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Simultaneous use of two concepts of equitable efficiency and efficiency in solving multi-objective optimization problems

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Abstract

The two requirements of impartiality and equitability expressed with the principle of transfers are fulfilled by all objective functions in equitable multi-objective optimization. However, in some practical situations, the decision-maker believes these requirements should only be satisfied by a subset of objective functions. To solve the problem in this paper, we first divide the set of objective functions into two subsets, the subset given by the decision maker and its complement. Then, we apply the concepts of equitable efficiency and efficiency for these two subsets, respectively. Furthermore, we apply the mean and inequality measures for these subsets of objective functions and present the new mean-equity models for solving the location problem. We investigate the relationship between 2-efficient solutions of the new mean-equity models and efficient solutions of the location problem.

Keywords: Efficiency, Equitable efficiency, Inequality measure, Location, Multi-objective optimization
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1 Introduction

It is well-known that any multi-objective optimization problem starts usually with an assumption that the criteria are incomparable, i.e., different criteria may have different units and physical interpretations. Many applications, however, arise from situations which present equitable criteria. Equitability is based on the assumption that the criteria are not only comparable (measured on a common scale) but also anonymous (impartial). The latter makes the distribution of outcomes among the criteria more important than the assignment of outcomes to specific criteria, and therefore models are equitable allocation of resources.

The equitable preference was first known as the generalized Lorenz dominance [6, 11]. Kostreva and Ogryczak [4] are the first ones who introduced the concept of equitability into multi-objective programming. They have shown equitable efficiency to be a refinement of Pareto efficiency by adding, to the reflexivity, strict monotonicity and transitivity of the Pareto preference order, the requirements of impartiality and satisfaction of the principle of transfers. Then Kostreva et al. [5] presented the theory of equitable efficiency in greater generality. They have developed scalarization approaches to generating equitably efficient solutions for linear and nonlinear multi-objective programs. Moreover, Ogryczak applied equitability to various problems such as location problems [14, 15, 16].

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This paper focuses on the fact that the requirements of impartiality and the principle of transfers are fulfilled by all objective functions in equitable multi-objective optimization. However, in some practical situations, the decision-maker believes these requirements should only be satisfied by a subset of objective functions. Let $\mathcal{E}$ be a subset of indices of objective functions that the decision maker intends to use the equitable concept for their objective functions. To solve the problem in this paper, we introduce the concept of $\mathcal{E}$-equitable efficiency. In fact, we use the equitable efficiency and efficiency concepts for the set $\mathcal{E}$ and its complement, respectively.

Equity, which implies fairness and justice, is a key performance indicator for locating public service facilities, [18]. It is usually quantified by inequality measures and equity maximization suggests that inequality is minimized. Inequality measures were primarily studied in economics [1, 17, 18]. Marsh and Schilling in [12] compiled twenty different measures proposed in the literature to gauge the level of equity in facility location alternatives. The simplest of these measures is based on the absolute measurement of the spread of outcomes or the measurement relative to the mean outcome, such as maximum absolute difference, mean absolute difference (Gini mean difference), maximum absolute deviation and mean absolute deviation, [9, 14].

Unfortunately, direct minimization of typical inequality measures is in contradiction with optimization of individual outcomes, [3]. To overcome this flaw, Mandell [10] introduced a bicriteria mean-equity model, which this model considers both the efficiency with the minimization of the mean outcome and the equity with the minimization of an inequality measure. This model still does not completely eliminate contradiction to the minimization of individual outcomes. Afterwards, Ogryczak [15] used the idea of combining the inequality measures with the mean itself into optimization criteria and proposed a bicriteria optimization problem. The model of Ogryczak, is useful to eliminate the contradiction to the minimization of individual outcomes, and it remains in harmony with both inequality minimization and minimization of distances. Moreover, Ogryczak introduced the concept of equitably α-consistent and stated sufficient conditions for the inequality measures to keep this concept. By this concept, he showed that every efficient solution of the bicriteria mean-equity model is an equitably efficient solution to the location problem.

Another purpose of this paper is to solve the location problem by generalizing the mean-equity models of Ogryczak's, [15]. We employ the mean and inequality measures for two subproblems of the location problem corresponding to these two subsets and present the new mean-equity models. Furthermore, we investigate the relationship between 2-efficient solutions of the new mean-equity models and $\mathcal{E}$-equitably efficient location problem, by introducing the equitable consistency property for the inequality measures.

The paper is organized as follows. We start with notations and preliminaries in section 2. In Section 3, the concept of $\mathcal{E}$-equitable efficiency is introduced by applying the equitable rational preference relations and the rational preference relations for outcomes corresponding to the set $\mathcal{E}$ and its complement, respectively. Also, the weighted sum scalarization approach is developed to generate these solutions. Furthermore, in section 3, we introduce a subset of the weakly efficient set called the 2-efficient set, which is useful for solving the location problem. In Section 4, the new mean-equity models are introduced to finding the $\mathcal{E}$-equitably efficient solutions to the location problem. Finally, the last section presents some conclusions.

2 Preliminaries and inequality measures

Throughout this paper, the following notations will be used. Let $\mathbb{R}^m$ be the Euclidean vector space and $y', y'' \in \mathbb{R}^m$. The notation $y' \leqq y''$ means that $y'_i \leq y''_i$ for $i = 1, \ldots, m$. Moreover, the symbol $y' < y''$ denotes $y'_i < y''_i$ for $i = 1, \ldots, m$, also the notation $y' \leq y''$ denotes $y' \leqq y''$ but $y'' \nleqq y'$.

Consider a decision problem defined as an optimization problem with m objective functions. Without restriction of generality, we can assume the objective functions are minimized. Hence, the problem can be formulated as follows:

$$\begin{aligned} &\min \left(f_1(x), f_2(x), \ldots, f_m(x)\right), \\ &\text{subject to } x \in X, \end{aligned} \tag{2.1}$$

where x stands a vector of decision variables which is selected from the feasible set X and $f(x) = (f_1(x), f_2(x), \ldots, f_m(x))$ is a vector function that maps the feasible set X into the objective (criterion) space $\mathbb{R}^m$. We refer to the elements of the objective space as outcome vectors. An outcome vector y is attainable if it expresses outcomes of a feasible solution, i.e., $y = f(x)$ for some $x \in X$. The set of all attainable outcome vectors will be denoted by $Y = f(X)$.

To make the multi-objective optimization model operational, one needs to assume some solution concepts specifying what it means to minimize multi-objective functions. The solution concepts are defined by the properties of the corresponding preference model. We can assume that solution concepts depend only on the evaluation of the outcome vectors. Thus, we can limit our considerations to the preference model in the objective space Y.

In the following, some basic concepts and definitions of preference relations are reviewed in [4]. Preferences are represented by a weak preference relation by $\preceq$, which allows us to compare pairs of outcome vectors y', y'' in the objective space Y. We say $y' \preceq y''$ if and only if "y' is at least as good as y''" or "y' is weakly preferred to y''". In other words, $y' \preceq y''$ means that the decision maker thinks that the outcome vector y' is at least as good as the outcome vector y''. From $\preceq$, we can derive two other important relations on Y.

Definition 2.1. Suppose that $y', y'' \in \mathbb{R}^m$. Let $\preceq$ be a relation of weak preference defined on $\mathbb{R}^m \times \mathbb{R}^m$. The strict preference relation, $\prec$, is defined by

$$y' \prec y'' \quad \Leftrightarrow \quad (y' \preceq y'' \textit{and not } y'' \preceq y'), \tag{2.2}$$

and the indifference relation, $\simeq$, is defined by

$$y' \simeq y'' \quad \Leftrightarrow \quad (y' \preceq y'' \textit{and } y'' \preceq y'). \tag{2.3}$$

Definition 2.2. Preference relations satisfying in the following axioms are called equitable rational preference relations:

1. Reflexivity: for all $y \in \mathbb{R}^m$, $y \preceq y$.
2. Transitivity: for all $y', y'', y''' \in \mathbb{R}^m$, $y' \preceq y''$ and $y'' \preceq y'''$ implies that $y' \preceq y'''$.
3. Monotonicity: for all $y \in \mathbb{R}^m$, $y - \epsilon e_i \prec y$ for all $\epsilon > 0$ where e_i denotes the i^{th} unit vector in $\mathbb{R}^m$, for all $i \in \{1, 2, \ldots, m\}$.
4. Impartial: for all $y \in \mathbb{R}^m$

$$(y_1, y_2, \ldots, y_m) \simeq (y_{\tau(1)}, y_{\tau(2)}, \ldots, y_{\tau(m)}),$$

where τ stands for an arbitrary permutation of components of y.

5. Principle of transfers: for all $y \in \mathbb{R}^m$ and for all $i, j \in \{1, 2, \ldots, m\}$

$$y_i > y_j \Rightarrow y - \epsilon e_i + \epsilon e_j \prec y,$$

where $0 < \epsilon < y_i - y_j$.

Note that a preference relation with the reflexivity, transitivity and monotonicity axioms, is called rational preference relation. The rational preference relations and the equitable rational preference relations allow us to define the equitable efficiency and efficiency concepts, respectively. The following definitions are given in [4].

Definition 2.3. Let $y', y'' \in Y$.

(i) We say that y' rationally dominates y'', and denote by $y' \prec_r y''$ if and only if $y' \preceq y''$ for all rational preference relations $\preceq$, and there is a rational preference relation $\preceq_1$ such that $y' \prec_1 y''$. An outcome vector y is called rationally nondominated if and only if there is not another outcome vector y' such that $y' \prec_r y$. Analogously, a feasible solution $x \in X$ said to be an efficient (or a Pareto optimal) solution of multi-objective problem (2.1) if and only if $y = f(x)$ is rationally nondominated. The set of all efficient solutions and the set of all nondominated points of problem (2.1) are denoted by X_E and Y_N, respectively.

(ii) We say that y' equitably dominates y'', and denote by $y' \prec_e y''$ if and only if $y' \preceq y''$ for all equitable rational preference relations $\preceq$ and there is an equitable rational preference relation $\preceq_2$ such that $y' \prec_2 y''$. An outcome vector y is called equitably nondominated if and only if there is not another outcome vector y' such that $y' \prec_e y$. Analogously, a feasible solution x is called an equitably efficient solution of multi-objective problem (2.1) if and only if $y = f(x)$ is equitably nondominated. The set of all equitably efficient solutions and the set of all equitably nondominated points of problem (2.1) are denoted by X_e and Y_{eN}, respectively.

Definition 2.4. Let $y \in \mathbb{R}^m$.

1. The function $\theta : \mathbb{R}^m \to \mathbb{R}^m$ is called an ordering map if and only if $\theta(y) = (\theta_1(y), \theta_2(y), \ldots, \theta_m(y))$, where $\theta_1(y) \geq \theta_2(y) \geq \ldots \geq \theta_m(y)$ in which $\theta_i(y) = y_{\tau(i)}$ for $i = 1, 2, \ldots, m$, and τ is a permutation of the set $\{1, 2, \ldots, m\}$.
2. The function $\overline{\theta} : \mathbb{R}^m \to \mathbb{R}^m$ is called a cumulative ordering map if and only if $\overline{\theta}(y) = (\overline{\theta}_1(y), \overline{\theta}_2(y), \ldots, \overline{\theta}_m(y))$, where $\overline{\theta}_i(y) = \sum_{j=1}^{i} \theta_j(y)$ for $i = 1, 2, \ldots, m$.

Table 1: consistency results

Measure	Formulation	The maximum value of α for Δ-boundedness of $\alpha\rho(y)$
Maximum absolute difference	$S(y) = \max_{i,j\in M} \lvert y_i - y_j\rvert$	$\frac{1}{m}$
Mean absolute difference	$D(y) = \frac{1}{2m^2}\sum_{i=1}^{m}\sum_{j=1}^{m} \lvert y_i - y_j\rvert$	1
Maximum absolute deviation	$R(y) = \max_{i\in M} \lvert y_i - \mu(y)\rvert$	$\frac{1}{m-1}$
Mean absolute deviation	$\delta(y) = \frac{1}{m}\sum_{i=1}^{m} \lvert y_i - \mu(y)\rvert$	$\frac{1}{2}$
Standard deviation	$\sigma(y) = \sqrt{\frac{1}{m}\sum_{i=1}^{m}(y_i - \mu(y))^2}$	$\frac{1}{\sqrt{m-1}}$
Maximum upper semideviation	$\Delta(y) = \max_{i\in M}(y_i - \mu(y))$	1
Mean absolute semideviation	$\bar{\delta}(y) = \frac{1}{m}\sum_{i=1}^{m}(y_i - \mu(y))_+$	1
Standard upper semideviation	$\bar{\sigma}(y) = \sqrt{\frac{1}{m}\sum_{i=1}^{m}(y_i - \mu(y))_+^2}$	1

Proposition 2.5 ([4], Propositions 1.1 and 2.3). For any two vectors $y', y'' \in Y$, we have

(i) $y' \preceq_r y'' \Leftrightarrow y' \leqq y'$ and $y' \prec_r y'' \Leftrightarrow y' \leq y''$;
(ii) $y' \preceq_e y'' \Leftrightarrow \overline{\theta}(y') \leqq \overline{\theta}(y'')$ and $y' \prec_e y'' \Leftrightarrow \overline{\theta}(y') \leq \overline{\theta}(y'')$.

As a consequence of part (i), we can state that a feasible solution $x \in X$ is an efficient (or a Pareto optimal) solution of the multi-objective problem (2.1), if and only if there is not $x' \in X$ such that $f_i(x') \leq f_i(x)$ for $i = 1, 2, \ldots, m$, where at least one inequality is strict.

The problem (2.1) can be considered as the generic location problem from a multi-criteria perspective, where X denotes the feasible set of location patterns (location decisions). There is given a set $\mathcal{M} = \{1, 2, \ldots, m\}$ of $\mathcal{M}$ clients (service recipients). Each client is represented by a specific point in the geographical space. The real value of the function $f_i(x)$ measures the outcome $y_i = f_i(x)$ of the location pattern x for client i. The outcomes can be measured as distance, travel time, the levels of clients dissatisfaction of locations, etc.

The equity issue has received increasing attention in recent years in location decisions, especially in the applications related to the public sector, where fair distributions of accessibility to the services should be guaranteed among users. Equity is usually quantified with the so-called inequality measures to be minimized. In facility location problem literature, a huge number of inequality measures have been proposed [3, 7, 8, 9, 10, 13, 14, 18]. Such measures have been formulated to capture the level of inequity of distribution, i.e., the higher the value, the less fair the distribution. Hence, in order to maximize the equity such measures should be minimized. In Table 1, a list of the most popular measures is reported. In the formulation of such measures, we will refer to the following notations: The mean outcome is $\mu(y) = \frac{1}{m}\sum_{i=1}^{m} y_i$ and $(.)_+$ denotes the non-negative part of a number.

It should be noted that the inequality measures used in economics are usually normalized by dividing the mean outcome. As a typical example of a relative inequality measure, we can mention the Gini coefficient $D(y)/\mu(y)$ which has been analyzed in the location context.

Let us denote by ρ an arbitrary inequality measure. One can easily verify that direct minimization of the inequality measures, i.e.

$$\min \rho(f(x)), \quad \text{subject to } x \in X,$$

contradicts the optimization of individual outcomes. To overcome this flaw, Mandell [10] introduced the following bicriteria mean-equity model

$$\min \left(\mu(f(x)), \rho(f(x))\right), \quad \text{subject to } x \in X.$$

Although this model considers both the efficiency with minimization of the mean outcome $\mu(f(x))$ and the equity with minimization of an inequality measure $\rho(f(x))$, it still does not completely eliminate contradiction to the minimization of individual outcomes. Hence, Ogryczak [15] used the idea of combining the inequality measures with the mean itself into optimization criteria and proposed the following problem

$$\begin{aligned}&\min\left(\mu(f(x)), \mu(f(x)) + \alpha\rho(f(x))\right),\\&\text{subject to } x \in X.\end{aligned} \tag{2.4}$$

He introduced the concept of equitably α-consistent and stated sufficient conditions for the inequality measures to keep this concept. Moreover, he showed that every efficient solution of the bicriteria problem (2.4) is an equitably efficient location. In the following, we will recall some of these conditions.

Definition 2.6. (i) We say the inequality measure ρ is convex, if

$$\rho(\lambda y' + (1-\lambda)y'') \leqslant \lambda\rho(y') + (1-\lambda)\rho(y''),$$

for any $y', y'' \in \mathbb{R}^m$ and $0 \leqslant \lambda \leqslant 1$. Moreover, the inequality measure ρ is strictly convex on equally distributed outcome vectors, if

$$\rho(\lambda y' + (1-\lambda)y'') < \lambda\rho(y') + (1-\lambda)\rho(y''), \quad \text{for } \; 0 < \lambda < 1,$$

for any two vectors $y' \neq y''$ but representing the same outcomes distribution as some y, i.e., $y' = (y'_{\tau'(1)}, \ldots, y'_{\tau'(m)})$ for some permutation τ' and $y'' = (y''_{\tau''(1)}, \ldots, y''_{\tau''(m)})$ for some permutation τ''.

(ii) The inequality measure ρ is positively homogeneous, if $\rho(\lambda y) = \lambda\rho(y)$ for positive real number λ and $y \in \mathbb{R}^m$.

(iii) Let $\rho(y) \geqslant 0$, $\alpha > 0$ and the inequality measure $\alpha\rho$ is defined by $(\alpha\rho)(y) = \alpha\rho(y)$. We say that inequality measure $\alpha\rho$ is Δ-bounded, if $\alpha\rho(y) \leqslant \Delta(y)$ for any $y \in \mathbb{R}^m$. This means that $\alpha\rho$ is upper bounded by the maximum upper deviation. Moreover, we say that $\alpha\rho$ is strictly Δ-bounded if $\alpha\rho(y) < \Delta(y)$ for any $y \in \mathbb{R}^m$ with $\Delta(y) > 0$.

It can be easily checked that the typical inequality measures in Table 1 are convex and positively homogeneous. For $\alpha > 0$, an inequality measure $\alpha\rho(y)$ satisfies the convexity and positive homogeneity conditions if these conditions hold for $\rho(y)$. As discussed in [15], we have

$$\begin{aligned}&D(y) \leqslant \Delta(y), \quad \overline{\sigma}(y) \leqslant \Delta(y), \quad \overline{\delta}(y) \leqslant \Delta(y), \quad \frac{1}{m}S(y) \leqslant \Delta(y), \quad \frac{1}{m-1}R(y) \leqslant \Delta(y),\\&\frac{1}{2}\delta(y) \leqslant \Delta(y), \quad \frac{1}{\sqrt{m-1}}\sigma(y) \leqslant \Delta(y),\end{aligned} \tag{2.5}$$

for any $y \in \mathbb{R}^m$. Hence, the measures D, $\overline{\sigma}$, $\overline{\delta}$, $\frac{1}{m}S$, $\frac{1}{m-1}R$, $\frac{1}{2}\delta$ and $\frac{1}{\sqrt{m-1}}\sigma$ are Δ-bounded. For any outcome vector y with $\Delta(y) > 0$, it concludes that at least one outcome y_i must be below the mean. Thus, we can deduce that the above inequality measures are strictly Δ-bounded. It is also obvious that the maximum absolute upper deviation Δ is Δ-bounded but it is not strictly Δ-bounded.

According to the relations (2.5), we can determine the maximum value of α for which the inequality measure $\alpha\rho(y)$ is Δ-bounded. For example, we have $\alpha S(y) \leqslant \Delta(y)$, for all $0 < \alpha \leqslant \frac{1}{m}$. Therefore, $\frac{1}{m}$ is the maximum value that $\alpha S(y)$ is Δ-bounded. The Δ-boundedness results for $\alpha\rho(y)$ are summarized for typical inequality measures in Table 1.

Theorem 2.7 ([15], Theorems 4-8). Let $\alpha > 0$ and $\rho(y) \geqslant 0$ be a convex and positively homogeneous inequality measure. Also, let $\alpha\rho$ be Δ-bounded. We have the following assertions:

(i) $\rho(y)$ is mean-complementary α-monotonous, i.e.

$$y' \leqq y' \Longrightarrow \mu(y') + \alpha\rho(y') \leqslant \mu(y'') + \alpha\rho(y''). \tag{2.6}$$

(ii) $\rho(y)$ is mean-complementary equitably α-consistent, i.e.

$$y' \preceq_e y'' \Longrightarrow \mu(y') + \alpha\rho(y') \leqslant \mu(y'') + \alpha\rho(y''). \tag{2.7}$$

If the inequality measure $\alpha\rho$ is strictly Δ-bounded, then

(i') $\rho(y)$ is mean-complementary strictly α-monotonous, i.e.

$$y' \leq y' \Longrightarrow \mu(y') + \alpha\rho(y') < \mu(y'') + \alpha\rho(y''). \tag{2.8}$$

Moreover, if $\rho(y)$ is strictly convex on equally distributed outcomes, then

(ii') $\rho(y)$ is mean-complementary equitably strongly α-consistent, i.e.

$$y' \prec_e y'' \Longrightarrow \mu(y') + \alpha\rho(y') < \mu(y'') + \alpha\rho(y''). \tag{2.9}$$

Theorem 2.8 ([15], Corollary 2). Let $\alpha > 0$ and $\rho(y) \geqslant 0$ be a convex and positively homogeneous inequality measure.

(i) If the inequality measure $\alpha\rho$ is Δ-bounded, then except for outcomes with identical values of $\mu(y)$ and $\rho(y)$, every efficient solution of the bicriteria problem (2.4) is an equitably efficient location.

(ii) If $\rho(y)$ is strictly convex on equally distributed outcomes and the inequality measure $\alpha\rho$ is strictly Δ-bounded, then every efficient solution of the problem (2.4) is an equitably efficient location.

3 $\mathcal{E}$-equitable efficiency

Let $\mathcal{E} \subset \mathcal{M}$ be a subset of indices of objective functions and $\mathcal{E}^c = \mathcal{M} - \mathcal{E}$ be the complement of $\mathcal{E}$. In this section, we intend to apply the concept of equitability for the objective functions $(f_j)_{j\in\mathcal{E}}$ and the concept of efficiency for the objective functions $(f_j)_{j\in\mathcal{E}^c}$. We will use the notations $f^{\mathcal{E}} = (f_j)_{j\in\mathcal{E}}$ and $f^{\mathcal{E}^c} = (f_j)_{j\in\mathcal{E}^c}$, and similarly $y^{\mathcal{E}} = (y_j)_{j\in\mathcal{E}}$ and $y^{\mathcal{E}^c} = (y_j)_{j\in\mathcal{E}^c}$, for any outcome vector $y = (y_1, \ldots, y_m)$.

First, let us define the $\mathcal{E}$-equitable dominance relation and the concepts of $\mathcal{E}$-equitable efficiency by equitable rational and rational preference relations.

Definition 3.1. Let $y', y'' \in Y$ and $\mathcal{E} \subset \mathcal{M}$. We say that y' $\mathcal{E}$-equitably dominates y'', and denote by $y' \prec_{\mathcal{E}} y''$ if and only if $y'^{\mathcal{E}} \preceq_1 y''^{\mathcal{E}}$ for all equitable rational preference relations $\preceq_1$, and $y'^{\mathcal{E}^c} \preceq_2 y''^{\mathcal{E}^c}$ for all rational preference relations $\preceq_2$. In addition, there is an equitable rational preference relation $\preceq_1$ or a rational preference relation $\preceq_2$ such that

$$y'^{\mathcal{E}} \prec_1 y''^{\mathcal{E}} \quad \text{or} \quad y'^{\mathcal{E}^c} \prec_2 y''^{\mathcal{E}^c},$$

respectively. An outcome vector y is called $\mathcal{E}$-equitably nondominated if and only if there is not another outcome vector y' such that $y' \prec_{\mathcal{E}} y$. Analogously, a feasible solution $x \in X$ is called an $\mathcal{E}$-equitably efficient solution of multi-objective problem (2.1) if and only if $y = f(x)$ is $\mathcal{E}$-equitably nondominated.

The set of all $\mathcal{E}$-equitably efficient solutions and the set of all $\mathcal{E}$-equitably nondominated points denoted by $X_{\mathcal{E}}$ and $Y_{\mathcal{E}N}$, respectively. Similar to the relation of $\mathcal{E}$-equitable dominance, we can define the relation of weak $\mathcal{E}$-equitable dominance, $\preceq_{\mathcal{E}}$. We say that $y' \preceq_{\mathcal{E}} y''$ if and only if $y'^{\mathcal{E}} \preceq_1 y''^{\mathcal{E}}$ for all equitable rational preference relations $\preceq_1$, and $y'^{\mathcal{E}^c} \preceq_2 y''^{\mathcal{E}^c}$ for all rational preference relations $\preceq_2$. Note that the relations $\prec_{\mathcal{E}}$ and $\preceq_{\mathcal{E}}$ satisfy the condition (2.2). Also, according to the condition (2.3), the indifference relation, $\simeq_{\mathcal{E}}$, is defined by

$$y' \simeq_{\mathcal{E}} y'' \Leftrightarrow (y' \preceq_{\mathcal{E}} y'' \quad \text{and} \quad y'' \preceq_{\mathcal{E}} y').$$

By Definitions 3.1, 2.3 and Proposition 2.5, we can state the $\mathcal{E}$-equitable dominance relation in terms of vector inequality on the outcome vectors.

Proposition 3.2. Let $y', y'' \in Y$ and $\mathcal{E} \subset \mathcal{M}$. We have

(i)

$$y' \preceq_{\mathcal{E}} y'' \;\Leftrightarrow\; \left(y'^{\mathcal{E}} \preceq_e y''^{\mathcal{E}} \quad \text{and} \quad y'^{\mathcal{E}^c} \preceq_r y''^{\mathcal{E}^c}\right)$$
$$\Leftrightarrow\; \left(\overline{\Theta}(y'^{\mathcal{E}}) \leqq \overline{\Theta}(y''^{\mathcal{E}}) \quad \text{and} \quad y'^{\mathcal{E}^c} \leqq y''^{\mathcal{E}^c}\right); \tag{3.1}$$

(ii) $y' \prec_{\mathcal{E}} y''$ if and only if the relation (3.1) is established and at least one of these inequalities holds strictly, i.e. $\overline{\Theta}(y'^{\mathcal{E}}) \leq \overline{\Theta}(y''^{\mathcal{E}})$ or $y'^{\mathcal{E}^c} \leq y''^{\mathcal{E}^c}$.

Remark 3.3. For $\mathcal{E} = \emptyset$ and $\mathcal{E} = \mathcal{M}$, we have Proposition 2.5.

The above results allow us to express $\mathcal{E}$-equitable efficiency for the problem (2.1) in terms of the standard efficiency for the multi-objective problem

$$\begin{aligned} &\min \left(\overline{\Theta}(f^{\mathcal{E}}(x)), f^{\mathcal{E}^c}(x)\right), \\ &\text{subject to } x \in X. \end{aligned} \tag{3.2}$$

Theorem 3.4. The feasible solution $x \in X$ is an efficient solution of problem (3.2) if and only if it is an $\mathcal{E}$-equitably efficient solution of problem (2.1).

Remark 3.5. If $\mathcal{E} = \mathcal{M}$, we have Corollary 2.2 from [4]. So, the feasible solution $x \in X$ is an efficient solution of the multi-objective problem

$$\begin{aligned} &\min \overline{\Theta}(f(x)) \\ &\text{subject to } x \in X, \end{aligned} \tag{3.3}$$

if and only if it is an equitably efficient solution of the problem (2.1).

It is noteworthy that in addition to the fact that the set of $\mathcal{E}$-equitably efficient solutions is contained within the set of Pareto optimal solutions, it also contains the set of equitably efficient solutions. Hereafter, the notation $|A|$ denotes the number of elements of the set A.

Theorem 3.6. We have $X_e \subset X_{\mathcal{E}} \subset X_E$, and hence $Y_{eN} \subset Y_{\mathcal{E}N} \subset Y_N$.

Proof . Let $x \in X_e$. If $x \notin X_{\mathcal{E}}$, then there exists a feasible solution $x' \in X$ such that

$$\sum_{j=1}^{k} \theta_j(f^{\mathcal{E}}(x')) \;\leqslant\; \sum_{j=1}^{k} \theta_j(f^{\mathcal{E}}(x)) \text{ (for } k = 1, \ldots, |\mathcal{E}|), \tag{3.4}$$
$$f_k^{\mathcal{E}^c}(x') \;\leqslant\; f_k^{\mathcal{E}^c}(x) \text{ (for } \; k \in \mathcal{E}^c), \tag{3.5}$$

where strict inequality holds at least once. Since $f(x) = (f^{\mathcal{E}}(x), f^{\mathcal{E}^c}(x))$, the definition of $\overline{\theta}_k$ allows us to consider only the following three cases.

Case (i): Let $\sum_{j=1}^{k} \theta_j(f(x')) = \sum_{j=1}^{k} \theta_j(f^{\mathcal{E}}(x'))$. In this case, the relation (3.4) follows that

$$\sum_{j=1}^{k} \theta_j(f(x')) \leqslant \sum_{j=1}^{k} \theta_j(f^{\mathcal{E}}(x)) \leqslant \sum_{j=1}^{k} \theta_j(f(x)).$$

Case (ii): There exists a subset $A \subset \mathcal{E}^c$ such that $|A| = k$ and $\sum_{j=1}^{k} \theta_j(f(x')) = \sum_{j \in A} f_k^{\mathcal{E}^c}(x')$. In this case, (3.5) concludes that

$$\sum_{j=1}^{k} \theta_j(f(x')) \leqslant \sum_{j \in A} f_k^{\mathcal{E}^c}(x) \leqslant \sum_{j=1}^{k} \theta_j(f(x)).$$

Case (iii): There exist a positive integer p and a subset $A \subset \mathcal{E}^c$ such that $p + |A| = k$ and $\sum_{j=1}^{k} \theta_j(f(x')) = \sum_{j=1}^{p} \theta_j(f^{\mathcal{E}}(x')) + \sum_{j\in A} f_k^{\mathcal{E}^c}(x')$. In this case, the relations (3.4) and (3.5) imply that

$$\sum_{j=1}^{k} \theta_j(f(x')) \leqslant \sum_{j=1}^{p} \theta_j(f^{\mathcal{E}}(x)) + \sum_{j\in A} f_k^{\mathcal{E}^c}(x) \leqslant \sum_{j=1}^{k} \theta_j(f(x)).$$

Therefore, we deduce that $f(x') \prec_e f(x)$ in all cases. This contradicts the equitable efficiency of x, hence $X_e \subset X_{\mathcal{E}}$. Because,

$$f(x') \leq f(x) \Longrightarrow \overline{\theta}(f(x')) \leq \overline{\theta}(f(x)),$$

it is evident that $X_{\mathcal{E}} \subset X_E$. □

In what follows we will use Theorem 3.4 and Proposition 3.2 to show that efficient solutions of the subproblems

$$\begin{aligned} &\min \overline{\Theta}(f^{\mathcal{E}}(x)) \\ &\text{subject to } x \in X, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} &\min f^{\mathcal{E}^c}(x)) \\ &\text{subject to } x \in X, \end{aligned} \tag{3.7}$$

reduce $\mathcal{E}$-equitably efficient solutions of the original problem (2.1).

Theorem 3.7. (i) Suppose that $\sum_{j\in\mathcal{E}} f_j$ is an injective function. If $x \in X$ is an efficient solution of problem (3.6), then it is an $\mathcal{E}$-equitably efficient solution of problem (2.1).

(ii) Suppose that the function f_k is injective for some $k \in \mathcal{E}^c$. If $x \in X$ is an efficient solution of problem (3.7), then it is an $\mathcal{E}$-equitably efficient solution of problem (2.1).

Since the set of efficient solutions of problem (3.2) is contained within the set of efficient solutions of problem (2.1), and the set of efficient solutions of problem (3.2) contains the set of efficient solutions of the problems (3.6) and (3.7), we can use efficient solutions of problem (3.2) to coordinate efficient solutions of these problems.

Scalarization is one of the most common approaches used to solve a multi-objective problem. Scalarizing functions are used to transform a given multi-objective problem into a single-objective optimization problem, by aggregating the objectives of a multi-objective problem into a single objective. The weighted sum method is one of the most common scalarizing techniques for finding efficient solutions to multi-objective problems. The relationships between the optimal solutions of this scalarization method and (weakly) efficient solutions of the multi-objective problems are investigated in [2]. Kostreva et al. [5] have proven every optimal solution of the weighted sum problem with strictly decreasing positive weights and ordering map $\theta(f(x))$, is an equitably efficient solution of the original multi-objective optimization problem. In the following, we construct an appropriately weighted sum to find $\mathcal{E}$-equitably efficient solutions. To do this, the next assertion is useful.

Proposition 3.8. Let $y' = (y'_1, \dots y'_m)$ and $y'' = (y''_1, \dots y''_m)$ be two vectors in $\mathbb{R}^m$ such that

$$\sum_{j=1}^{i} y'_j \leq \sum_{j=1}^{i} y''_j \qquad (i = 1, \dots, m), \tag{3.8}$$

where strict inequality holds at least once. If $(w_1, \dots, w_m) \in \mathbb{R}^m$ is a strictly decreasing vector and positive, i.e. $w_1 > \ldots > w_m > 0$, then

$$\sum_{j=1}^{m} w_j y'_j < \sum_{j=1}^{m} w_j y''_j.$$

Proof . Put $w_{m+1} = 0$. Since $w_j > w_{j+1}$ for $j = 1, \dots, m$, by applying Abel summation, we have

$$\sum_{j=1}^{m} w_j y'_j = \sum_{j=1}^{m} (w_j - w_{j+1}) \sum_{k=1}^{j} y'_k \leq \sum_{j=1}^{m} (w_j - w_{j+1}) \sum_{k=1}^{j} y''_k = \sum_{j=1}^{m} w_j y''_j.$$

Since at least one of the inequalities in (3.8) is strict, it is clear that the above inequality strictly holds. □

Theorem 3.9. Let $\lambda = (\lambda^{\mathcal{E}}, \lambda^{\mathcal{E}^c}) \in R^m$, $\lambda > 0$ and $\lambda^{\mathcal{E}}$ be a strictly decreasing vector. Then the optimal solution of the problem

$$\min_{x \in X} \left\{ \sum_{k \in \mathcal{E}} \lambda_k \theta_k (f^{\mathcal{E}}(x)) + \sum_{k \in \mathcal{E}^c} \lambda_k f_k(x) \right\} \tag{3.9}$$

is an $\mathcal{E}$-equitably efficient solution of problem (2.1).

Proof . Suppose that x is not a $\mathcal{E}$-equitably efficient solution of problem (2.1). Then a feasible vector x' must exist such that

$$\overline{\theta}(f^{\mathcal{E}}(x')) \leqq \overline{\theta}(f^{\mathcal{E}}(x)) \quad \text{and} \quad f^{\mathcal{E}^c}(x') \leqq f^{\mathcal{E}^c}(x), \tag{3.10}$$

where strict inequality holds at least once. According to Proposition 3.8, we have

$\sum_{i \in \mathcal{E}} \lambda_i \theta_i (f^{\mathcal{E}}(x')) \leqslant \sum_{i \in \mathcal{E}} \lambda_i \theta_i (f^{\mathcal{E}}(x))$. On the other hand, the second part of the relation (3.10) implies that $\sum_{i \in \mathcal{E}^c} \lambda_i f_i(x') \leqslant \sum_{i \in \mathcal{E}^c} \lambda_i f_i(x)$. Since at least one of the above two inequalities is strict, we deduce that

$$\sum_{k \in \mathcal{E}} \lambda_k \theta_k (f^{\mathcal{E}}(x')) + \sum_{k \in \mathcal{E}^c} \lambda_k f_k(x') < \sum_{k \in \mathcal{E}} \lambda_k \theta_k (f^{\mathcal{E}}(x)) + \sum_{k \in \mathcal{E}^c} \lambda_k f_k(x),$$

which this contradicts the optimality of x for problem (3.9). □

Remark 3.10. It should be noted that Theorem 3.9 becomes Theorem 2 from [5] and Proposition 3.9 from [2], when $\mathcal{E} = \mathcal{M}$ and $\mathcal{E} = \emptyset$, respectively.

At the end of this section, we introduce a subset of the weakly efficient set, called the 2-efficient set, which is useful for solving the location problem.

Definition 3.11. Let $y', y'' \in Y$. We say that y' 2-dominates y'' and write $y' \leq_2 y''$ if and only if $y'_j \leqslant y''_j$ for all $j \in M$ and there exist $j_1, j_2 \in M$ such that $y'_{j_k} < y''_{j_k}$ for $k = 1, 2$. An outcome vector y is called 2-nondominated if and only if there is not another outcome vector y' such that $y' \leq_2 y$. Analogously, a feasible solution $x \in X$ is called an 2-efficient solution of multi-objective problem (2.1) if and only if $y = f(x)$ is 2-nondominated. The set of all 2-efficient solutions and the set of all 2-nondominated points are denoted by X_{2E} and Y_{2N}, respectively.

Let us recall the definition of weak efficiency. A feasible solution $\hat{x} \in X$ is called weakly efficient, if there is no other $x \in X$ such that $f(x) < f(\hat{x})$. If we denote the set of all weakly efficient solutions by X_{WE}, then one can easily conclude that

$$X_E \subset X_{2E} \subset X_{WE}. \tag{3.11}$$

Put

$$\begin{aligned} \mathbb{R}^m_{\geq_2} &= \{d \in \mathbb{R}^m : d_j \geqslant 0 \text{ for all } j \in M \text{ and } d_{j_k} > 0 \text{ for some two } j_1, j_2 \in M\}, \\ \mathbb{R}^m_{\geqq} &= \{d \in \mathbb{R}^m : d_j \geqslant 0 \; for \; all \; j \in M\}, \end{aligned}$$

it is worth to mention the feasible solution $\hat{x} \in X$ is 2-efficient if and only if

$$(f(\hat{x}) - \mathbb{R}^m_{\geq_2}) \cap f(X) = \emptyset.$$

The following example is given to illustrate the concept of 2-efficiency.

Example 3.12. Let $X = [0,1]\times[0,1]\times[0,1]$, and $f(x) = x$ and $Y = X$. Since

$$\mathbb{R}^3_{\geq_2} = \mathbb{R}^3_{\geqq} - \{\{(d_1,0,0)\colon d_1 \geqslant 0\} \cup \{(0,d_2,0)\colon d_2 \geqslant 0\} \cup \{(0,0,d_3)\colon d_3 \geqslant 0\}\},$$

we obtain

$$\begin{aligned} X_E &= \{(0,0,0)\}, \\ X_{2E} &= \{(x_1,0,0)\colon 0 \leqslant x_1 \leqslant 1\} \cup \{(0,x_2,0)\colon 0 \leqslant x_2 \leqslant 1\} \cup \{(0,0,x_3)\colon 0 \leqslant x_3 \leqslant 1\}, \\ X_{WE} &= \{(x_1,x_2,0)\colon 0 \leqslant x_1,x_2 \leqslant 1\} \cup \{(x_1,0,x_3)\colon 0 \leqslant x_1,x_3 \leqslant 1\} \\ &\quad \cup \{(0,x_2,x_3)\colon 0 \leqslant x_2,x_3 \leqslant 1\}. \end{aligned}$$

These results confirm the validity of the relation (3.11).

4 Inequality measures and $\mathcal{E}$-equitably efficient locations

According to the conventions of the previous section, suppose that $\mathcal{E} \subset \mathcal{M}$, $f^{\mathcal{E}}(x) = (f_j(x))_{j\in\mathcal{E}}$ and $f^{\mathcal{E}^c}(x) = (f_j(x))_{j\in\mathcal{E}^c}$. In this section, we are interested in the issue of equity by minimization of the inequality measures of objective functions, $f^{\mathcal{E}}(x)$ and $f^{\mathcal{E}^c}(x)$. For this purpose, we introduce the following problem

$$\begin{aligned} &\min \left(\rho(f^{\mathcal{E}}(x)), \rho(f^{\mathcal{E}^c}(x))\right), \\ &\text{subject to } x \in X. \end{aligned} \tag{4.1}$$

Unfortunately, we can easily verify that the minimization of (4.1) contradicts the minimization of individual outcomes in (2.1). This can be illustrated by the simple example of a discrete location problem.

Example 4.1. Let us consider a single facility location problem with three clients (C_1, C_2 and C_3) and three potential locations (P_1, P_2 and P_3). Assume that

$$C_1 = (10,0), C_2 = (6,8), C_3 = (\sqrt{84},4), P_1 = (10,-1), P_2 = (4,8), P_3 = (0,0),$$

represent the position of clients and potential locations in the Cartesian coordinate system. The distances between several clients and potential locations, in terms of kilometers, are as follows:

	C_1	C_2	C_3
P_1	1	9.84	5.07
P_2	10	2	6.53
P_3	10	10	10

Hence, the potential locations generate the outcome vectors $y^1 = (1, 9.84, 5.07)$, $y^2 = (10, 2, 6.53)$ and $y^3 = (10, 10, 10)$, respectively. For $\mathcal{E} = \{1,2\}$ and $\mathcal{E}^c = \{3\}$, we have

$$\begin{aligned} &y^{1\mathcal{E}} = (1, 9.84),\ y^{1\mathcal{E}^c} = (5.07),\ y^{2\mathcal{E}} = (10, 2), \\ &y^{2\mathcal{E}^c} = (6.53),\ y^{3\mathcal{E}} = (10, 10),\ y^{3\mathcal{E}^c} = (10), \end{aligned}$$

and

$$\overline{\theta}(y^{1\mathcal{E}}) = (9.84, 10.84),\ \overline{\theta}(y^{2\mathcal{E}}) = (10, 12),\ \overline{\theta}(y^{3\mathcal{E}}) = (10, 20).$$

Since $y^1 \leq y^3$, $y^2 \leq y^3$ and $y^1 \leq_{\mathcal{E}} y^2 \leq_{\mathcal{E}} y^3$, we deduce that $Y_N = \{y^1, y^2\}$ and $Y_{\mathcal{E}N} = \{y^1\}$. On the other hand, it is easy to check that $\rho(y^{i\mathcal{E}}) > 0$ $(i = 1,2)$ and $\rho(y^{3\mathcal{E}}) = 0$ for any inequality measures ρ of Table 1. Hence the third location pattern y^3, is nondominated for the problem (4.1).

Similar to the idea proposed by Ogryczak in [15] for equitable efficiency, to overcome the flaws of direct minimization of inequality measures of subproblems, we present the following problem

$$\min\left(\mu(f(x)), \mu(f^{\mathcal{E}}(x)) + \alpha\rho(f^{\mathcal{E}}(x)), \mu(f^{\mathcal{E}^c}(x)) + \alpha\rho(f^{\mathcal{E}^c}(x))\right), \quad \text{subject to } x \in X, \tag{4.2}$$

where $\alpha > 0$. The model takes into account both the efficiency with minimization of the mean outcome $\mu(f(x))$ and the equity with minimization of the mean outcome and inequality measure weighted sum of $f^{\mathcal{E}}(x)$ and $f^{\mathcal{E}^c}(x)$. It is valuable to know that (4.2) becomes the model (2.4), when $\mathcal{E} = \mathcal{M}$.

Theorem 4.2. Let $\alpha > 0$ and $\rho(y) \geqslant 0$ be a convex and positively homogeneous inequality measure.

(i) If the inequality measure $\alpha\rho$ is Δ-bounded, then except for outcomes with identical values of $\mu(y)$ and $|\mathcal{E}|\rho(y^{\mathcal{E}}) + |\mathcal{E}^c|\rho(y^{\mathcal{E}^c})$, every efficient solution of the problem (4.2) is an $\mathcal{E}$-equitably efficient location. In particular, every efficient solution of the problem (2.4) is an equitably efficient location.

(ii) If $\rho(y)$ is strictly convex on equally distributed outcomes and the inequality measure $\alpha\rho$ is strictly Δ-bounded, then every 2-efficient solution of the problem (4.2) is an $\mathcal{E}$-equitably efficient location. In particular, every weakly efficient solution of the problem (2.4) is an equitably efficient location.

Proof . (i) Suppose that $\hat{x} \in X$ is an efficient solution of the problem (4.2) and it is not an $\mathcal{E}$-equitably efficient location. Due to Proposition 3.2, there exists a feasible solution $x \in X$ such that

$$\overline{\Theta}(f^{\mathcal{E}}(x)) \leqq \overline{\Theta}(f^{\mathcal{E}}(\hat{x})) \quad \text{and} \quad f^{\mathcal{E}}(x) \leqq f^{\mathcal{E}}(\hat{x}),$$

where $\overline{\Theta}(f^{\mathcal{E}}(x)) \leq \overline{\Theta}(f^{\mathcal{E}}(\hat{x}))$ or $f^{\mathcal{E}}(x) \leq f^{\mathcal{E}}(\hat{x})$. Hence $\mu(f(x)) \leqslant \mu(f(\hat{x}))$. Now, by applying Theorem 2.7, we obtain

$$\mu(f^{\mathcal{E}}(x)) + \alpha\rho(f^{\mathcal{E}}(x)) \leqslant \mu(f^{\mathcal{E}}(\hat{x})) + \alpha\rho(f^{\mathcal{E}}(\hat{x})), \tag{4.3}$$

$$\mu(f^{\mathcal{E}^c}(x)) + \alpha\rho(f^{\mathcal{E}^c}(x)) \leqslant \mu(f^{\mathcal{E}^c}(\hat{x})) + \alpha\rho(f^{\mathcal{E}^c}(\hat{x})). \tag{4.4}$$

If $\mu(f(x)) < \mu(f(\hat{x}))$ or at least one of the above inequalities is strict, then $\hat{x}$ cannot be an efficient solution of the problem (4.2), which is a contradiction. Otherwise, we have $\mu(f(x)) = \mu(f(\hat{x}))$ and

$$\mu(f^{\mathcal{E}}(x)) + \alpha\rho(f^{\mathcal{E}}(x)) = \mu(f^{\mathcal{E}}(\hat{x})) + \alpha\rho(f^{\mathcal{E}}(\hat{x})), \text{ and}$$
$$\mu(f^{\mathcal{E}^c}(x)) + \alpha\rho(f^{\mathcal{E}^c}(x)) = \mu(f^{\mathcal{E}^c}(\hat{x})) + \alpha\rho(f^{\mathcal{E}^c}(\hat{x})).$$

The equality

$$|\mathcal{E}|\mu(f^{\mathcal{E}}(x)) + |\mathcal{E}^c|\mu(f^{\mathcal{E}^c}(x)) = m\mu(f(x)), \tag{4.5}$$

implies that $|\mathcal{E}|\rho(f^{\mathcal{E}}(x)) + |\mathcal{E}^c|\rho(f^{\mathcal{E}^c}(x)) = |\mathcal{E}|\rho(f^{\mathcal{E}}(\hat{x})) + |\mathcal{E}^c|\rho(f^{\mathcal{E}^c}(\hat{x}))$. This contradicts the assumption of the theorem, so this situation does not occur.

(ii) Let $\hat{x} \in X$ be a 2-efficient solution of the problem (4.2). Similar to the proof of part (i), if $\hat{x} \notin X_{\mathcal{E}}$, there is a feasible solution $x \in X$ such that $\mu(f(x)) < \mu(f(\hat{x}))$ and at least one of the inequalities (4.3) or (4.4), is strict. Thus $\hat{x}$ cannot be a 2-efficient solution of (4.2), which is a contradiction. □

By taking the weighted sum of the first and second criteria in problem (4.2), according to (4.5), we obtain the bicriteria optimization problem

$$\min\left(\mu(f(x)), \mu(f(x)) + \frac{\alpha}{m}\left(|\mathcal{E}|\rho(f^{\mathcal{E}}(x)) + |\mathcal{E}^c|\rho(f^{\mathcal{E}^c}(x))\right)\right), \quad \textit{subject to } x \in X. \tag{4.6}$$

It should be noted that, the problem (4.6) is converted to the problem (2.4), when $\mathcal{E} = \mathcal{M}$.

Table 2: Values of inequality measures for Example 4.1

y	$\mu(y)$	$S(y)$	$D(y)$	$R(y)$	$\delta(y)$	$\sigma(y)$	$\Delta(y)$	$\overline{\delta}(y)$	$\overline{\sigma}(y)$
y^1	5.3	8.84	1.96	4.54	3.03	3.61	4.54	1.51	2.62
$y^{1\mathcal{E}}$	5.42	8.84	2.21	4.42	4.42	4.42	4.42	2.21	3.12
$y^{1\mathcal{E}^c}$	5.07	0	0	0	0	0	0	0	0
y^2	6.18	8	1.78	4.18	2.78	3.28	3.82	1.39	1.92
$y^{2\mathcal{E}}$	6	8	2	4	4	4	4	2	2.83
$y^{2\mathcal{E}^c}$	6.53	0	0	0	0	0	0	0	0
y^3	10	0	0	0	0	0	0	0	0
$y^{3\mathcal{E}}$	10	0	0	0	0	0	0	0	0
$y^{3\mathcal{E}^c}$	10	0	0	0	0	0	0	0	0

Proposition 4.3. (i) Every (weakly) efficient solution of the problem (4.6) is (weakly) efficient for the problem (4.2).

(ii) Except for outcomes with identical values of $\mu(f^{\mathcal{E}}(x)) + \alpha\rho(f^{\mathcal{E}}(x))$ and $\mu(f^{\mathcal{E}^c}(x)) + \alpha\rho(f^{\mathcal{E}^c}(x))$, every efficient solution of the bicriteria problem (4.6) is 2-efficient for the problem (4.2).

Due to Proposition 4.3, the following corollary holds.

Corollary 4.4. Let $\alpha > 0$ and $\rho(y) \geqslant 0$ be a convex and positively homogeneous inequality measure.

(i) If the inequality measure $\alpha\rho$ is Δ-bounded, then except for outcomes with identical values of $\mu(y)$ and $|\mathcal{E}|\rho(y^{\mathcal{E}})+|\mathcal{E}^c|\rho(y^{\mathcal{E}^c})$, every efficient solution of the bicriteria problem (4.6) is an $\mathcal{E}$-equitably efficient location.

(ii) If $\rho(y)$ is strictly convex on equally distributed outcomes and the inequality measure $\alpha\rho$ is strictly Δ-bounded, then except for outcomes with identical values of $\mu(f^{\mathcal{E}}(x))+\alpha\rho(f^{\mathcal{E}}(x))$ and $\mu(f^{\mathcal{E}^c}(x))+\alpha\rho(f^{\mathcal{E}^c}(x))$, every weakly efficient solution of the bicriteria problem (4.6) is an $\mathcal{E}$-equitably efficient location.

For $0 < \lambda < \alpha$, we have

$$(1-\frac{\lambda}{\alpha})\mu(y) + \frac{\lambda}{\alpha}\left(\mu(y) + \frac{\alpha}{m}\left(|\mathcal{E}|\rho(y^{\mathcal{E}}) + |\mathcal{E}^c|\rho(y^{\mathcal{E}^c})\right)\right) = \mu(y) + \frac{\lambda}{m}\left(|\mathcal{E}|\rho(y^{\mathcal{E}}) + |\mathcal{E}^c|\rho(y^{\mathcal{E}^c})\right),$$

hence Corollary 4.2 allow us to express the following assertion.

Corollary 4.5. Let $0 < \lambda < \alpha$ and $\rho(y) \geqslant 0$ be a convex and positively homogeneous inequality measure. If the inequality measure $\alpha\rho$ is Δ-bounded, then except for outcomes with identical values of $\mu(y)$ and $|\mathcal{E}|\rho(y^{\mathcal{E}})+|\mathcal{E}^c|\rho(y^{\mathcal{E}^c})$, every optimal solution of the problem

$$\begin{aligned} &\min\left\{\mu(f(x)) + \frac{\lambda}{m}\left(|\mathcal{E}|\rho(f^{\mathcal{E}}(x)) + |\mathcal{E}^c|\rho(f^{\mathcal{E}^c}(x))\right)\right\}, \\ &\textit{subject to } \ x \in X. \end{aligned} \tag{4.7}$$

is an $\mathcal{E}$-equitably efficient location.

To illustrate further these results, let us consider Example 4.1. Recall that $Y_{\mathcal{E}} = \{y^1\}$ and the outcome vector y^1 is a $\mathcal{E}$-equitably nondominated point of the location problem. We have calculated the values of inequality measures of the outcomes y^i, $y^{i\mathcal{E}}$ and $y^{i\mathcal{E}^c}$ for $i = 1, 2, 3$ in Table 2.

The assumptions of the part (ii) of Theorem 4.2 are satisfied by $D, \overline{\sigma}, \overline{\delta}$, $(\alpha = 1)$. Hence, one can easily check that

$$\begin{aligned} \left(\mu(y^1), \mu(y^{1\mathcal{E}}) + \rho(y^{1\mathcal{E}}), \mu(y^{1\mathcal{E}^c}) + \rho(y^{1\mathcal{E}^c})\right) \quad &\leq_2 \left(\mu(y^2), \mu(y^{2\mathcal{E}}) + \rho(y^{2\mathcal{E}}), \mu(y^{2\mathcal{E}^c}) + \rho(y^{2\mathcal{E}^c})\right) \\ &\leq_2 \left(\mu(y^3), \mu(y^{3\mathcal{E}}) + \rho(y^{3\mathcal{E}}), \mu(y^{3\mathcal{E}^c}) + \rho(y^{3\mathcal{E}^c})\right), \end{aligned}$$

for these inequality measures. Thus y^1 is a 2-nondominated point of the problem (4.2), which confirms the validity of Theorem 4.2. On the other hand, the outcome vectors y^2, y^3 are two nondominated points of the problem

$$\begin{aligned}&\min\left(\mu(f(x)), \mu(f^{\mathcal{E}}(x)) + S(f^{\mathcal{E}}(x)), \mu(f^{\mathcal{E}^c}(x)) + S(f^{\mathcal{E}^c}(x))\right),\\&\text{subject to } x \in X,\end{aligned}$$

because the inequality measure S is not Δ-bounded. This shows that the part (i) of Theorem 4.2 is not true for S and $\alpha = 1$. However, for α and values smaller than α, according to Table 2, we have

$$\begin{aligned}\left(\mu(y^1), \mu(y^{1\mathcal{E}}) + \alpha\rho(y^{1\mathcal{E}}), \mu(y^{1\mathcal{E}^c}) + \alpha\rho(y^{1\mathcal{E}^c})\right) \quad &\leq_2 \left(\mu(y^2), \mu(y^{2\mathcal{E}}) + \alpha\rho(y^{2\mathcal{E}}), \mu(y^{2\mathcal{E}^c}) + \alpha\rho(y^{2\mathcal{E}^c})\right)\\&\leq_2 \left(\mu(y^3), \mu(y^{3\mathcal{E}}) + \alpha\rho(y^{3\mathcal{E}}), \mu(y^{3\mathcal{E}^c}) + \alpha\rho(y^{3\mathcal{E}^c})\right),\end{aligned}$$

for the inequality measures S, R, δ, σ. Thus, the outcome vector y^1 is a 2-nondominated point of the problem (4.2).

5 Conclusion

In this paper, we simultaneously applied the equitable efficiency and efficiency concepts for a multi-objective optimization problem by introducing the concept of $\mathcal{E}$-equitable efficiency. Moreover, we studied some theoretical and practical aspects of the $\mathcal{E}$-equitably efficient solutions and showed that the set of $\mathcal{E}$-equitably efficient solutions is contained within the set of efficient solutions for the same problem. Therefore, considering models with the $\mathcal{E}$-equitable efficiency relieves some of the burdens from the decision maker by shrinking the solution set.

To provide an application of the concepts discussed, we decomposed the multi-objective location problem into two subproblems and applied the mean and inequality measures to these subproblems. The new mean-equity models introduced by this paper take into account both the efficiency with minimization of the mean outcome and the equity with minimization of the sum of the mean outcome and the inequality measure for these two subproblems.

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