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On the relationships between cubic transmutations and the mixture of failure distributions of k-out-of-3 systems

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Abstract

In this study, at first a new polynomial rank transmutation is proposed. Then, a new cubic rank transmutation is introduced by simplifying the set of transmutation parameters in order to improve its usefulness in statistical modeling. The probability density function, moment generating function, survival and hazard rate functions of cubic rank transmuted distribution are examined. Moreover, existence of stochastic, hazard rate and likelihood ratio orderings are investigated with respect to the baseline distribution. The processes of estimating of the parameters with maximum likelihood and EM algorithm are mentioned. The modeling performance of the distribution is showed by considering real data sets obtained from different areas.

Keywords: polynomial rank transmutation, cubic rank transmutation, quadratic rank transmutation, three-component-systems, transmuted distribution family
2020 MSC: Primary 05C38, 15A15; Secondary 05A15, 15A18

1 Introduction

In this study, we inspire the quadratic rank transmutation map (QRTM) proposed by [24]. The mapping is given as

$$u \to u + \lambda u\,(1-u) \tag{1.1}$$

where $u \in [0,1]$ and $\lambda \in [-1,1]$. Using this transmutation many distributions have been derived and still continue to be derived. Beside this, there are also some studies on the modifications of the QRTM. Some of the pioneering works on proposing modified QRTM can be given as follows: [8] proposed a new Weibull distribution by using exponentiated QRTM. [3] generated a new distribution family by considering exponentiated distribution as the baseline distribution. [17] studied a new distribution by taking the baseline distribution as exponentiated exponential distribution. [9] introduced transmuted exponentiated modified Weibull distribution, and [4] introduced transmuted exponentiated Lomax distribution. The last three studies can be seen as a special case of [3]. [15] introduced a new transmutation map by adding extra two parameters to get more flexible distribution. Then, [16] introduced a new Lindley distribution by using this new transmutation map approach. [7] introduced a kind of generalization of QRTM by considering sum of k- dimensional vector of transmutation parameters. There are two similar studies which are the generalized transmuted G family by [20] and generalized transmuted Weibull distribution by [21]. Also, by taking into account recent works,

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[23] introduced a new distribution named as transmuted generalized Gamma distribution. They use QRTM to generate this distribution family.

In this study, a new polynomial rank transmutation is proposed additionally to [25]. Since the parameter set is still complex, a new cubic rank transmutation is introduced in the light of the idea behind QRTM. In our study, since an extra transmutation parameter is added, the distribution has become more flexible. Statistical and reliability properties of the cubic rank transmuted distribution are examined in general. In particular, we examine some mathematical and statistical properties by taking the baseline distribution as exponential distribution. In order to demonstrate the usefulness of the gained flexibility of the distribution family, illustrative examples are given on the real data sets in the application section.

2 Motivation

[25] proposed polynomial rank transmutation map to demonstrate Skew-kurtotic transmutations. *Figure 3* of [25] indicates that admissible parameter region. However this region is quite complex structure, the points on *Figure 5* of them show some special cases related to family of order statistics up to 3-sized sample. Under the leadership of this idea, we propose a new polynomial rank transmutation to get simpler structure of parameter region. Let $G(u)$ stand for the polynomial rank transmutation defined on $[0,1]$. Then, we have

$$G(u) = u + \lambda_1 u\left(1-u\right) + \lambda_2 u^2\left(1-u\right) \tag{2.1}$$

with $G(0) = 0$ and $G(1) = 1$. Note that, λ_1 and λ_2 are the transmutation parameters. Parameter region will be defined with following discussion. Since G should be non-decreasing, non-negativity of the first derivative of G with respect to u is examined. Thus, the shape of the parameter region is determined. By calling this derivative with g, we have

$$g\left(u,\lambda_1,\lambda_2\right) = -3\lambda_2 u^2 - 2u\left(\lambda_1-\lambda_2\right) + \left(1+\lambda_1\right). \tag{2.2}$$

Non-negativity of $g\left(u,\lambda_1,\lambda_2\right)$ at the end-points, namely the inequalities $g\left(0,\lambda_1,\lambda_2\right) = 1+\lambda_1 \geq 0$ and $g\left(1,\lambda_1,\lambda_2\right) = 1-\lambda_1-\lambda_2 \geq 0$ both requires that

$$\lambda_1 \geq -1 \qquad \lambda_1+\lambda_2 \leq 1. \tag{2.3}$$

From these two inequalities, it is clear that $\lambda_2 \leq 2$. When the eq. (2.2) is taken into account, $g\left(u,\lambda_1,\lambda_2\right)$ is a concave function for $\lambda_2 \in (0,2]$. As long as the inequality (2.3) is valid, $g\left(u,\lambda_1,\lambda_2\right)$ will take non-negative values. For $\lambda_2 \leq 0$, we will investigate the sufficient conditions on non-negativity of $g\left(u,\lambda_1,\lambda_2\right)$. In this case, $g\left(u,\lambda_1,\lambda_2\right)$ has a minimum point since it is a convex function. If this minimum point is within $(0,1)$, the value at that point of the function $g\left(u,\lambda_1,\lambda_2\right)$ must be positive. Accordingly, the minimum point is obtained by taking the derivative of the eq. (2.2) and equating them to zero as follows:

$$g'\left(u,\lambda_1,\lambda_2\right) = -6\lambda_2 u - 2\left(\lambda_1-\lambda_2\right) = 0 \Rightarrow u^* = \frac{-\left(\lambda_1-\lambda_2\right)}{3\lambda_2}. \tag{2.4}$$

Then, the value of $g\left(u,\lambda_1,\lambda_2\right)$ at u^* must satisfy

$$g\left(\frac{-\left(\lambda_1-\lambda_2\right)}{3\lambda_2},\lambda_1,\lambda_2\right) = \frac{\lambda_1^2+\lambda_1\lambda_2+\lambda_2\left(3+\lambda_2\right)}{3\lambda_2} \geq 0. \tag{2.5}$$

Hence, it is necessary to say that the value of the numerator in (2.5) is non-positive. If this statement given by the numerator is considered as a second order polynomial of λ_1, the roots are given by

$$\lambda_{1_{1,2}} = \frac{-\lambda_2 \pm \sqrt{-3\lambda_2\left(\lambda_2+4\right)}}{2}. \tag{2.6}$$

Here, we can say that the condition $-4 \leq \lambda_2$ must also occur in order for the roots to be real valued. Thus, under the condition $-4 \leq \lambda_2 < 0$, we have bounds for λ_1 as follows:

$$\frac{-\lambda_2 - \sqrt{-3\lambda_2\left(\lambda_2+4\right)}}{2} \leq \lambda_1 \leq \frac{-\lambda_2 + \sqrt{-3\lambda_2\left(\lambda_2+4\right)}}{2}. \tag{2.7}$$

For these bounds, the numerator in (2.5) has a negative sign. This leads to the following conclusion: The range of λ_1 is as in (2.7) for $\lambda_2 \in [-4, 0)$. However, the minimum value of the lower bound in (2.7) can be -1, while the maximum value of the upper bound can be 3. From this we can say that the range of λ_1 is $[-1, 3]$. Thus, combining this results, the parameter region for (λ_1, λ_2) appears as shown in the Figure 1.

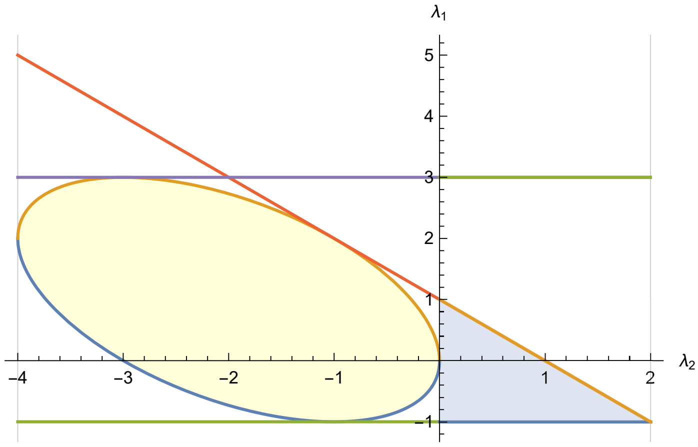

Figure 1: Valid parameter set (ellipsoid and triangle on the right-side)

By considering this parameter set of (λ_1, λ_2), many well defined distributions are generated from the eq. (2.1) with the baseline distribution F. Now, let's get a map of the integer values of the pair (λ_2, λ_1) to see the known distributions tabulated in Table 1.

Table 1: Some generated distributions according to special cases for the parameter values

λ_2	λ_1	Some Generated Distributions	λ_2	λ_1	Some Generated Distributions
-4	2	$4F^3 - 6F^2 + 3F$	-1	0	$F^3 - F^2 + F$
-3	0	$3F^3 - 3F^2 + F$	-1	1	$F^3 - 2F^2 + 2F$
-3	1	$3F^3 - 4F^2 + 2F$	-1	2	$F^3 - 3F^2 + 3F*$
-3	2	$3F^3 - 5F^2 + 3F$	0	-1	F^2*
-3	3	$3F^3 - 6F^2 + 4F$	0	0	$F*$
-2	0	$2F^3 - 2F^2 + F$	0	1	$2F - F^2*$
-2	1	$2F^3 - 3F^2 + 2F$	1	-1	$2F^2 - F^3*$
-2	2	$2F^3 - 4F^2 + 3F$	1	0	$F^2 + F - F^3*$
-1	-1	F^3*	2	-1	$3F^2 - 2F^3*$

The distributions specified by the star in Table 1 are described below how they correspond to some known failure distributions.

Let $X_{r:n}$ be the rth order statistic in a sample of size n. By noting that, for $\lambda_1 = -1, \lambda_2 = -1$ generated distribution indicates the failure distribution of the lifetime of three-component parallel system, namely, this distribution indicates the distribution of the random variable $X_{3:3} = \max\{X_1, X_2, X_3\}$ where X_1, X_2 and X_3 are independent and identically distributed as F. Similarly, for $\lambda_1 = 2, \lambda_2 = -1$ generated distribution indicates the distribution of the random variable $X_{1:3} = \min\{X_1, X_2, X_3\}$. For $\lambda_1 = -1, \lambda_2 = 1$ generated distribution indicates the distribution of $\max\{X_1, \min\{X_2, X_3\}\}$. For $\lambda_1 = 0, \lambda_2 = 1$ generated distribution indicates the distribution of $\min\{X_1, \max\{X_2, X_3\}\}$. For $\lambda_1 = -1, \lambda_2 = 2$ generated distribution indicates the failure distribution of the lifetime of the three-out- of- two system, namely, this distribution indicates the distribution of the random variable $X_{2:3} = \max\{\min\{X_1, X_2\}, \min\{X_1, X_3\}, \min\{X_2, X_3\}\}$. On the other hand, for $\lambda_1 = -1, \lambda_2 = 0$ generated distribution indicates the failure distribution of the lifetime of the two-component parallel system, namely distribution of $X_{2:2} = \max\{X_1, X_2\}$. For $\lambda_1 = 1, \lambda_2 = 0$ generated distribution indicates the failure distribution of the lifetime of the two-component series system, namely distribution of $X_{1:2} = \min\{X_1, X_2\}$.

In this case, in addition to the known distributions introduced by the quadratic transmutation, more informative distribution functions occur. However, the set of the transformation parameters of the proposed cubic transmutation is still complicated. In order to eliminate of this complexity, by referring to the concept of reliability evaluation of coherent system by using signature (see, [10, 11]), we come up with an idea inspired by both works of [24] and [29] as follows:

$$\Pr\left(X_{2:2} \leq t\right) = \Pr\left(\max\left\{X_1, X_2\right\} \leq t\right) = F^2\left(t\right)$$

and

$$\Pr\left(X_{1:2} \leq t\right) = \Pr\left(\min\left\{X_1, X_2\right\} \leq t\right) = 2F\left(t\right) - F^2\left(t\right)$$

where $F\left(t\right)$ indicates the failure distribution of the component lifetime, namely, $\Pr\left(X_1 \leq t\right) = F\left(t\right)$. Hence there exists a stochastic ordering relation such as $X_{1:2} \prec_{st} X \prec_{st} X_{2:2}$. In this case, these three failure distributions can be ordered as $F^2\left(t\right) \leq F\left(t\right) \leq 2F\left(t\right) - F^2\left(t\right)$. From the latter inequality, we can say that $F(t)$ is represented by a convex combination of $2F\left(t\right) - F^2\left(t\right)$ and $F^2\left(t\right)$ where the value of the combination parameter is $\frac{1}{2}$. On the other hand, it is possible to obtain many distributions besides F. Let G stand for the distribution obtained by this convex combination. Then, for $\delta \in [0, 1]$, we have

$$\begin{aligned} G\left(t\right) =& \delta\left(2F\left(t\right) - F^2\left(t\right)\right) + \left(1-\delta\right)\left(F^2\left(t\right)\right) \\ =& 2\delta F\left(t\right) + \left(1 - 2\delta\right)F^2\left(t\right). \end{aligned} \tag{2.8}$$

Here, the combination parameter is reparametrized by taking $\delta = \frac{1+\lambda}{2}$ to attain quadratic rank transmutation. Now, the new parameter λ takes the values in $[-1, 1]$. As can be seen immediately, $\lambda = 0$ corresponds to $\delta = \frac{1}{2}$. In eq. (2.8), substituting δ by λ, we have

$$G\left(t\right) = \left(1+\lambda\right)F\left(t\right) - \lambda F^2\left(t\right). \tag{2.9}$$

The above expression is the quadratic rank transmutation proposed by [24]. Now, we concentrate on 3-component systems with similar thinking. Let X_1, X_2 and X_3 be independent random variables distributed as F. Let $X_{r:3}$ denote rth order statistic of $\left(X_1, X_2, X_3\right)$ with corresponding distribution $F_{r:3}$. Then we have

$$F_{3:3}\left(t\right) = \Pr\left(X_{3:3} \leq t\right) = \Pr\left(\max\left\{X_1, X_2, X_3\right\} \leq t\right) = F^3\left(t\right) \tag{2.10}$$

$$\begin{aligned} F_{2:3}\left(t\right) &= \Pr\left(X_{2:3} \leq t\right) \\ &= \Pr\left(\max\left\{\min\left\{X_1, X_2\right\}, \min\left\{X_1, X_3\right\}, \min\left\{X_2, X_3\right\}\right\} \leq t\right) \\ &= 3F^2\left(t\right) - 2F^3\left(t\right) \end{aligned} \tag{2.11}$$

$$\begin{aligned} F_{1:3}\left(t\right) &= \Pr\left(X_{1:3} \leq t\right) = \Pr\left(\min\left\{X_1, X_2, X_3\right\} \leq t\right) \\ &= 3F\left(t\right) - 3F^2\left(t\right) + F^3\left(t\right). \end{aligned} \tag{2.12}$$

According to [29], the properties $F_{3:3} \leq F_{2:3} \leq F_{1:3}$ and $F = \frac{1}{3}F_{3:3} + \frac{1}{3}F_{2:3} + \frac{1}{3}F_{1:3}$. are hold. In other words, F can be represented by a convex combination of $F_{1:3}$, $F_{2:3}$ and $F_{3:3}$. On the other hand, there is also an ordering for F such that $F_{3:3} \leq F \leq F_{1:3}$. If $F_{2:3}$ is also included in this ordering, we have for $F \geq \frac{1}{2}$, $F_{3:3} \leq F \leq F_{2:3} \leq F_{1:3}$ and for $F < \frac{1}{2}$, $F_{3:3} \leq F_{2:3} \leq F \leq F_{1:3}$. Hence, we can suggest a convex combination to cover both ordering situations. Our aim is to determine exactly where F is. In this case, we can write the following convex combination obtained by $F_{1:3}$ and $F_{2:3}$, called as G^*.

$$G^* = \delta_1 F_{1:3} + \left(1 - \delta_1\right)F_{2:3} \tag{2.13}$$

where $\delta_1 \in [0, 1]$. Now, let's write a convex combination between G^* and $F_{3:3}$. Denoting this convex combination by G, we have

$$G = \delta_2 G^* + \left(1 - \delta_2\right)F_{3:3} \tag{2.14}$$

where $\delta_2 \in [0, 1]$. Combining with the equations (2.13) and (2.14), we obtain G as

$$G = \delta_2\delta_1 F_{1:3} + \delta_2\left(1 - \delta_1\right)F_{2:3} + \left(1 - \delta_2\right)F_{3:3}. \tag{2.15}$$

If the notation F is used for the representation of $F_{r:3}$, and rearranging with respect to polynomial degree of F, the following expression is obtained:

$$\begin{aligned} G &= \delta_2\delta_1\left(3F - 3F^2 + F^3\right) + \delta_2\left(1 - \delta_1\right)\left(3F^2 - 2F^3\right) + \left(1 - \delta_2\right)F^3 \\ &= 3\delta_1\delta_2 F + 3\delta_2\left(1 - 2\delta_1\right)F^2 + \left(1 - 3\delta_2 + 3\delta_1\delta_2\right)F^3. \end{aligned} \tag{2.16}$$

Undoubtedly, G is a distribution function. However, reparameterization is made on the model in order to achieve the similar structure of the quadratic rank transmutation. Now, by taking $w_1 = \delta_1\delta_2$ and $w_2 = \delta_2 - \delta_1\delta_2$, eq. (2.16) can be rewritten as follows:

$$G = 3w_1 F + 3\left(w_2 - w_1\right) F^2 + \left(1 - 3w_2\right) F^3 \tag{2.17}$$

where $w_1, w_2 \in [0,1]$. In eq. (2.17), by the reparametrizating as $w_1 = \frac{1+\lambda_1}{3}$ and $w_2 = \frac{1+\lambda_2}{3}$, we have

$$G = \left(1 + \lambda_1\right) F + \left(\lambda_2 - \lambda_1\right) F^2 - \lambda_2 F^3 \tag{2.18}$$

where $\lambda_1, \lambda_2 \in [-1,2]$. Since $\delta_2 = w_1 + w_2$, the parameter set is also constrained by the condition $\lambda_1 + \lambda_2 \leq 1$. Consequently, the parameter set of λ_1 and λ_2 is presented in a simpler form than the parameter region given in Figure 1. This transmutation defined in eq. (2.18) is called as cubic rank transmutation and transformed distribution G is named as CRT-F.

As can be seen immediately, CRT-F defines a quadratic rank transmuted distribution at $\lambda_2 = 0$, and $\lambda_1 = \lambda_2 = 0$ gives the baseline distribution F. For this reason, CRT-F can be seen as a generalized form of QRT. The parameter set of λ_1 and λ_2, which is defined as $\{(\lambda_1,\lambda_2) : \lambda_1, \lambda_2 \in [-1,2], \lambda_1 + \lambda_2 \leq 1\}$ can be figure out in Figure 2.

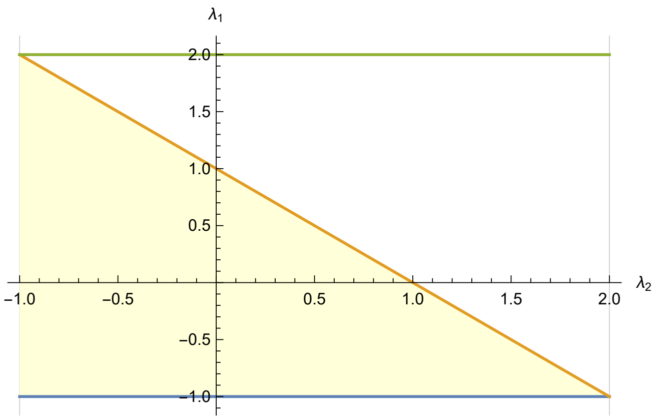

Figure 2: Region of the parameter set of CRT-F

Now, referring to the integer values of λ_1 and λ_2, we can determine the generated distribution functions by the Table 2.

Table 2: Identifications of CRT-F distribution for special values of transmutation parameters

λ_1	λ_2	CRT-F	Identification
-1	-1	F^3	Distribution of $T_{3:3}$
-1	0	F^2	Distribution of $T_{2:2}$
-1	1	$2F^2 - F^3$	Distribution of $\max\{X_1, \min\{X_2, X_3\}\}$
-1	2	$3F^2 - 2F^3$	Distribution of $T_{2:3}$
0	-1	$F^3 - F^2 + F$	$\frac{1}{3}F_{1:3} + \frac{2}{3}F_{3:3}$
0	0	F	Baseline Distribution
0	1	$F^2 + F - F^3$	Distribution of $\min\{X_1, \max\{X_2, X_3\}\}$
1	-1	$F^3 - 2F^2 + 2F$	$\frac{2}{3}F_{1:3} + \frac{1}{3}F_{3:3}$
1	0	$2F - F^2$	Distribution of $T_{1:2}$
2	-1	$F^3 - 3F^2 + 3F$	Distribution of $T_{1:3}$

Identifications given in Table 2 show that *Table 1* of [25] is included by CRT-F according to special choices of transmutation parameters. Note that, by taking into account the parameter set of (2.18), the distribution family CRT-F is different as compared with the families proposed by [12] and [22].

2.1 Probability Density Function of CRT Random Variable

The pdf of the CRT random variable can be obtained from using the equations (2.18) and (2.15) as follows:

$$\begin{aligned} g(t) &= f(t)\left[(1+\lambda_1)+2(\lambda_2-\lambda_1)F(t)-3\lambda_2F^2(t)\right] \\ &= (1+\lambda_1)f(t)+(\lambda_2-\lambda_1)f_{2:2}(t)-\lambda_2 f_{3:3}(t) \\ &= \left(\frac{1+\lambda_1}{3}\right)f_{1:3}(t)+\left(\frac{1+\lambda_2}{3}\right)f_{2:3}(t)+\left(\frac{1-\lambda_1-\lambda_2}{3}\right)f_{3:3}(t) \end{aligned} \tag{2.19}$$

2.2 Moment Generating Function of CRT Random Variable

The mgf of the CRT random variable can be obtained by using the eq. (2.19) as

$$\begin{aligned} M_T(v) &= (1+\lambda_1)M_X(v)+(\lambda_2-\lambda_1)M_{X_{2:2}}(v)-\lambda_2 M_{X_{3:3}}(v) \\ &= \left(\frac{1+\lambda_1}{3}\right)M_{X_{1:3}}(v)+\left(\frac{1+\lambda_2}{3}\right)M_{X_{2:3}}(v)+\left(\frac{1-\lambda_1-\lambda_2}{3}\right)M_{X_{3:3}}(v) \end{aligned} \tag{2.20}$$

and kth raw moments of CRT random variable can be obtained as

$$E\left[T^k\right]=(1+\lambda_1)E\left[X^k\right]+(\lambda_2-\lambda_1)E\left[X_{2:2}^k\right]-\lambda_2 E\left[X_{3:3}^k\right]. \tag{2.21}$$

2.3 Survival Function of CRT Random Variable

The survival function obtained from (2.18) is given by

$$\begin{aligned} \bar{G}(t) &= (1+\lambda_1)\bar{F}(t)+(\lambda_2-\lambda_1)\bar{F}(t)(1+F(t))-\lambda_2\bar{F}(t)\left(1+F(t)+F^2(t)\right) \\ &= \bar{F}(t)\left[1-\lambda_1F(t)-\lambda_2F^2(t)\right] \end{aligned} \tag{2.22}$$

2.4 Hazard Rate Function of CRT Random Variable

One of the important characteristics of the life distributions is hazard rate or failure rate function. Using the equations (2.19) and (2.22) the hazard rate of CRT-F is

$$\begin{aligned} r_G(t) &= r_F(t)\left[1+\frac{\lambda_1+(2\lambda_2-\lambda_1)F(t)-2\lambda_2F^2(t)}{1-\lambda_1F(t)-\lambda_2F^2(t)}\right] \\ &= r_F(t)\left[3+\frac{-2+\lambda_1+(\lambda_1+2\lambda_2)F(t)}{1-\lambda_1F(t)-\lambda_2F^2(t)}\right] \\ &= r_F(t)\left[1+\bar{F}(t)\frac{\lambda_1+2\lambda_2F(t)}{1-\lambda_1F(t)-\lambda_2F^2(t)}\right] \end{aligned} \tag{2.23}$$

with $r_G(0)=(1+\lambda_1)r_F(0)$ and $r_G(\infty)=r_F(\infty)$. Following proposition and theorem provide monotonicity properties of the hazard rate function of CRT random variable.

Proposition 2.1. The following requirements relate to the monotonicity property of $r_G(t)$ according to the baseline hazard rate.

(a) If $r_F(t)$ increases in t with $\frac{d}{dt}f(t)\geq 0$, and $\lambda_1,\lambda_2\geq 0$, $r_G(t)$ increases in t.

(b) If $r_F(t)$ increases in t with $\frac{d}{dt}f(t)\geq 0$, and $\lambda_2<0$ and $\lambda_1\geq 2\sqrt{-\lambda_2}$, $r_G(t)$ increases in t.

(c) If $r_F(t)$ increases in t with $\frac{d}{dt}f(t)\leq 0$, and $\lambda_2\geq 0$ and $\lambda_1\leq -2\lambda_2$, $r_G(t)$ increases in t.

(d) If $r_F(t)$ increases in t with $\frac{d}{dt}f(t)\leq 0$, and $\lambda_2\leq 0$ and $\lambda_1\leq -2\sqrt{-\lambda_2}$, $r_G(t)$ increases in t.

(e) If $r_F(t)$ decreases in t, and $-0.5\leq\lambda_2\leq 0$ and $-2\lambda_2\leq\lambda_1\leq\sqrt{-2\lambda_2}$, $r_G(t)$ decreases in t.

Proof . From 2.22 hazard rate function of $G(t)$ can be written as

$$r_G(t) = r_F(t) + \frac{d}{dt}\left[-\log\left(1 - \lambda_1 F(t) - \lambda_2 F^2(t)\right)\right]. \tag{2.24}$$

The first derivation of $r_G(t)$ depends on the derivation of hazard rate function of the baseline distribution and the second derivation of $-\log\left(1-\lambda_1 F(t) - \lambda_2 F^2(t)\right)$. Hence, the second derivative of the right hand side statement in (2.24) is given by

$$f'(t)\frac{\lambda_1 + 2\lambda_2 F(t)}{1-\lambda_1 F(t) - \lambda_2 F^2(t)} + f^2(t)\frac{2\lambda_2\left(1-\lambda_1 F(t) - \lambda_2 F^2(t)\right) + \left(\lambda_1 + 2\lambda_2 F(t)\right)^2}{\left(1-\lambda_1 F(t) - \lambda_2 F^2(t)\right)^2} \tag{2.25}$$

(a) Positivity of (2.25) is provided. Hence G is also an IHR (Increasing hazard rate) distribution. Monotonicity is satisfied.

(b) Positivity of first summand in (2.25) is satisfied with $\lambda_1 \geq -2\lambda_2$. Besides, the fact that positivity of the second summand depends on positivity of the expression $2\lambda_2\left(1-\lambda_1 u - \lambda_2 u^2\right) + \left(\lambda_1 + 2\lambda_2 u\right)^2$ where $u \in [0,1]$. Expanding this statement as $2\lambda_2^2 u^2 + 2\lambda_1\lambda_2 u + \lambda_1^2 + 2\lambda_2$, then this statement is a convex function of u on $(0,1)$. In this case, positivity of the value of the function at the minimum point is sufficient for the second summand to be positive. This function reaches the minimum value of $2\lambda_2 + \frac{\lambda_1^2}{2}$ at $u^* = -\frac{\lambda_1}{2\lambda_2}$. Therefore, it is ensured that the minimum value is positive as long as the condition $\lambda_1^2 \geq -4\lambda_2$ is satisfied. Combining this condition with $\lambda_1 \geq -2\lambda_2$, the result should not be $\lambda_1 < 0$. Hence, $\lambda_1 \in \left[2\sqrt{-\lambda_2}, 2\right]$ is provided.

(c) Proof is similar to (b)

(d) Proof is similar to (b)

(e) Negativity of first summand in (2.25) is satisfied with $\lambda_1 \geq -2\lambda_2$. Furthermore, negativity of the second summand depends on negativity of the endpoints of the function $\rho(u) = 2\lambda_2^2 u^2 + 2\lambda_1\lambda_2 u + \lambda_1^2 + 2\lambda_2$, $u \in [0,1]$ since $\rho(u)$ is a convex function of u. Here, both $\rho(0) = \lambda_1^2 + 2\lambda_2 \leq 0$ and $\rho(1) = 2\lambda_2^2 + 2\lambda_1\lambda_2 + \lambda_1^2 + 2\lambda_2 \leq 0$ must be satisfied. This results yields that first condition is $\lambda_1 \leq \sqrt{-2\lambda_2}$ and the second is $0 \leq \lambda_1 \leq -\lambda_2 + \sqrt{-\lambda_2(2+\lambda_2)}$. By combining these conditions with $\lambda_1 \geq -2\lambda_2$, bounds for λ_1 can be achieved as $\max\{0, -2\lambda_2\} \leq \lambda_1 \leq \min\left\{\sqrt{-2\lambda_2}, -\lambda_2 + \sqrt{-\lambda_2(2+\lambda_2)}\right\}$ and then $-2\lambda_2 \leq \lambda_1 \leq \sqrt{-2\lambda_2}$ is provided.

□

The following theorem relates to the monotonicity of the hazard rate according to λ_1 and λ_2.

Theorem 2.2. $r_G(t;\Theta)$ is a nondecreasing function in λ_1 and λ_2. Here Θ is a parameter set and defined by $\left\{\left(\underset{\sim}{\theta}, \lambda_1, \lambda_2\right) :\in \mathbb{R}^k, \lambda_1, \lambda_2 \in [-1,2], \lambda_1 + \lambda_2 \leq 1\right\}$.

Proof . From the last equality in (2.23) the sign of $\frac{\partial r_G(t;\Theta)}{\partial \lambda_j}$ is related to the sign of $\frac{\partial}{\partial \lambda_j}\psi(\lambda_1,\lambda_2)$ for $j = 1,2$ where $\psi(\lambda_1,\lambda_2) = \left(\frac{\lambda_1 + 2\lambda_2 u}{1-\lambda_1 u - \lambda_2 u^2}\right)$. Hence, we have the following expressions:

$$\frac{\partial}{\partial \lambda_1}\psi(\lambda_1,\lambda_2) = \frac{1+\lambda_2 u^2}{\left(1-\lambda_1 u - \lambda_2 u^2\right)^2} \geq 0$$

and

$$\frac{\partial}{\partial \lambda_2}\psi(\lambda_1,\lambda_2) = \frac{u(2-\lambda_2 u)}{\left(1-\lambda_1 u - \lambda_2 u^2\right)^2} \geq 0.$$

□

Following theorem is about stochastic, hazard rate and likelihood ratio orderings between CRT random variable, T and the baseline random variable, X.

Theorem 2.3. Following orderings hold by T and X according to transmutation parameters λ_1 and λ_2.

(a) $\lambda_2 \leq 0$ and $\lambda_1 + \lambda_2 \in [0,1]$ or $\lambda_2 > 0$ and $\lambda_1 > 0 \Rightarrow T$ is stochastically smaller than X , i.e., $T \leq_{st} X$,

(b) $\lambda_2 \le 0$ and $\lambda_1 \le 0$ or $\lambda_2 > 0$ and $\lambda_1 + \lambda_2 \in [-2, 0) \Rightarrow X$ is stochastically smaller than T , i.e., $X \le_{st} T$,

(c) $\lambda_2 \le 0$ and $\lambda_1 \ge -2\lambda_2$ or $\lambda_2 > 0$ and $\lambda_1 > 0 \Rightarrow T$ is smaller than X in the hazard rate order, i.e., $T \le_{hr} X$,

(d) $\lambda_2 \le 0$ and $\lambda_1 \le 0$ or $\lambda_2 > 0$ and $\lambda_1 < -2\lambda_2 \Rightarrow X \le_{hr} T$,

(e) $\lambda_2 \le 0$ and $\lambda_1 \le \lambda_2$ or $\lambda_2 \in \left(0, \frac{1}{2}\right]$ and $\lambda_1 < -2\lambda_2 \Rightarrow X$ is smaller than T in the likelihood ratio order, i.e., $X \le_{lr} T$,

(f) $\lambda_2 \ge 0$ and $\lambda_1 \ge \lambda_2$ or $\lambda_2 < 0$ and $\lambda_1 > -2\lambda_2 \Rightarrow T \le_{lr} X$.

Proof .

(a) From the equation (2.22) the sign of $\lambda_1 + \lambda_2 u$ indicates this ordering is valid or not. Hence, for $\lambda_2 \le 0$ positivity of $\lambda_1 + \lambda_2 u$ which depends on $\lambda_1 + \lambda_2 \ge 0$ guarantiates $\bar{F} \ge \bar{G}$. For $\lambda_2 > 0$, positivity of $\lambda_1 + \lambda_2 u$ implies $\lambda_1 > 0$. This case quarantiates also $\bar{F} \ge \bar{G}$.

(b) For $\lambda_2 \le 0$ negativity of $\lambda_1 + \lambda_2 u$ which depends on $\lambda_1 \le 0$ guarantiates $\bar{F} \le \bar{G}$. For $\lambda_2 > 0$, negativity of $\lambda_1 + \lambda_2 u$ implies $\lambda_1 + \lambda_2 < 0$. This case quarantiates also $\bar{F} \le \bar{G}$.

(c) From the equation (2.24), the sign of $\lambda_1 + 2\lambda_2 u$ indicates this ordering is valid or not. For $\lambda_2 \le 0$ positivity of $\lambda_1 + 2\lambda_2 u$ which depends on $\lambda_1 \ge -2\lambda_2$ i.e. $\lambda_1 + \lambda_2 \in [-\lambda_2, 1]$ guarantiates $r_G \ge r_F$. For $\lambda_2 > 0$, positivity of $\lambda_1 + 2\lambda_2 u$ implies $\lambda_1 > 0$. This case quarantiates also $r_G \ge r_F$.

(d) Proof is similar to (c).

(e) From the equation (2.19), monotonicity of $\frac{g(t)}{f(t)}$ can be determined by checking of the sign of $\lambda_2 - \lambda_1 - 3\lambda_2 u$. For $\lambda_2 \le 0$ positivity of $\lambda_2 - \lambda_1 - 3\lambda_2 u$ which depends on $\lambda_2 \ge \lambda_1$ guarantiates that $\frac{g(t)}{f(t)}$ is an increasing function in t . Similarly, for the case $\lambda_2 > 0$, $\lambda_1 < -2\lambda_2$ guarantiates increasingness of $\frac{g(t)}{f(t)}$.

(f) Proof is similar to (e).

□

3 Generating a Random Number from CRT Distribution

In order to generate a random number from CRT-F, let us recall that CRT-F is represented by a mixed distribution as in eq. (2.15). Here mixing components are lifetime distribution of 3-out-of-3-system, lifetime distribution of 2-out-of-3-system and lifetime distribution of 1-out-of-3-system, respectively. Mixing weights are respectively $\frac{1+\lambda_1}{3}$, $\frac{1+\lambda_2}{3}$ and $\frac{1-(\lambda_1+\lambda_2)}{3}$ where $\lambda_1, \lambda_2 \in [-1, 2]$ and $\lambda_1 + \lambda_2 \in [-2, 1]$. According to this, we have the following steps to generate a random number from CRT-F:

Step 1: Three random numbers X_1, X_2, X_3 are independently generated from the baseline distribution F.

Step 2: The random number U is generated from the uniform distribution on $(0, 1)$.

Step 3: If $U \le \frac{1+\lambda_1}{3}$, a random number T from G is $\min\{X_1, X_2, X_3\}$.

Step 4: If $U \le \frac{2+\lambda_1+\lambda_2}{3}$, T is $\max\{\min\{X_1, X_2\}, \min\{X_1, X_3\}, \min\{X_2, X_3\}\}$, otherwise T is $\max\{X_1, X_2, X_3\}$.

Following section is considered the special case of the baseline distribution as an exponential distribution. Pdf, mgf, raw moments, expected value, variance, skewness and kurtosis are examined. Some reliability characteristics of CRT-E distribution such as survival, hazard rate and mean residual life functions are discussed. Maximum likelihood estimation process and Expectation-Maximization (EM) algorithm are proposed for the estimations of the parameters of the CRT-E distribution.

4 Special Case: Exponential Baseline

In this section, we assume that the baseline distribution for cubic transformation is exponential distribution. A brief description of this special distribution, the graphs of the pdf and the hazard rate are given.

4.1 Pdf of *CRT-E* Random Variable

Let us take the baseline distribution as $F(t) = 1 - e^{-\theta t}$, $t > 0$, $\theta > 0$. Then, using (2.19) we have the pdf of *CRT-E* distribution as follows:

$$\begin{aligned} g(t;\lambda_1,\lambda_2,\theta) &= \theta e^{-\theta t}\left[1-\lambda_1-\lambda_2+2(\lambda_1+2\lambda_2)e^{-\theta t}-3\lambda_2 e^{-2\theta t}\right] \\ &= (1-\lambda_1-\lambda_2)\,\theta e^{-\theta t}+(\lambda_1+2\lambda_2)\,2\theta e^{-2\theta t}+(-\lambda_2)\,3\theta e^{-3\theta t} \\ &= (1-\lambda_1-\lambda_2)\,f_{Exp(\theta)}(t)+(\lambda_1+2\lambda_2)\,f_{Exp(2\theta)}(t)+(-\lambda_2)\,f_{Exp(3\theta)}(t) \end{aligned} \tag{4.1}$$

where, $\theta > 0$, $\lambda_1, \lambda_2 \in [-1, 2]$, $\lambda_1 + \lambda_2 \leq 1$. The latter impression above says that the pdf can be expressed as the arbitrary sums of the exponential pdfs with the parameters $\theta, 2\theta$ and 3θ. A random variable T having the above pdf is called as CRT-E random variable. Figure 3 illustrates some of the possible shapes of the pdf of a CRT-E random variable for the selected values of the parameters λ_1 and λ_2 for $\theta = 1$.

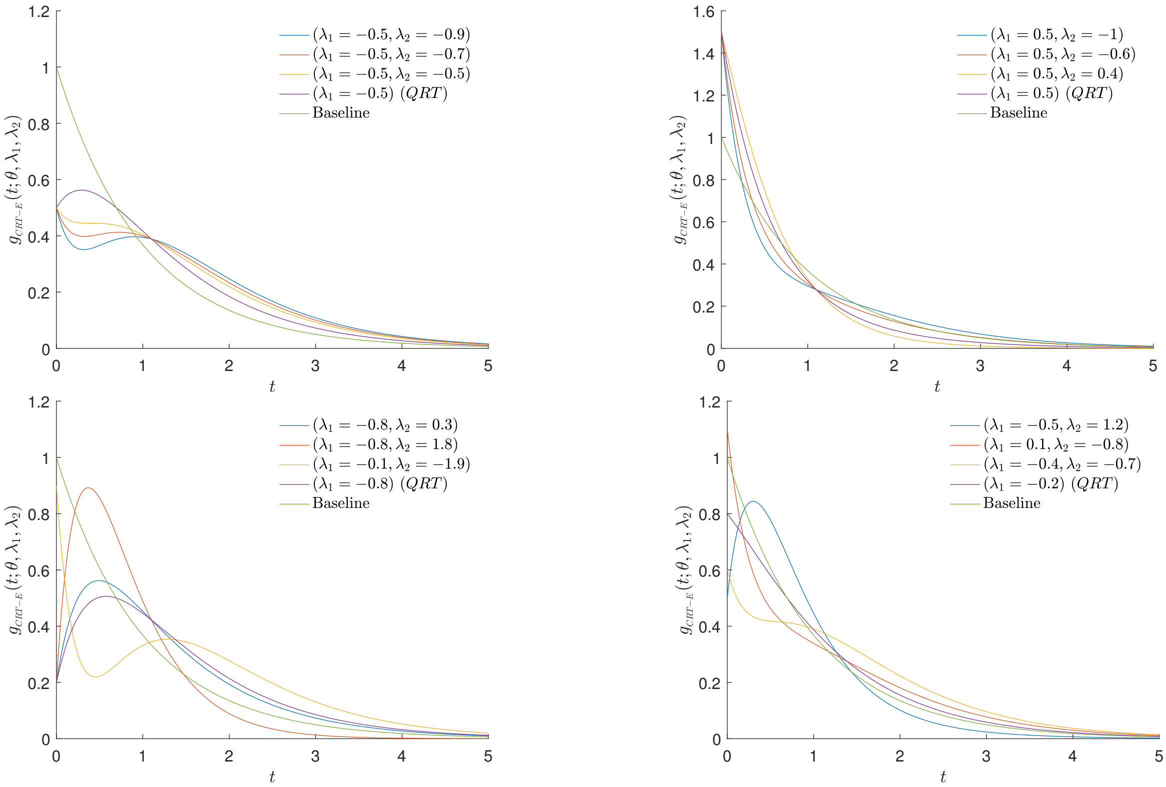

Figure 3: Some possible shapes of pdf of CRT-E random variable for $\theta = 1$

4.2 Moment Generating Function and Raw Moments of *CRT-E* Random Variable

Using the form of the pdf given in (4.1), we have the moment generating function and kth raw moments respectively as follows:

$$M_T(v) = (1-\lambda_1-\lambda_2)\frac{\theta}{\theta - v} + (\lambda_1+2\lambda_2)\frac{2\theta}{2\theta - v} - \lambda_2\frac{3\theta}{3\theta - v}, v < \theta$$

$$E\left[T^k\right] = \Gamma(k+1)\left[(1-\lambda_1-\lambda_2)\frac{1}{\theta^k} + (\lambda_1+2\lambda_2)\frac{1}{2^k\theta^k} - \lambda_2\frac{1}{3^k\theta^k}\right] \tag{4.2}$$

4.3 Expected Value, Variance, Skewness and Kurtosis of of CRT-E Random Variable

First moment can be obtained by using (4.2) as

$$E\left[T\right]=\left[\left(1-\lambda_{1}-\lambda_{2}\right)\frac{1}{\theta}+\left(\lambda_{1}+2\lambda_{2}\right)\frac{1}{2\theta}-\lambda_{2}\frac{1}{3\theta}\right]=\frac{1}{\theta}-\lambda_{1}\frac{1}{2\theta}-\lambda_{2}\frac{1}{3\theta}.$$

When $\lambda_1 < -\frac{2}{3}\lambda_2$, CRT-E random variable has more long life expectancy than the exponential random variable. The variance of CRT-E random variable can be obtained as

$$Var\left(T\right)=\frac{1}{\theta^{2}}-\frac{4(1+\lambda_{2})^{2}+12\left(1+\lambda_{1}\right)\lambda_{2}+9(1+\lambda_{1})^{2}-13}{36\theta^{2}}.$$

Furthermore, by using some algebraic operations, $Var(T)$ lies in an interval $[\frac{1}{9\theta^2}, \frac{53}{36\theta^2}]$. Now, the skewness coefficient of the CRT-E will be compared with the exponential distribution. The coefficient of the skewness of the exponential distribution is $\gamma = 2$, and the skewness coefficient of the CRT-E is obtained as follows:

$$\gamma_{CRT-E}=\frac{2\left(216-27\lambda_{1}^{3}-114\lambda_{2}-60\lambda_{2}^{2}-8\lambda_{2}^{3}-27\lambda_{1}^{2}\left(3+2\lambda_{2}\right)-9\lambda_{1}\left(9+16\lambda_{2}+4\lambda_{2}^{2}\right)\right)}{\left[-9\lambda_{1}^{2}-6\lambda_{1}\left(3+2\lambda_{2}\right)-4\left(-9+5\lambda_{2}+\lambda_{2}^{2}\right)\right]^{\frac{3}{2}}}$$

The lower and upper bounds for γ_{CRT-E} can be obtained as $1.3610 \leq \gamma_{CRT-E} \leq 4.0549$ by taking into account critical points $(\lambda_1, \lambda_2) = (-0.5697, -1)$ and $(\lambda_1, \lambda_2) = (1.8809, -1)$. According to this bound, CRT-E is a positively skewed distribution. CRT-E is both positive and negative skewed relative to exponential distribution. The coefficient of the kurtosis of CRT-E can be given as

$$\kappa_{CRT-E}=-6\left[\frac{1}{2}+\frac{\left(810\lambda_{1}^{2}+1176\lambda_{1}\lambda_{2}+1620\lambda_{1}+392\lambda_{2}^{2}+1960\lambda_{2}-2592\right)}{\left(9\lambda_{1}^{2}+6\lambda_{1}\left(3+2\lambda_{2}\right)+4\left(-9+5\lambda_{2}+\lambda_{2}^{2}\right)\right)^{2}}\right].$$

Accordingly, lower bound is $\kappa_{CRT-E} = 6.0017$ for $(\lambda_1, \lambda_2) = (-0.4108, -1)$ and upper bound is $\kappa_{CRT-E} = 33.1793$ for $(\lambda_1, \lambda_2) = (1.9209, -1)$. Since the kurtosis coefficient of the exponential distribution is 9, the CRT-E distribution exhibits sometimes a tinner tailed distribution, and sometimes a fatter tailed distribution according to exponential distribution.

4.4 Survival, Hazard Rate and Mean Residual Life Functions of CRT-E Random Variable

From the equation (2.22), the survival function of the CRT-E random variable is obtained as

$$\bar{G}\left(t\right)=e^{-\theta t}\left[1-\lambda_{1}-\lambda_{2}+\left(\lambda_{1}+2\lambda_{2}\right)e^{-\theta t}-\lambda_{2}e^{-2\theta t}\right]$$

where, $\theta > 0,\ \lambda_1, \lambda_2 \in [-1, 2]\,,\ \lambda_1 + \lambda_2 \leq 1$. Using the equation (2.23) the hazard rate of CRT-E is given by

$$r_{G}\left(t\right)=\theta\left[3-\frac{2\left(1-\lambda_{1}-\lambda_{2}\right)+\left(\lambda_{1}+2\lambda_{2}\right)e^{-\theta t}}{\left(1-\lambda_{1}-\lambda_{2}+\left(\lambda_{1}+2\lambda_{2}\right)e^{-\theta t}-\lambda_{2}e^{-2\theta t}\right)}\right].$$

Figure 4 indicates some possible shapes of the hazard rate function of the CRT-E random variable with respect to some values of the parameters λ_1 and λ_2 for $\theta = 1$.

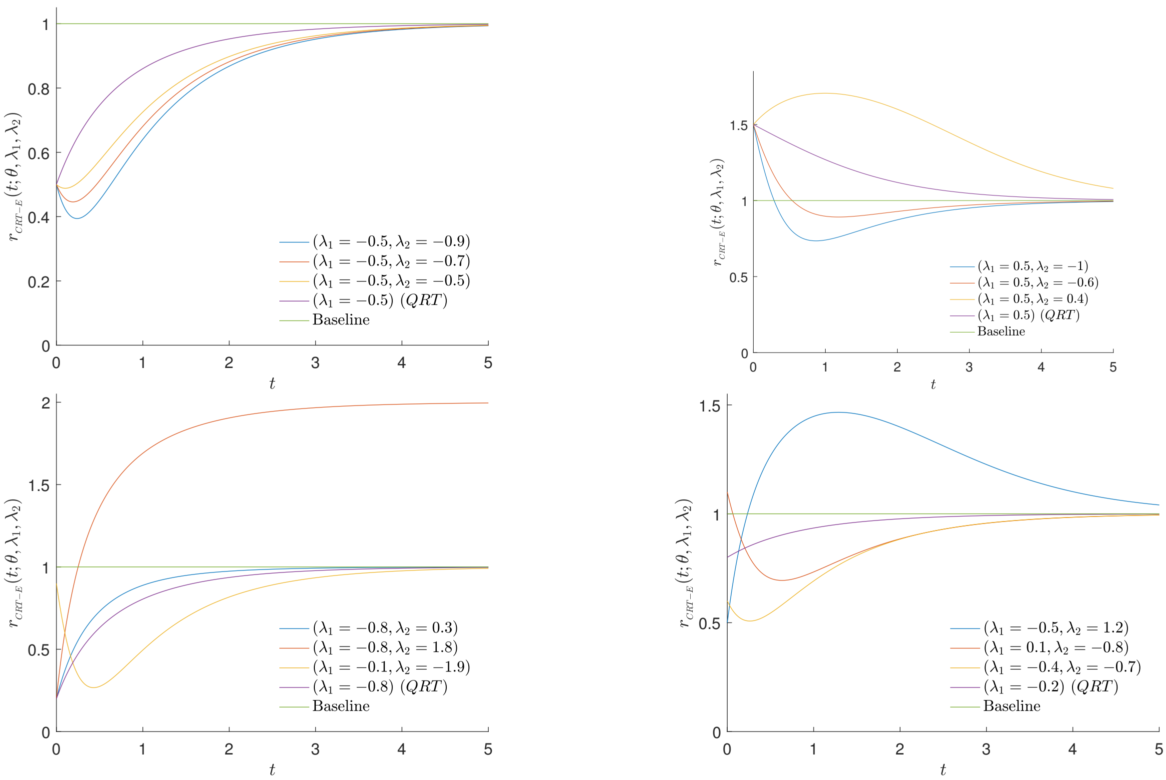

Figure 4: Some possible shapes of the hazard rate of CRT-E random variable for $\theta = 1$

Other important characteristic of the lifetime random variable is the mean residual life function (mrl) which is defined by

$$m(t) = E\left[T - t\,|T > t\right] = \frac{\int_t^\infty \bar{G}(z)dz}{\bar{G}(t)} = \frac{\int_t^\infty zg(z)dz}{\bar{G}(t)} - t.$$

The mrl function for the CRT-E random variable is given as

$$m(t) = \frac{1}{\theta}\left[1 + \frac{-3\left(\lambda_1 + 2\lambda_2\right)e^{-\theta t} + 4\lambda_2 e^{-2\theta t}}{6\left[1 - \lambda_1 - \lambda_2 + \left(\lambda_1 + 2\lambda_2\right)e^{-\theta t} - \lambda_2 e^{-2\theta t}\right]}\right].$$

with $\lim_{t\to\infty} m(t) = \frac{1}{\theta}$.

4.5 MLE of the Parameters of the CRT-E Distribution

Let $t_1, t_2, ..., t_n$ be a random sample of size n from CRT-E distribution. Log-likelihood function of CRT-E can be written with respect to the parameter set $(\lambda_1, \lambda_2, \theta)$ as follows:

$$\begin{aligned}\ell\left(\lambda_1, \lambda_2, \theta\right) &= \sum_{j=1}^{n} \log\left(g\left(t_j; \lambda_1, \lambda_2, \theta\right)\right) \\ &= n\log\left(\theta\right) - \theta\sum_{j=1}^{n} t_j + \sum_{j=1}^{n}\log\left(1 - \lambda_1 - \lambda_2 + 2\left(\lambda_1 + 2\lambda_2\right)e^{-\theta t_j} - 3\lambda_2 e^{-2\theta t_j}\right)\end{aligned}$$

We have to maximize ℓ with the constraints $\lambda_1 + \lambda_2 \leq 1$ and $\lambda_1, \lambda_2 \in [-1, 2]$. Accordingly, partial derivatives of ℓ on λ_1, λ_2 and θ can be given by

$$\frac{\partial\ell\left(\lambda_1, \lambda_2, \theta\right)}{\partial\lambda_1} = -\sum_{j=1}^{n}\left(\frac{1 - 2e^{-\theta t_j}}{1 - \lambda_1 - \lambda_2 + 2\left(\lambda_1 + 2\lambda_2\right)e^{-\theta t_j} - 3\lambda_2 e^{-2\theta t_j}}\right)$$

$$\frac{\partial \ell(\lambda_1,\lambda_2,\theta)}{\partial \lambda_2} = -\sum_{j=1}^{n}\left(\frac{\left(1-e^{-\theta t_j}\right)\left(1-3e^{-\theta t_j}\right)}{1-\lambda_1-\lambda_2+2(\lambda_1+2\lambda_2)e^{-\theta t_j}-3\lambda_2 e^{-2\theta t_j}}\right)$$

$$\frac{\partial \ell(\lambda_1,\lambda_2,\theta)}{\partial \theta} = \frac{n}{\theta} - 3\sum_{j=1}^{n} t_j + 2\sum_{j=1}^{n} t_j\left(\frac{1-\lambda_1-\lambda_2+(\lambda_1+2\lambda_2)e^{-\theta t_j}}{1-\lambda_1-\lambda_2+2(\lambda_1+2\lambda_2)e^{-\theta t_j}-3\lambda_2 e^{-2\theta t_j}}\right).$$

The maximum likelihood estimations of $(\theta, \lambda_1, \lambda_2)$ is obtained by equating these equations to zero, and solving this nonlinear system of equations. It is usually more convenient to use constrained nonlinear optimization algorithms to maximize the log-likelihood function numerically.

4.6 EM Algorithm for the Estimations of the Parameters of the CRT-E Distribution

According to equation (2.19), pdf of CRT-E random variable can be written as

$$g(t;\theta,\rho_1,\rho_2,\rho_3) = \rho_1 f_1(t;\theta) + \rho_2 f_2(t;\theta) + \rho_3 f_3(t;\theta) \tag{4.3}$$

where ρ_i's are positive and $\rho_1+\rho_2+\rho_3 = 1$, and $f_1(t;\theta) = 3F^2(t;\theta)f(t;\theta)$, $f_2(t;\theta) = 6F(t;\theta)(1-F(t;\theta))f(t;\theta)$, $f_3(t;\theta) = 3(1-F(t;\theta))^2 f(t;\theta)$.

The constrained log likelihood function can be written with respect to the parameter set $(\rho_1,\rho_2,\rho_3,\theta)$ as follows:

$$\ell(\rho_1,\rho_2,\rho_3,\theta) = \sum_{j=1}^{n}\log\left(\rho_1 f_1(t_j;\theta) + \rho_2 f_2(t_j;\theta) + \rho_3 f_3(t_j;\theta)\right) - \varepsilon(\rho_1+\rho_2+\rho_3-1).$$

Accordingly, equating the partial derivatives of ℓ on ρ_1, ρ_2, ρ_3, ε and θ to zero, we have

$$\frac{\partial \ell(\rho_1,\rho_2,\rho_3,\theta,\varepsilon)}{\partial \rho_1} = 0 \Rightarrow \sum_{j=1}^{n}\left(\frac{f_1(t_j;\theta)}{g(t;\theta,\rho_1,\rho_2,\rho_3)}\right) = \varepsilon$$

$$\frac{\partial \ell(\rho_1,\rho_2,\rho_3,\theta,\varepsilon)}{\partial \rho_2} = 0 \Rightarrow \sum_{j=1}^{n}\left(\frac{f_2(t_j;\theta)}{g(t;\theta,\rho_1,\rho_2,\rho_3)}\right) = \varepsilon$$

$$\frac{\partial \ell(\rho_1,\rho_2,\rho_3,\theta,\varepsilon)}{\partial \rho_3} = 0 \Rightarrow \sum_{j=1}^{n}\left(\frac{f_3(t_j;\theta)}{g(t;\theta,\rho_1,\rho_2,\rho_3)}\right) = \varepsilon$$

$$\frac{\partial \ell(\rho_1,\rho_2,\rho_3,\theta,\varepsilon)}{\partial \varepsilon} = 0 \Rightarrow \rho_1+\rho_2+\rho_3 = 1$$

$$\frac{\partial \ell(\rho_1,\rho_2,\rho_3,\theta,\varepsilon)}{\partial \theta} = \sum_{j=1}^{n}\left(\begin{array}{c} \frac{\rho_1 f_1(t_j;\theta)\frac{\partial \log f_1(t_j;\theta)}{\partial\theta}}{g(t;\theta,\rho_1,\rho_2,\rho_3)} + \\ \frac{\rho_2 f_2(t_j;\theta)\frac{\partial \log f_2(t_j;\theta)}{\partial\theta}}{g(t;\theta,\rho_1,\rho_2,\rho_3)} + \\ \frac{\rho_3 f_3(t_j;\theta)\frac{\partial \log f_3(t_j;\theta)}{\partial\theta}}{g(t;\theta,\rho_1,\rho_2,\rho_3)} \end{array}\right) = 0.$$

The first three equations respectively are multiplied by ρ_m,$(m = 1,2,3)$ and then $\varepsilon = n$ is obtained by summing up them. The probability that the jth observation comes from the mth component is denoted by $\frac{\rho_m f_m(t_j;\theta)}{g(t;\theta,\rho_1,\rho_2,\rho_3)} = P(m\,|t_j) = P_{mj}$ and taking into account fourth equation above we have

$$\hat{\rho}_m = \frac{1}{n}\sum_{j=1}^{n} P(m\,|t_j),\ m = 1,2,\ \ \hat{\rho}_3 = 1-(\hat{\rho}_1+\hat{\rho}_2).$$

In this case, the fifth equation becomes as follows:

$$\frac{\partial \ell}{\partial \theta} = \sum_{j=1}^{n}\left(\frac{\partial \log f_1(t_j;\theta)}{\partial\theta}P_{1j} + \frac{\partial \log f_2(t_j;\theta)}{\partial\theta}P_{2j} + \frac{\partial \log f_3(t_j;\theta)}{\partial\theta}P_{3j}\right) = 0. \tag{4.4}$$

Now, we find the respective derivative expressions as follows:

$$\frac{\partial \log f_1(t_j;\theta)}{\partial\theta}=\frac{1}{\theta}-3t_j+\frac{2t_j}{1-e^{-\theta t_j}},\quad \frac{\partial \log f_2(t_j;\theta)}{\partial\theta}=\frac{1}{\theta}-3t_j+\frac{t_j}{1-e^{-\theta t_j}}\quad \frac{\partial \log f_3(t_j;\theta)}{\partial\theta}=\frac{1}{\theta}-3t_j$$

If these equations are written in the expression (4.4), we get the following equation

$$\frac{1}{\theta}-3\bar{t}+\frac{2}{n}\sum_{j=1}^{n}\frac{t_j}{1-e^{-\theta t_j}}P_{1j}+\frac{1}{n}\sum_{j=1}^{n}\frac{t_j}{1-e^{-\theta t_j}}P_{2j}=0. \tag{4.5}$$

Iterative procedure starts with setting initials $\rho_1^{(0)}$, $\rho_2^{(0)}$ and $\theta^{(0)}$. $P_{mj}^{(0)}$ is calculated according to the given initials. Then, new $\rho_1^{(1)}$ and $\rho_2^{(1)}$ are calculated. $\theta^{(1)}$ is solved from the eq. (4.5) by using $P_{mj}^{(0)}$. Thus, $\ell^{(1)}(\rho_1,\rho_2,\rho_3,\theta)$ is obtained. According to the iteratively obtained parameter set $(\rho_1,\rho_2,\rho_3,\theta)^{(1)}$, $P_{mj}^{(1)}$ is calculated. Using the new recurrence value for the parameter set, $\ell^{(2)}(\rho_1,\rho_2,\rho_3,\theta)$ is obtained. These operations will continue until the difference $\ell^{(k)}-\ell^{(k-1)}$ is smaller than the desired value $\zeta>0$. Thus, cubic transmutation parameters are estimates as $\hat{\lambda}_m=3\hat{\rho}_m-1,\ m=1,2$ and $\hat{\lambda}_3=1-\hat{\lambda}_1-\hat{\lambda}_2$.

5 Real Data Applications

In this section, modeling success of the CRT-E distribution is considered primarily by exponential distribution and quadratic transmuted exponential distribution (Data Set 1). In addition, a comparison is made with the appropriate distributions obtained by considering the application section of Matlab distribution fitting (Data Set 2). Moreover, modeling examples in the literature are compared with CRT-E for both data sets. In order to see the modeling performance of the CRT-E, we use the results of Kolmogorov Simirnov goodness-of -fit test and the Akaike Information Criterion (AIC) which is calculated by using ML estimates of the model parameters.

Data Set 1. (Coal mine accidents data) The data set contains the time intervals (in days) between coal mine accidents caused death of 10 or more men which is given by 378, 96, 59, 108, 54, 275, 498, 228, 217, 19, 156, 36, 124, 61, 188, 217, 78, 49, 271, 120, 329, 47, 15, 50, 1, 233, 113, 17, 131, 208, 275, 330, 129, 31, 120, 13, 28, 32, 1205, 182, 517, 20, 312, 1630, 215, 203, 189, 22, 23, 644, 255, 1613, 66, 171, 29, 11, 176, 345, 61, 151, 467, 195, 54, 291, 145, 217, 137, 55, 20, 78, 361, 871, 224, 326, 4, 75, 7, 4, 93, 81, 99, 312, 48, 566, 1312, 369, 364, 18, 15, 59, 286, 326, 354, 123, 390, 348, 338, 37, 1357, 72, 315, 114, 275, 58, 457, 72, 745, 336, 19. Firstly, this data set was obtained by [14]. There were lots of models on this data set such as [1] and [13]. They suggested to use Exponential-Geometric (EG) (K-S= 0.0761) and Exponential-Poisson (EP) (K-S= 0.0625) distributions, respectively. On the other hand, [28] have proposed two-component mixed exponential distribution (2MED) (K-S=0.0578) for modeling this data set. According to Table 3, CRT-E distribution has the best fit amongst the other distributions (including the EG, EP and 2MED).

Table 3: ML estimates, K-S statistics and AIC values for Exponential, T-E and CRT-E Distributions (Coal Mine Accidents Data)

Model	**Parameter Estimations**	**K-S (p value)**	**-2LL**	**AIC**
Exponential	$\hat{\theta}=0.0042$	0.0906 (0.3129)	1413.6	1415.6
Transmuted-Exponential	$\hat{\theta}=0.0028$ $\hat{\lambda}=0.6722$	0.0627 (0.7603)	1408.2	1412.2
CRT-E	$\hat{\theta}=0.0019$ $\hat{\lambda}_1=1.7548$ $\hat{\lambda}_2=-1.0000$	0.0574 (0.8449)	1405.3	1411.3

Data Set 2. (Wheaton river flood data) The data consist of the exceedances of flood peaks (in m^3/s) of the Wheaton River near Carcross in Yukon Territory, Canada. The data consist of 72 exceedances for the years 1958–1984, rounded to one decimal place: 1.7, 2.2, 14.4, 1.1, 0.4, 20.6, 5.3, 0.7, 13.0, 12.0, 9.3, 1.4, 18.7, 8.5, 25.5, 11.6, 14.1, 22.1, 1.1, 2.5, 14.4, 1.7, 37.6, 0.6, 2.2, 39.0, 0.3, 15.0, 11.0, 7.3, 22.9, 1.7, 0.1, 1.1, 0.6, 9.0, 1.7, 7.0, 20.1, 0.4, 14.1, 9.9, 10.4, 10.7, 30.0, 3.6, 5.6, 30.8, 13.3, 4.2, 25.5, 3.4, 11.9, 21.5, 27.6, 36.4, 2.7, 64.0, 1.5, 2.5, 27.4, 1.0, 27.1, 20.2, 16.8, 5.3, 9.7, 27.5, 2.5, 27.0, 1.9, 2.8. Firstly, these data were analyzed by [6]. Later on, Beta-Pareto (BP) distribution was applied to these data by [2]. [5] proposed Kumaraswamy Pareto (Kw-P) distribution. [18] made a comparison between Pareto (P) and Transmuted Pareto (TP) distribution. They showed that better model is the transmuted Pareto distribution.

[26] have proposed Weibull-Pareto (WP) distribution and made a comparison with Beta Exponentiated Pareto (BEP) distribution. [19] have proposed a different type of Weibull-Pareto (NWP) distribution. Exponential Modified Discrete Lindley (EMDL) distribution (K-S=0.1162, AIC=507.6) has been applied to these data by [27]. We fit data to CRT-E, Transmuted-Exponential distributions and additionally Dagum and Weibull distributions. According to the model selection criteria (AIC) tabulated in Table 4, CRT-E takes the first place amongst 5 proposed models.

Table 4: ML estimates, K-S statistics and AIC values for Dagum, Weibull, Exponential, T-E and CRT-E Distributions (Wheaton River Flood Data)

Model	**Parameter Estimations**	**K-S (p value)**	**-2LL**	**AIC**
Dagum	$\hat{\theta} = 30.2023$ $\hat{\alpha} = 4.7939$ $\hat{k} = 0.1265$	0.1107 (0.317)	502.3845	508.39
Weibull	$\hat{\theta} = 11.6322$ $\hat{\beta} = 0.9012$	0.105 (0.377)	502.996	507
Exponential	$\hat{\theta} = 0.0819$	0.1422 (0.098)	504.26	506.26
Transmuted-Exponential	$\hat{\theta} = 0.0755$ $\hat{\lambda} = 0.1668$	0.131 (0.155)	503.93	507.93
CRT-E	$\hat{\theta} = 0.0945$ $\hat{\lambda}_1 = 0.3856$ $\hat{\lambda}_2 = -1.0000$	0.074 (0.610)	497.75	503.75

6 Conclusion

In this article, we propose a new version of polynomial rank transmutation. Since the parameter set is still complex, a new cubic rank transmutation is introduced in the light of the idea behind QRTM technique. We obtain the probability density and hazard functions of the CRT-F. We derive explicit expressions for the moments and the moment generating function. The new distribution has an increasing, a decreasing, and non-monotonic hazard rate function for lifetime data. Some orderings such as stochastic, hazard rate and likelihood ratio orderings are investigated between the distribution and its baseline distribution. A simple algorithm is given to generate a random number from CRT-F. Furthermore, we consider that exponential distribution is the baseline distribution. We obtain the corresponding analytical shapes of the probability density and hazard functions of the CRT-E. We derive explicit expressions for the moments and the moment generating function. Some discussions on expected value, variance, the coefficients of skewness and kurtosis of CRT-E with respect to exponential distribution are given. The method of MLE and EM algorithm to estimate parameters of CRT-E are also discussed. The usefulness of the new model is illustrated by two applications of real data using MLE. Based on goodness-of-fit measure and AIC, we conclude that the CRT-E distribution provides best fits to the coal mine accidents data and Wheaton river flood data compared with other alternatives. We conclude that the proposed cubic transmuted distribution family may attract wider applications in the analysis of real data.

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