 0.1$ [4].

According to the Equation (4.2), the price of stock 2, $S_2(t)$ is computed for a series of 1-day time periods. Considers 251 trading days in a year. So, stock 2 is simulated where, $\mu = 0.3$ and $\sigma = 0.1$. Additionally, according to Equation (4.3), the stock price of stock 1 can be obtained as

$$S_1(t_i) = S_2(t_i)e^{x(t_i)}. \tag{4.16}$$

For obtaining the stock price paths for stock 1 and 2, the above simulations was run for one trading year(251 days). The results has been shown in the Figure 2.

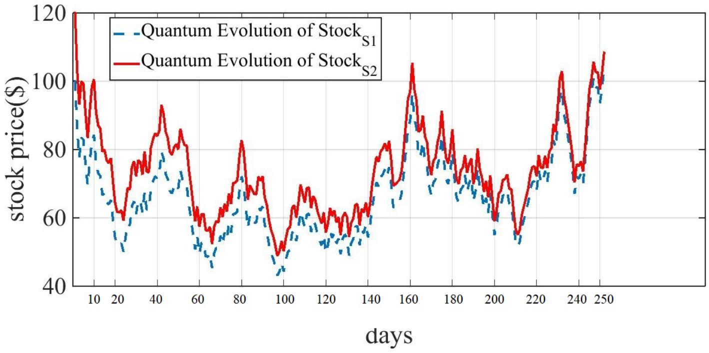

Figure 2: Time series of stock 1 and 2 by Quantum Supervisory Fuzzy Control method.

Figure 2 depicts the time series of stock 1 and 2. Therefore, we can identify the path dynamics of the two stocks and adopt it in describing the wealth value dynamics. According to (3.6) and theorem.3.1 the hybrid direct fuzzy quantum(HDFQ) controller is implemented . Therefore, HDFQ pairs-trading strategy can be computed via data corresponding to stock 1, stock 2, the spread between the pair of stocks and the corresponding parameter values. The simulated values of the HDFQ pairs-trading strategy are shown in Figure (3).

Figure3 shows that the obtained quantum control signal is a smooth signal without any chattering. Also, the amplitude of the HDFQ control signal is bounded by the interval $(0, 1)$, therefore the control cost is low, in comparing with other methods. More information about this simulation is presented in Table 1. By adopting the obtained HDFQ pairs-trading strategy, we can construct a portfolio of a risk-free asset with price S_0 and the pair of stocks with price

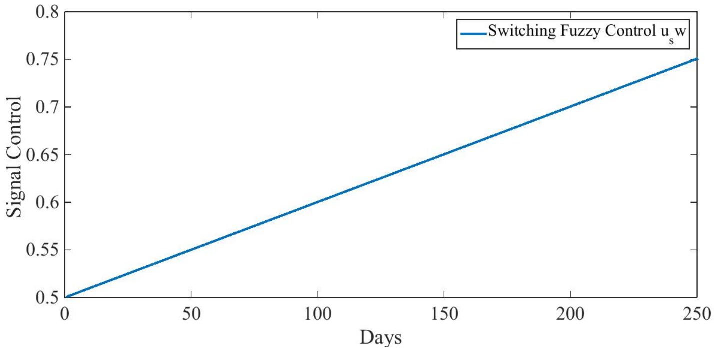

Figure 3: pairs-trading strategy by HDFQC method.

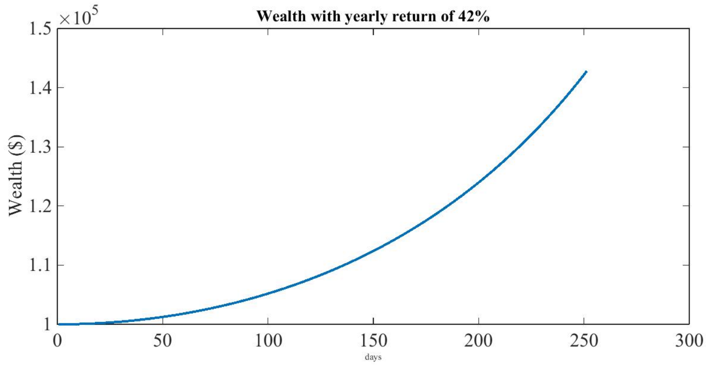

Figure 4: Wealth evolution by HDFQC method.

S_1 and S_2. Based on the pairs-trading strategy, the quantum portfolio is traded and a wealth value is produced in every trading day. With the above simulated data and according to the Equation (4.8) the wealth values simulation is illustrated in the Figure 4.

Figure 4 shows that, the value of the portfolio increases from 1000\$ to 14240\$ in one year. Indeed, this represents a yearly return over of 42% for the portfolio based on the HDFQ pairs-trading strategy. So, by adopting the HDFQ pairs-trading strategy, we can obtain a portfolio with high return. Now, the stochastic control approach [17] is implemented for two stocks 1 and 2 and it's results are compared with the proposed HDFQC method results. For this purpose, we assume that price process of stock 2 follows a random process with mean 0 and standard deviation 1 [17]. Also, the spread between the two stocks is defined as Equation (4.3). So, the spread follows a random process with mean 0 and standard deviation 1. The correlation coefficient between the two processes is set 0.19. For corresponding parameters values, the simulation of the stock price paths for stock 1 and 2 is run in the one year (251 trading days). Figure 5 shows the logarithm values of data for stock 1 and 2.

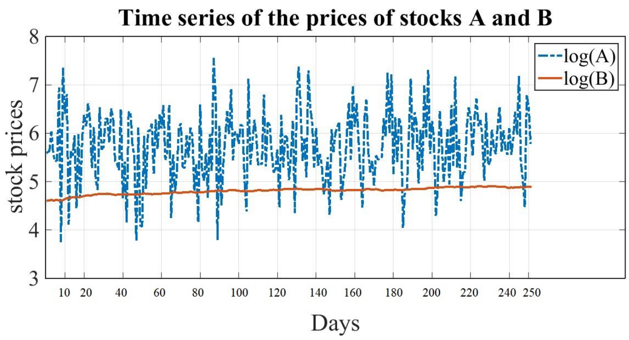

Figure 5: Time series of stock 1 and 2 by [17] .

Additionally, the optimal pairs-trading strategy can be computed by using the above identified data. The simulation of the optimal pairs-trading strategy based on the presented method in [17], is shown in Figure 6.

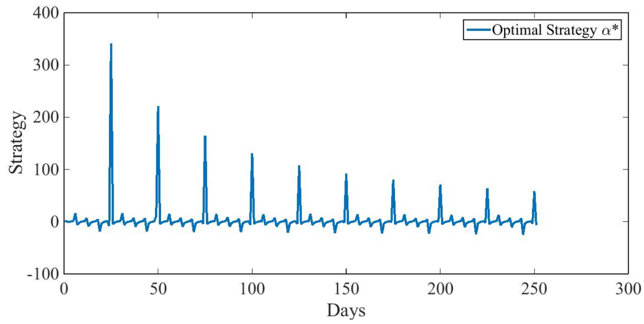

Figure 6: Optimal Stochastic Pairs-Trading by [17].

Figure 6 shows that the obtained control signal is a non-smooth signal with severe chattering. As in compared, Figure 3 indicates in the control signal of HDFQC method has not any chattering and this indicates an important advantage of the proposed method. Additionally, according to Equation (4.8) and pairs-trading strategies, the wealth values are obtained and the results of simulation are shown in Figure 7.

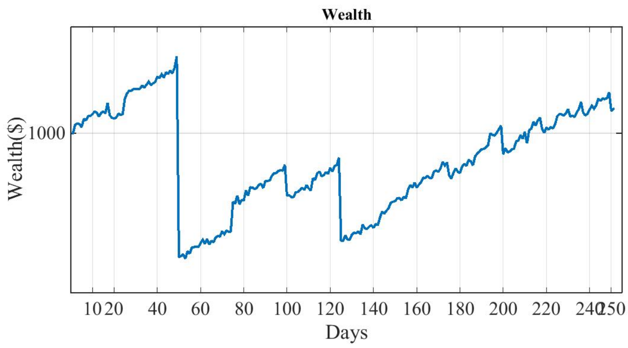

Figure 7: Wealth evolution by method[17].

Figure 7 shows that value of the portfolio increases from 1000$ to 1380$ in one year. Indeed, this represents a yearly return of over 38% for the portfolio based on the optimal pairs-trading strategy. Figure7 shows that the wealth value of the portfolio deviates from increasing procedure.

5 Table

More comparisons of the performance of the presented HDFQC method and [17] are presented in table 1.

Table 1: Simulation results of pairs trading problem by HDFQC and [17] methods .

Item / method	Amplitude of Control Signal	return of portfolio
HDFQC	(0,1)	%42
[17] OSC	[-50,350]	%38

6 Conclusion

In this paper, the Hybrid Direct Fuzzy Quantum(HDFQ) control for a class of quantum stochastic systems was designed. Since the system dynamics was governed by the Quantum Stochastic Differential Equation(QSDE), therefore the HDFQ control method was designed such that the existence and uniqueness condition of the solution of the QSDE, $\|\hat{H}|\phi_t\rangle\| < 1$, and stability of the system were guaranteed. The HDFQ controller was comprised of two parts, fuzzy control part and fuzzy-quantum control part that according to values of the quantum variable $|\phi_t\rangle$, each part performed the control action. The fuzzy-quantum control law and adaptive rules of the fuzzy control system parameters were obtained based on the Lyaponuv technique. For facilitating the HDFQ controller algorithm, one theorem was expressed and proved. The simulation results are illustrated through simulating pairs trading problem. The simulation results indicate the performance improvement of the presented pair trading strategies and the the advantages of the proposed method.

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