Int. J. Nonlinear Anal. Appl. 16 (2025) 3, 115–123
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2024.31494.4646

Fixed points in tricomplex valued S-metric spaces with applications

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(Communicated by Ali Farajzadeh)

Abstract

The concept of tricomplex valued S-metric space is introduced in this article, and some properties are derived. Also, some fixed point results of contraction maps satisfying various types of rational inequalities are proved for tricomplex valued S-metric spaces. Moreover, an example is given to illustrate our main result. Furthermore, an existence theorem for the unique solution to the linear system of equations is obtained by using our main result.

Keywords: Complex number, Partial order, S-metric space, Nonsingular
2020 MSC: Primary 54E40; Secondary 54H25

1 Introduction

Let $\mathbb{C}_0$ and $\mathbb{C}_1$ be the set of all real and complex numbers respectively. Bicomplex numbers are defined by C. Segre [10] as: $z = \mho_1 + \mho_2 i_1 + \mho_3 i_2 + \mho_4 i_1 i_2$, where $\mho_1, \mho_2, \mho_3, \mho_4 \in \mathbb{C}_0$, and the independent units i_1, i_2 are such that ${i_i}^2 = {i_2}^2 = -1$, and $i_1 i_2 = i_2 i_1$. We denote the set of bicomplex numbers $\mathbb{C}_2$ is defined as:

$$\mathbb{C}_2 = \{z : z = \mho_1 + \mho_2 i_1 + \mho_3 i_2 + \mho_4 i_1 i_2, \mho_1, \mho_2, \mho_3, \mho_4 \in \mathbb{C}_0\},$$

i.e., $\mathbb{C}_2 = \{z : z = l_1 + i_2 l_2, l_1, l_2 \in \mathbb{C}_1\}$, where $l_1 = \mho_1 + \mho_2 i_1 \in \mathbb{C}_1$ and $l_2 = \mho_3 + \mho_4 i_1 \in \mathbb{C}_1$. Tricomplex numbers are defined by G. B. Price [6] as: $\varsigma = \mho_1 + \mho_2 i_1 + \mho_3 i_2 + \mho_4 j_1 + \mho_5 i_3 + \mho_6 j_2 + \mho_7 j_3 + \mho_8 i_4$, where $\mho_1, \mho_2, \mho_3, \mho_4, \mho_5, \mho_6, \mho_7, \mho_8 \in \mathbb{C}_0$, and the independent units $i_1, i_2, i_3, i_4, j_1, j_2, j_3$ are such that ${i_1}^2 = {i_2}^2 = {i_3}^2 = {i_4}^2 = -1$, $i_4 = i_1 j_3 = i_1 i_2 i_3$, $j_2 = i_1 i_3 = i_3 i_1, {j_2}^2 = 1, j_1 = i_1 i_2 = i_2 i_1$ and ${j_1}^2 = {j_3}^2 = 1$. We denote the set of tricomplex numbers $\mathbb{C}_3$ is defined as:

$$\mathbb{C}_3 = \{\varsigma : \varsigma = \mho_1 + \mho_2 i_1 + \mho_3 i_2 + \mho_4 j_1 + \mho_5 i_3 + \mho_6 j_2 + \mho_7 j_3 + \mho_8 i_4, \mho_1, \mho_2, \mho_3, \mho_4, \mho_5, \mho_6, \mho_7, \mho_8 \in \mathbb{C}_0\},$$

i.e., $\mathbb{C}_3 = \{\varsigma : \varsigma = z_1 + i_3 z_2, z_1, z_2 \in \mathbb{C}_2\}$, where $z_1 = \mho_1 + \mho_2 i_2 \in \mathbb{C}_2$ and $z_2 = \mho_3 + \mho_4 i_2 \in \mathbb{C}_2$. If $\varsigma = z_1 + i_3 z_2, \nu = w_1 + i_3 w_2 \in \mathbb{C}_3$, then the sum is $\varsigma \pm \nu = (z_1 + i_3 z_2) \pm (w_1 + i_3 w_2) = (z_1 \pm w_1) + i_3(z_2 \pm w_2)$ and the product is

$$\varsigma.\nu = (z_1 + i_3 z_2).(w_1 + i_3 w_2) = (z_1 w_1 - z_2 w_2) + i_3(z_1 w_2 + z_2 w_1).$$

Let $0, 1, e_1 = 1 + j_3/2, e_2 = 1 - j_3/2$ be four idempotent elements in $\mathbb{C}_3$ such that $e_1 + e_2 = 1$ and $e_1 e_2 = 0$. Any $\varsigma = z_1 + i_3 z_2 \in \mathbb{C}_3$ can be uniquely be expressed as the combination of e_1 and e_2, i.e., $\varsigma = z_1 + i_3 z_2 =$

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Received: August 2023 *Accepted:* February 2024

$(z_1 - i_2 z_2)e_1 + (z_1 + i_2 z_2)e_2$. This representation of ς is called the idempotent representation with respect to the idempotent components $\varsigma_1 = (z_1 - i_2 z_2)e_1$ and $\varsigma_2 = (z_1 + i_2 z_2)e_2$.

An element $\varsigma = z_1 + i_3 z_2 \in \mathbb{C}_3$ is called invertible if there exists an element ν in $\mathbb{C}_3$ such that $\varsigma\nu = 1$ where ν is called inverse of ς. An element in $\mathbb{C}_3$ is called nonsingular element if it has an inverse in $\mathbb{C}_3$ and an element in $\mathbb{C}_3$ is called singular element if it does not have an inverse in $\mathbb{C}_3$.

An element $\nu = w_1 + i_2 w_2 \in \mathbb{C}_3$ is nonsingular if and only if $|{w_1}^2 + {w_2}^2| \neq 0$ and singular if and only if $|{w_1}^2 + {w_2}^2| = 0$. The inverse of ν is defined as $\nu^{-1} = \frac{w_1 - i_2 w_2}{{w_1}^2 + {w_2}^2}$. The norm $||.||$ of $\mathbb{C}_3$ is a positive real valued function and $||.|| : \mathbb{C}_3 \to {\mathbb{C}_0}^+$ by

$$
\begin{aligned}
||\varsigma|| &= ||z_1 + i_3 z_2|| \\
&= \{|z_1|^2 + |z_2|^2\}^{\frac{1}{2}} \\
&= \left[\frac{|(z_1 - i_2 z_2)|^2 + |(z_1 + i_2 z_2)|^2}{2}\right]^{\frac{1}{2}} \\
&= ({\mho_1}^2 + {\mho_2}^2 + {\mho_3}^2 + {\mho_4}^2 + {\mho_5}^2 + {\mho_6}^2 + {\mho_7}^2 + {\mho_8}^2)^{\frac{1}{2}},
\end{aligned}
$$

where $\varsigma = \mho_1 + \mho_2 i_1 + \mho_3 i_2 + \mho_4 j_1 + \mho_5 i_3 + \mho_6 j_2 + \mho_7 j_3 + \mho_8 i_4 = z_1 + i_3 z_2 \in \mathbb{C}_3$.

Define a partial order $\precsim_{i_3}$ on $\mathbb{C}_3$ as follows. For $\varsigma = z_1 + i_3 z_2, \nu = w_1 + i_3 w_2 \mathbb{C}_3$. $\varsigma \precsim_{i_3} \nu$ if and only if $z_1 \precsim_{i_2} w_1$, and $z_2 \precsim_{i_2} w_2$. It follows that $\varsigma \precsim_{i_3} \nu$ if one of the following conditions is fulfilled:

(i) $z_1 = w_1$, $z_2 = w_2$,

(ii) $z_1 \prec_{i_3} w_1$, $z_2 = w_2$,

(iii) $z_1 = w_1$, $z_2 \prec_{i_3} w_2$,

(iv) $z_1 \prec_{i_3} w_1$, $z_2 \prec_{i_3} w_2$.

In particular we will write $\varsigma \precnsim_{i_3} \nu$ if $\varsigma \precsim_{i_3} \nu$ and $\varsigma \neq \nu$ and one of (ii),(iii), and (iv) is fulfilled, and we will write $\varsigma \prec_{i_3} \nu$ if only (iv) is fulfilled. Note that

(I) $\varsigma \precsim_{i_3} \nu \Rightarrow ||\varsigma|| \leq ||\nu||$,

(II) $||\varsigma + \nu|| \leq ||\varsigma|| + ||\nu||$,

(III) $||a\varsigma|| = |a|||\varsigma||$, where a is a non negative real number,

(IV) $||\varsigma\nu|| \leq 2||\varsigma||||\nu||$, and the equality holds only when atleast one of ς and ν is nonsingular,

(V) $||\varsigma^{-1}|| = ||\varsigma||^{-1}$ if ς is a nonsingular,

(VI) $||\frac{\varsigma}{\nu}|| = \frac{||\varsigma||}{||\nu||}$, if ν is a nonsingular.

Complex valued metric spaces were first discussed by A. Azam et al. in [1]. The notion of bicomplex valued metric spaces was introduced by J. Choi et al in [2]. G. Mani et al. proposed the idea of tricomplex valued metric spaces in their paper [3]; various properties were derived, and common fixed point results for mappings meeting a rational inequality were demonstrated. See [11, 15, 16] for a recent publication on fixed point theory in complex, bicomplex, and tricomplex valued metric spaces, respectively.

Definition 1.1. [1] Assuming that $G \neq \emptyset$ is a set. The mapping $d : G \times G \to \mathbb{C}_3$ is said to be a tricomplex valued metric if

(i) $0 \precsim_{i_3} d(\beth, \aleph)$, $\forall\ \beth, \aleph \in G$,

(ii) $d(\beth, \aleph) = 0$ if and only if $\beth = \aleph$ in G,

(iii) $d(\beth, \aleph) = d(\aleph, \beth)$, $\forall\ \beth, \aleph \in G$,

(iv) $d(\beth, \aleph) \precsim_{i_3} d(\beth, \wp) + d(\wp, \aleph)$, $\forall\ \beth, \wp, \aleph \in G$.

The pair (G, d) is called a tricomplex valued metric space.

The concept of S-metric space was introduced by S. Sedghi et al in [9]. A S-metric is a real valued mapping on G^3, for some set $G \neq \emptyset$, where the map represents the perimeter of the triangle. There are many articles on fixed point theory in S-metric spaces; see [5, 7, 8, 12, 13, 14].

Definition 1.2. [9] Assuming that $G \neq \emptyset$ is a set. The mapping $S : G^3 \to [0, \infty)$ is said to be a S-metric if

(1) $S(\daleth, \aleph, \rho) \geq 0$, for all $\daleth, \aleph, \rho \in G$ and

(2) $S(\daleth, \aleph, \rho) = 0$ if and only if $\daleth = \aleph = \rho$, for all $\daleth, \aleph, \rho \in G$; and

(3) $S(\daleth, \aleph, \rho) \leq S(\daleth, \daleth, \sigma) + S(\aleph, \aleph, \sigma) + S(\rho, \rho, \sigma)$, for all $\daleth, \aleph, \rho, \sigma \in G$.

The pair (G, S) is called a S-metric space.

Recently, N. M. Mlaika [4] introduced the notion of complex valued S-metric space to derive common fixed point results. In this paper, we extend the codomain of complex valued S-metric to tricomplex numbers, and we present a new definition of tricomplex valued S-metric space that generalizes the notion of complex valued metric space, and S-metric space. Also, we derive some properties of tricomplex valued S-metric spaces. Moreover, we prove some fixed point results for contraction maps satisfying various types of rational inequalities in tricomplex valued S-metric space. Moreover, we provide an example to illustrate our main result. Furthermore, we prove an existence theorem for the unique solution to the linear system of equations by using our main result.

2 Tricomplex Valued S-Metric Spaces

In this section, we introduce tricomplex valued S-metric space and some of its properties to derive results on fixed point theory.

Definition 2.1. Assuming that $G \neq \emptyset$ is a set. The mapping $S : G^3 \to \mathbb{C}_3$ is said to be a tricomplex valued S-metric if

(1) $0 \precsim_{i_3} S(\daleth, \aleph, \rho)$, for all $\daleth, \aleph, \rho \in G$ and

(2) $S(\daleth, \aleph, \rho) = 0$ if and only if $\daleth = \aleph = \rho$, for all $\daleth, \aleph, \rho \in G$; and

(3) $S(\daleth, \aleph, \rho) \precsim_{i_3} S(\daleth, \daleth, \sigma) + S(\aleph, \aleph, \sigma) + S(\rho, \rho, \sigma)$, for all $\daleth, \aleph, \rho, \sigma \in G$.

The pair (G, S) is called a tricomplex valued S-metric space(or, TVSMS).

Example 2.2. Let $G = (0, \infty)$ and $S(\daleth, \aleph, \rho) = (i_2 i_3)(|\daleth - \rho| + |\aleph - \rho|)$. Then (G, S) is a TVSMS.

Definition 2.3. Assuming that (G, S) is a TVSMS and $\{\daleth_n\}$ is a sequence in G.

(i) $(\daleth_n)_{n=1}^{\infty}$ converges to a point $\daleth$(or $(\daleth_n)_{n=1}^{\infty} \to \daleth$) if and only if for every $c \in \mathbb{C}_3$ with $0 \prec_{i_3} c$, there exists $n_0 \in \mathbb{N}$(Natural numbers) such that $S(\daleth_n, \daleth_n, \daleth) \prec_{i_3} c$, $\forall\ n \geq n_0$.

(ii) $\{\daleth_n\}$ is called a Cauchy sequence, if for each $c \in \mathbb{C}_3$ with $0 \prec_{i_3} c$, there is an $n_0 \in \mathbb{N}$ such that $S(\daleth_n, \daleth_n, \daleth_m) \prec_{i_3} c$, for all $n, m \geq n_0$.

Definition 2.4. Assuming that (G, S) is a TVSMS. Then G is said to be complete if every Cauchy sequence is convergent in G.

Lemma 2.5. Assuming that (G, S) is a TVSMS. Then a sequence $(\daleth_n)_{n=1}^{\infty}$ converges to a point $\daleth$ if and only if $||S(\daleth_n, \daleth_n, \daleth)|| \to 0$ as $n \to \infty$.

Proof . Let $(\beth_n)_{n=1}^{\infty}$ be a sequence, and $(\beth_n)_{n=1}^{\infty} \to \beth \in G$. For $\epsilon \in \mathbb{C}_0$ with $\epsilon > 0$, let $c = \frac{\epsilon}{\sqrt{8}} + i_1\frac{\epsilon}{\sqrt{8}} + i_2\frac{\epsilon}{\sqrt{8}} + j_1\frac{\epsilon}{\sqrt{8}} + i_3\frac{\epsilon}{\sqrt{8}} + j_2\frac{\epsilon}{\sqrt{8}} + j_3\frac{\epsilon}{\sqrt{8}} + i_4\frac{\epsilon}{\sqrt{8}}$. For every $c \in \mathbb{C}_3$ with $0 \prec_{i_3} c$, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, $S(\beth_n, \beth_n, \beth) \prec_{i_3} c$.

$$||S(\beth_n, \beth_n, \beth)|| < ||c|| = \epsilon, \ \forall\ n \geq n_0.$$

equation

It follows that $||S(\beth_n, \beth_n, \beth)|| \to 0$ as $n \to \infty$. Conversely, Assuming that $||S(\beth_n, \beth_n, \beth)|| \to 0$ as $n \to \infty$. Then given $c \in \mathbb{C}_3$ with $0 \prec_{i_3} c$, there exists $\delta \in \mathbb{C}_0$, with $\delta > 0$ such that for $z \in \mathbb{C}_3$

$$||z|| < \delta \Rightarrow z \prec_{i_3} c$$

For this δ, there exists $n_0 \in \mathbb{N}$ such that

$$||S(\beth_n, \beth_n, \beth)|| < \delta, \ \forall\ n \geq n_0.$$

This means that $S(\beth_n, \beth_n, \beth) \prec_{i_3} c, \forall n \geq n_0$. Hence $\beth_n \to \beth \in G$. □

Lemma 2.6. Assuming that (G, S) is a TVSMS. If a sequence $(\beth_n)_{n=1}^{\infty}$ converges to $\beth$ and a sequence $(\aleph_n)_{n=1}^{\infty}$ converges to $\aleph$, then $S(\beth_n, \beth_n, \aleph_n) \to S(\beth, \beth, \aleph)$ as $n \to \infty$.

Proof . Let $(\beth_n)_{n=1}^{\infty} \to \beth \in H$, and $(\aleph_n)_{n=1}^{\infty} \to \aleph \in G$. For $\epsilon \in \mathbb{C}_0$ with $\epsilon > 0$, let $c = \frac{\epsilon}{\sqrt{8}} + i_1\frac{\epsilon}{\sqrt{8}} + i_2\frac{\epsilon}{\sqrt{8}} + j_1\frac{\epsilon}{\sqrt{8}} + i_3\frac{\epsilon}{\sqrt{8}} + j_2\frac{\epsilon}{\sqrt{8}} + j_3\frac{\epsilon}{\sqrt{8}} + i_4\frac{\epsilon}{\sqrt{8}}$. For every $c \in \mathbb{C}_3$ with $0 \prec_{i_3} c$, there exist $n_0, n_1 \in \mathbb{N}$ such that, for all $n \geq n_0$, we have $S(\beth_n, \beth_n, \beth) \prec_{i_3} \frac{c}{4}$, and for all $n \geq n_1$, we have $S(\aleph_n, \aleph_n, \aleph) \prec_{i_3} \frac{c}{4}$, then for all $n \geq n_2 = \max\{n_0, n_1\}$,

$$\begin{aligned} S(\beth, \beth, \aleph) &\precsim_{i_3} 2S(\beth, \beth, \beth_n) + 2S(\aleph, \aleph, \aleph_n) + S(\aleph_n, \aleph_n, \beth_n) \\ &\prec c + S(\aleph_n, \aleph_n, \beth_n) \end{aligned}$$

implies

$$S(\beth, \beth, \aleph) - S(\beth_n, \beth_n, \aleph_n) \precsim_{i_3} c,$$
$$||S(\beth, \beth, \aleph) - S(\beth_n, \beth_n, \aleph_n)|| \leq ||c|| = \epsilon,$$

and hence $S(\beth_n, \beth_n, \aleph_n) \to S(\beth, \beth, \aleph)$ as $n \to \infty$. □

Lemma 2.7. Assuming that (G, S) is a TVSMS. Then $\{\beth_n\}$ is a Cauchy sequence if and only if $||S(\beth_n, \beth_n, \beth_{n+m})|| \to 0$ as $n \to \infty$.

Proof . Let $\{\beth_n\}$ is a Cauchy sequence. For $\epsilon \in \mathbb{C}_0$ with $\epsilon > 0$, let $c = \frac{\epsilon}{\sqrt{8}} + i_1\frac{\epsilon}{\sqrt{8}} + i_2\frac{\epsilon}{\sqrt{8}} + j_1\frac{\epsilon}{\sqrt{8}} + i_3\frac{\epsilon}{\sqrt{8}} + j_2\frac{\epsilon}{\sqrt{8}} + j_3\frac{\epsilon}{\sqrt{8}} + i_4\frac{\epsilon}{\sqrt{8}}$. For every $c \in \mathbb{C}_3$ with $0 \prec_{i_3} c$, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, $S(\beth_n, \beth_n, \beth_{n+m}) \prec_{i_3} c$.

$$||S(\beth_n, \beth_n, \beth_{n+m})|| < ||c|| = \epsilon, \ \forall\ n \geq n_0.$$

It follows that $||S(\beth_n, \beth_n, \beth_{n+m})|| \to 0$ as $n \to \infty$. Conversely, Assuming that $||S(\beth_n, \beth_n, \beth_{n+m})|| \to 0$ as $n \to \infty$. Then given $c \in \mathbb{C}_3$ with $0 \prec_{i_3} c$, there exists $\delta \in \mathbb{C}_0$, with $\delta > 0$ such that for $z \in \mathbb{C}_3$

$$||z|| < \delta \Rightarrow z \prec_{i_3} c$$

For this δ, there exists $n_0 \in \mathbb{N}$ such that

$$||S(\beth_n, \beth_n, \beth_{n+m})|| < \delta, \ \forall\ n \geq n_0.$$

This means that $S(\beth_n, \beth_n, \beth_{n+m}) \prec_{i_3} c, \forall n \geq n_0$. Therefore $\{\beth_n\}$ is a Cauchy sequence. □

Lemma 2.8. Assuming that S is a tricomplex valued S-metric on G, then $S(\beth, \beth, \aleph) = S(\aleph, \aleph, \beth)$, $\forall\ \beth, \aleph \in G$.

Proof . By the definition of tricomplex valued S-metric, we have $S(\beth, \beth, \aleph) \prec_{i_3} 2S(\beth, \beth, \beth) + S(\aleph, \aleph, \beth)$. In view of $S(\beth, \beth, \beth) = 0$, we find that $S(\beth, \beth, \aleph) \prec_{i_3} S(\aleph, \aleph, \beth)$. Similarly, we find $S(\aleph, \aleph, \beth) \prec_{i_3} S(\beth, \beth, \aleph)$. It follows that $S(\beth, \beth, \aleph) = S(\aleph, \aleph, \beth)$. □

3 Main Results

In this section, we shall prove some fixed point theorems for different types of contraction mappings satisfying rational inequalities on TVSMS.

Theorem 3.1. Assuming that (G,S) is a complete TVSMS with nonsingular $1+S(\beth,\beth,\aleph)$ and $||1+S(\beth,\beth,\aleph)||\neq 0$, whenever $\beth,\aleph\in G$. If a map $f:G\to G$ satisfies $S(f(\aleph),f(\aleph),f(\beth))\precsim_{i_3}\lambda S(\beth,\beth,\aleph)+\frac{\eta S(\beth,\beth,f(\beth))S(f(\aleph),f(\aleph),\aleph)}{1+S(\beth,\beth,\aleph)}$, for all $\beth,\aleph\in G$, whenever $\lambda,\eta\in[0,1)$ with $\lambda+2\eta<1$, then the function f has a unique fixed point(or, UFP).

Proof . Let $\beth_0\in G$, and $\beth_1=f(\beth_0)$. Suppose $\beth_{n+1}=f(\beth_n)$, whenever $n\in\mathbb{N}$. For every $n\in\mathbb{N}$, from

$$\begin{aligned}
S(\beth_n,\beth_n,\beth_{n-1}) &= S(f(\beth_{n-1}),f(\beth_{n-1}),f(\beth_{n-2}))\\
&\precsim_{i_3} \lambda S(\beth_{n-1},\beth_{n-1},\beth_{n-2})+\frac{\eta S(\beth_{n-2},\beth_{n-2},f(\beth_{n-2}))S(f(\beth_{n-1}),f(\beth_{n-1}),\beth_{n-1})}{1+S(\beth_{n-1},\beth_{n-1},\beth_{n-2})}\\
&= \lambda S(\beth_{n-1},\beth_{n-1},\beth_{n-2})+\frac{\eta S(\beth_{n-2},\beth_{n-2},\beth_{n-1})S(\beth_n,\beth_n\beth_{n-1})}{1+S(\beth_{n-1},\beth_{n-1},\beth_{n-2})}\\
||S(\beth_n,\beth_n,\beth_{n-1})|| &\leq ||\lambda S(\beth_{n-1},\beth_{n-1}\beth_{n-2})+\frac{\eta S(\beth_{n-2},,\beth_{n-2},\beth_{n-1})S(\beth_n,\beth_n,\beth_{n-1})}{1+S(\beth_{n-1},\beth_{n-1},\beth_{n-2})}||\\
&\leq \lambda||S(\beth_{n-1},\beth_{n-1},\beth_{n-2})||+2\eta||S(\beth_n,\beth_n,\beth_{n-1})||,
\end{aligned}$$

so that

$$||S(\beth_n,\beth_n,\beth_{n-1})||\leq\frac{\lambda}{1-2\eta}||S(\beth_{n-1},\beth_{n-1},\beth_{n-2})||,$$

Hence, by applying $\alpha=\frac{\lambda}{1-2\eta}$, we get

$$||S(\beth_n,\beth_n\beth_{n-1})||\leq\alpha^n||S(\beth_1,\beth_1,\beth_0)||.$$

For every $m,n\in\mathbb{N}$,

$$\begin{aligned}
S(\beth_n,\beth_n,\beth_m) &\precsim_{i_3} S(\beth_n,\beth_n,\beth_{n+1})+S(\beth_n,\beth_n,\beth_{n+1})+S(\beth_m,\beth_m,\beth_{n+1})\\
&\precsim_{i_3} 2S(\beth_n,\beth_n,\beth_{n+1})+S(\beth_{n+1},\beth_{n+1},\beth_m)\\
&\precsim_{i_3} 2(\alpha^n+...+\alpha^{m-1})S(\beth_1,\beth_1,\beth_0),\\
&\precsim_{i_3} 2(\frac{\alpha^n}{1-\alpha})S(\beth_1,\beth_1,\beth_0),\ \text{if } m>n,
\end{aligned}$$

$$||S(\beth_n,\beth_n,\beth_m)||\leq 2(\frac{\alpha^n}{1-\alpha})||S(\beth_1,\beth_1,\beth_0)||,\ \text{if } m>n.$$

By $\alpha\in(0,1)$, $||S(\beth_n,\beth_n,\beth_m)||\to 0$, as $n,m\to\infty$, we determine that $\{\beth_n\}$ is a Cauchy sequence. Since (G,S) is complete, $\{\beth_n\}$ converges to a point $\wp\in G$. By Lemma 2.6, $f(\beth_n)=\beth_{n+1}\to\wp\in G$ as $n\to\infty$ implies $S(f(\wp),f(\wp),f(\beth_n))\to S(f(\wp),f(\wp),\wp)$ as $n\to\infty$. Moreover, by taking the limit from

$$S(f(\wp),f(\wp),f(\beth_n))\precsim_{i_3}\lambda S(\beth_n,\beth_n,\wp)+\frac{\eta S(\beth_n,\beth_n,f(\beth_n))S(f(\wp),f(\wp),\wp)}{1+S(\beth_n,\beth_n,\wp)}$$

we obtain

$$||S(f(\wp),f(\wp),f(\beth_n))||\leq\lambda||S(\beth_n,\beth_n,\wp)||+\frac{\eta||S(\beth_n,\beth_n,f(\beth_n)S(f(\wp),f(\wp),\wp)||}{||1+S(\beth_n,\beth_n,\wp)||},$$

as $n\to\infty$, we get $S(f(\wp),f(\wp),\wp)=0$. Therefore $f(\wp)=\wp$. Hence $\wp$ is a fixed point of f. If, in addition, $f(\rho)=\rho$ for some another fixed point ρ of f, then

$$\begin{aligned}
S(\wp,\wp,\rho)=S(f(\wp),f(\wp),f(\rho)) &\precsim_{i_3} \lambda S(\wp,\wp,\rho)+\frac{\eta S(\wp,\wp,f(\wp))S(f(\rho),f(\rho),\rho)}{1+S(\wp,\wp,\rho)}\\
&\precsim_{i_3} \lambda S(\wp,\wp,\rho).
\end{aligned}$$

Therefore $||S(\wp,\wp,\rho)||=0$ and it is implies that $\rho=\wp$. Hence f has a UFP. □

If we choose $\eta=0$ in Theorem 3.1, then we obtain Corollary 3.2.

Corollary 3.2. Assuming that (G,S) is a complete TVSMS. If a map $f:G\to G$ satisfies $S(f(\aleph),f(\aleph),f(\beth))\precsim_{i_3}\lambda S(\beth,\beth,\aleph)$, for all $\beth,\aleph\in G$, whenever $\lambda\in[0,1)$, then the function f has a unique fixed point(or, UFP).

The last Corollary 3.2 is Theorem 3.1 in [9] for s-metric spaces.

Example 3.3. Assuming that $G=\{0,\frac{1}{2},2\}$ and $S(\beth,\aleph,\rho)=(1+i_3)(|\beth-\rho|+|\aleph-\rho|)$, where $\beth,\aleph,\rho\in G$. Then (G,S) is a complete TVSMS. Define a map $f:(G,S)\to(G,S)$ by $f(0)=0$, $f(\frac{1}{2})=0$, and $f(2)=\frac{1}{2}$. Then, f satisfies the inequality $S(f(\aleph),f(\aleph),f(\beth))\precsim_{i_3}\lambda S(\beth,\beth,\aleph)+\frac{\eta S(\beth,\beth,f(\beth))S(f(\aleph),f(\aleph),\aleph)}{1+S(\beth,\beth,\aleph)}$ for $\lambda=\frac{1}{3}$ and $\eta=\frac{1}{6}$. By Theorem 3.1, f has a UFP zero in G.

Theorem 3.4. Assuming that (G,S) is a complete TVSMS with nonsingular $1+S(\beth,\beth,\aleph)$ and $||1+S(\beth,\beth,\aleph)||\neq 0$, whenever $\beth,\aleph\in G$. If a map $f:(G,S)\to(G,S)$ satisfies $S(f(\aleph),f(\aleph),f(\beth))\precsim_{i_3}\lambda[S(\beth,\beth,f(\beth))+S(f(\aleph),f(\aleph),\aleph)]+\frac{\eta S(\beth,\beth,f(\beth))S(f(\aleph),f(\aleph),\aleph)}{1+S(\beth,\beth,\aleph)}$, for all $\beth,\aleph\in G$, whenever $\lambda,\eta\in[0,1)$ with $2\lambda+2\eta<1$, then the function $f:G\to G$ has a UFP.

Proof . Let $\beth_0\in G$, and $\beth_1=f(\beth_0)$. Suppose $\beth_{n+1}=f(\beth_n)$, whenever $n\in\mathbb{N}$. For every $n\in\mathbb{N}$, from

$$\begin{aligned}
S(\beth_n,\beth_n,\beth_{n-1}) &= S(f(\beth_{n-1}),f(\beth_{n-1}),f(\beth_{n-2}))\\
&\precsim_{i_3} \lambda[S(\beth_{n-1},\beth_{n-1},f(\beth_{n-1}))+S(\beth_{n-2},\beth_{n-2},f(\beth_{n-2}))]\\
&\quad+\frac{\eta S(\beth_{n-2},\beth_{n-2},f(\beth_{n-2}))S(f(\beth_{n-1}),f(\beth_{n-1}),\beth_{n-1})}{1+S(\beth_{n-1},\beth_{n-1},\beth_{n-2})}\\
&= \lambda[S(\beth_{n-1},\beth_{n-1},\beth_n)+S(\beth_{n-2},\beth_{n-2},\beth_{n-1})]+\frac{\eta S(\beth_{n-2},\beth_{n-2},\beth_{n-1})S(\beth_n,\beth_n\beth_{n-1})}{1+S(\beth_{n-1},\beth_{n-1},\beth_{n-2})}\\
||S(\beth_n,\beth_n,\beth_{n-1})|| &\leq ||[S(\beth_{n-1},\beth_{n-1},\beth_n)+S(\beth_{n-2},\beth_{n-2},\beth_{n-1})]\\
&\quad+\frac{\eta S(\beth_{n-2},,\beth_{n-2},\beth_{n-1})S(\beth_n,\beth_n,\beth_{n-1})}{1+S(\beth_{n-1},\beth_{n-1},\beth_{n-2})}||\\
&\leq \lambda||[S(\beth_{n-1},\beth_{n-1},\beth_n)+S(\beth_{n-2},\beth_{n-2},\beth_{n-1})]||+2\eta||S(\beth_n,\beth_n,\beth_{n-1})||,
\end{aligned}$$

we conclude that

$$||S(\beth_n,\beth_n,\beth_{n-1})||\leq\frac{\lambda}{1-\lambda-2\eta}||S(\beth_{n-1},\beth_{n-1},\beth_{n-2})||,$$

Hence, by applying $\alpha=\frac{\lambda}{1-\lambda-2\eta}$, we get

$$||S(\beth_n,\beth_n,\beth_{n-1})||\leq\alpha^n||S(\beth_{n-1},\beth_{n-1},\beth_{n-2})||$$

For every $m,n\in\mathbb{N}$,

$$\begin{aligned}
S(\beth_n,\beth_n,\beth_m) &\precsim_{i_3} S(\beth_n,\beth_n,\beth_{n+1})+S(\beth_n,\beth_n,\beth_{n+1})+S(\beth_m,\beth_m,\beth_{n+1})\\
&\precsim_{i_3} 2S(\beth_n,\beth_n,\beth_{n+1})+S(\beth_{n+1},\beth_{n+1},\beth_m)\\
&\precsim_{i_3} 2(\alpha^n+...+\alpha^{m-1})S(\beth_1,\beth_1,\beth_0),\\
&\precsim_{i_3} 2(\frac{\alpha^n}{1-\alpha})S(\beth_1,\beth_1,\beth_0),\ \text{if } m>n,
\end{aligned}$$

$$||S(\beth_n,\beth_n,\beth_m)||\leq 2(\frac{\alpha^n}{1-\alpha})||S(\beth_1,\beth_1,\beth_0)||,\ \text{if } m>n,$$

By $\alpha\in(0,1)$, $||S(\beth_n,\beth_n,\beth_m)||\to 0$, as $n,m\to\infty$, we determine that $\{\beth_n\}$ is a Cauchy sequence. Since (G,S) is complete, $\{\beth_n\}$ converges to a point $\wp\in G$. By Lemma 2.6, $f(\beth_n)=\beth_{n+1}\to\wp\in G$ as $n\to\infty$ implies $S(f(\wp),f(\wp),f(\beth_n))\to S(f(\wp),f(\wp),\wp)$ as $n\to\infty$. Moreover, by taking the limit from

$$S(f(\wp),f(\wp),f(\beth_n))\quad\precsim_{i_3}\quad\lambda[S(\beth_n,\beth_n,f(\beth_n))+S(f(\wp),f(\wp),\wp)]+\frac{\eta S(\beth_n,\beth_n,f(\beth_n))S(f(\wp),f(\wp),\wp)}{1+S(\beth_n,\beth_n,\wp)}$$

we obtain

$$||S(f(\wp), f(\wp), f(\beth_n))|| \leq \lambda[||S(\beth_n, \beth_n, f(\beth_n)) + S(\wp, \wp, f(\wp))||] + \frac{\eta||S(\beth_n, \beth_n, f(\beth_n)S(f(\wp), f(\wp), \wp)||}{||1 + S(\beth_n, \beth_n, \wp)||},$$

as $n \to \infty$, we get $S(f(\wp), f(\wp), \wp) = 0$. Therefore $f(\wp) = \wp$. Hence $\wp$ is a fixed point of f. If, in addition, $f(\rho) = \rho$ for some another fixed point ρ of f, then

$$S(\wp, \wp, \rho) = S(f(\wp), f(\wp), f(\rho)) \precsim_{i_3} \lambda[S(\wp, \wp, f(\wp)) + S(f(\rho), f(\rho), \rho)] + \frac{\eta S(\wp, \wp, f(\wp))S(f(\rho), f(\rho), \rho)}{1 + S(\wp, \wp, \rho)}$$

Therefore $||S(\wp, \wp, \rho)|| = 0$ and it is implies that $\rho = \wp$. Hence f has a UFP. □

If we choose $\eta = 0$ in Theorem 3.4, then we obtain Corollary 3.5.

Corollary 3.5. Assuming that (G, S) is a complete TVSMS. If a map $f : (G, S) \to (G, S)$ satisfies $S(f(\aleph), f(\aleph), f(\beth)) \precsim_{i_3} \lambda[S(\beth, \beth, f(\beth)) + S(f(\aleph), f(\aleph), \aleph)]$, for all $\beth, \aleph \in G$, whenever $\lambda \in [0, \frac{1}{2})$, then the function f has a UFP.

The last Corollary 3.5 is Corollary 2.8 in [8] for s-metric spaces.

Theorem 3.6. Assuming that (G, S) is a complete TVSMS with nonsingular $1 + S(\beth, \beth, f(\beth)) + S(f(\aleph), f(\aleph), \aleph)$ and $||1 + S(\beth, \beth, f(\beth)) + S(f(\aleph), f(\aleph), \aleph)|| \neq 0$, whenever $\beth, \aleph \in G$. If a map $f : G \to G$ satisfies $S(f(\aleph), f(\aleph), f(\beth)) \precsim_{i_3} \lambda[S(\beth, \beth, \aleph) + S(\beth, \beth, f(\beth)) + S(f(\aleph), f(\aleph), \aleph)] + \frac{\eta S(\beth, \beth, f(\beth))S(f(\aleph), f(\aleph), \aleph)}{1 + S(\beth, \beth, f(\beth)) + S(f(\aleph), f(\aleph), \aleph)}$, for all $\beth, \aleph \in G$, whenever $\lambda, \eta \in [0, 1)$ with $3\lambda + 2\eta < 1$, then the function f has a UFP.

Proof . Let $\beth_0 \in G$, and $\beth_1 = f(\beth_0)$. Suppose $\beth_{n+1} = f(\beth_n)$, whenever $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, from

$$\begin{aligned}
S(\beth_n, \beth_n, \beth_{n-1}) &= S(f(\beth_{n-1}), f(\beth_{n-1}), f(\beth_{n-2})) \\
&\precsim_{i_3} \lambda[S(\beth_{n-1}, \beth_{n-1}, \beth_{n-2}) + S(\beth_{n-1}, \beth_{n-1}, f(\beth_{n-1})) + S(\beth_{n-2}, \beth_{n-2}, f(\beth_{n-2}))] \\
&\quad + \frac{\eta S(\beth_{n-2}, \beth_{n-2}, f(\beth_{n-2}))S(f(\beth_{n-1}), f(\beth_{n-1}), \beth_{n-1})}{1 + S(\beth_{n-2}, \beth_{n-2}, f(\beth_{n-2})) + S(f(\beth_{n-1}), f(\beth_{n-1}), \beth_{n-1})} \\
&= \lambda[S(\beth_{n-1}, \beth_{n-1}, \beth_{n-2}) + S(\beth_{n-1}, \beth_{n-1}, \beth_n) + S(\beth_{n-2}, \beth_{n-2}, \beth_{n-1})] \\
&\quad + \frac{\eta S(\beth_{n-2}, \beth_{n-2}, \beth_{n-1})S(\beth_n, \beth_n \beth_{n-1})}{1 + S(\beth_{n-2}, \beth_{n-2}, \beth_{n-1}) + S(\beth_n, \beth_n, \beth_{n-1})} \\
||S(\beth_n, \beth_n, \beth_{n-1})|| &\leq ||\lambda[S(\beth_{n-1}, \beth_{n-1}, \beth_{n-2}) + S(\beth_{n-1}, \beth_{n-1}, \beth_n) + S(\beth_{n-2}, \beth_{n-2}, \beth_{n-1})] \\
&\quad + \frac{\eta S(\beth_{n-2}, , \beth_{n-2}, \beth_{n-1})S(\beth_n, \beth_n, \beth_{n-1})}{1 + S(\beth_{n-2}, \beth_{n-2}, \beth_{n-1}) + S(\beth_n, \beth_n, \beth_{n-1})}|| \\
&\leq \lambda||[S(\beth_{n-1}, \beth_{n-1}, \beth_{n-2}) + S(\beth_{n-1}, \beth_{n-1}, \beth_n) + S(\beth_{n-2}, \beth_{n-2}, \beth_{n-1})]|| \\
&\quad + 2\eta||S(\beth_n, \beth_n, \beth_{n-1})||,
\end{aligned}$$

we conclude that

$$||S(\beth_n, \beth_n, \beth_{n-1})|| \leq \frac{2\lambda}{1 - \lambda - 2\eta}||S(\beth_{n-1}, \beth_{n-1}, \beth_{n-2})||,$$

Hence, by applying $\alpha = \frac{\lambda}{1-\lambda-2\eta}$, we get

$$||S(\beth_n, \beth_n, \beth_{n-1})|| \leq \alpha^n||S(\beth_{n-1}, \beth_{n-1}, \beth_{n-2})||$$

For every $m, n \in \mathbb{N}$,

$$\begin{aligned}
S(\beth_n, \beth_n, \beth_m) &\precsim_{i_3} S(\beth_n, \beth_n, \beth_{n+1}) + S(\beth_n, \beth_n, \beth_{n+1}) + S(\beth_m, \beth_m, \beth_{n+1}) \\
&\precsim_{i_3} 2S(\beth_n, \beth_n, \beth_{n+1}) + S(\beth_{n+1}, \beth_{n+1}, \beth_m) \\
&\precsim_{i_3} 2(\alpha^n + ... + \alpha^{m-1})S(\beth_1, \beth_1, \beth_0), \\
&\precsim_{i_3} 2(\frac{\alpha^n}{1 - \alpha})S(\beth_1, \beth_1, \beth_0), \text{ if } m > n,
\end{aligned}$$

$$||S(\beth_n,\beth_n,\beth_m)|| \leq 2(\frac{\alpha^n}{1-\alpha})||S(\beth_1,\beth_1,\beth_0)||, \text{ if } m>n.$$

By $\alpha \in (0,1)$, $||S(\beth_n,\beth_n,\beth_m)|| \to 0$, as $n,m \to \infty$, we determine that $\{\beth_n\}$ is a Cauchy sequence. Since (G,S) is complete, $\{\beth_n\}$ converges to a point $\wp \in G$. By Lemma 2.6, $f(\beth_n) = \beth_{n+1} \to \wp \in G$ as $n \to \infty$ implies $S(f(\wp),f(\wp),f(\beth_n)) \to S(f(\wp),f(\wp),\wp)$ as $n \to \infty$. Moreover, by taking the limit from

$$S(f(\wp),f(\wp),f(\beth_n)) \precsim_{i_3} \lambda[S(\beth_n,\beth_n,\wp)+S(\beth_n,\beth_n,f(\beth_n))+S(f(\wp),f(\wp),\wp)]+\frac{\eta S(\beth_n,\beth_n,f(\beth_n))S(f(\wp),f(\wp),\wp)}{1+S(\beth_n,\beth_n,f(\beth_n))+S(f(\wp),f(\wp),\wp)}$$

we obtain

$$\begin{aligned}||S(f(\wp),f(\wp),f(\beth_n))|| &\leq \lambda[||S(\beth_n,\beth_n,\wp)+S(\beth_n,\beth_n,f(\beth_n))+S(\wp,\wp,f(\wp))||] \\ &+\frac{\eta||S(\beth_n,\beth_n,f(\beth_n)S(f(\wp),f(\wp),\wp)||}{||1+S(\beth_n,\beth_n,f(\beth_n))+S(f(\wp),f(\wp),\wp)||},\end{aligned}$$

as $n \to \infty$, we get $S(f(\wp),f(\wp),\wp)=0$. Therefore $f(\wp)=\wp$. Hence $\wp$ is a fixed point of f.
If, in addition, $f(\rho)=\rho$ for some another fixed point ρ of f, then

$$S(\wp,\wp,\rho)=S(f(\wp),f(\wp),f(\rho)) \precsim_{i_3} \lambda[S(\wp,\wp,\rho)+S(\wp,\wp,f(\wp))+S(f(\rho),f(\rho),\rho)]+\frac{\eta S(\wp,\wp,f(\wp))S(f(\rho),f(\rho),\rho)}{1+S(\wp,\wp,f(\wp))+S(f(\rho),f(\rho),\rho)}$$

Therefore $||S(\wp,\wp,\rho)||=0$ and it is implies that $\rho=\wp$. Hence f has a UFP. □

4 Applications

Using main theorem 3.1, we prove an existence theorem for the unique solution of the linear system of equations.

Theorem 4.1. Assuming that $G=\mathbb{C}^n$ is TVSMS with the metric $S(\beth,\aleph,\rho)=\sum_{j=1}^{n}(1+i_3)(|\beth-\rho|+|\aleph-\rho|)$, whenever $\beth,\aleph,\rho \in G$. If $\sum_{j=1}^{n}|\lambda_{ij}| \precsim_{i_2} \lambda<1$, whenever $i=1,2,...,n$, then the linear system

$$\begin{cases} b_1 = a_{11}\beth_1 + a_{12}\beth_2 + ... + a_{1n}\beth_n \\ b_2 = a_{21}\beth_1 + a_{22}\beth_2 + ... + a_{2n}\beth_n \\ \vdots \\ b_n = a_{n1}\beth_1 + a_{n2}\beth_2 + ... + a_{nn}\beth_n \end{cases}$$

of n linear equations in n unknown has a unique solution.

Proof . Define $f: G \to G$ by $f(\beth)=A\beth+b$, whenever $\beth=(\beth_1,\beth_2,...,\beth_n)\in\mathbb{C}^n, b=(b_1,b_2,...,b_n)\in\mathbb{C}^n$, and

$$A=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

Now,

$$\begin{aligned} S(f\beth,f\beth,f\aleph,) &= \sum_{j=1}^{n}(1+i_3)(|\lambda_{ij}(\beth-\aleph)|+|\lambda_{ij}(\beth-\aleph)|) \\ &\precsim_{i_3} \sum_{j=1}^{n}|\lambda_{ij}|\left[\sum_{j=1}^{n}(1+i_3)(|\beth-\aleph|+|\beth-\aleph|)\right] \\ &\precsim_{i_3} \lambda\sum_{j=1}^{n}(1+i_3)(|\beth-\aleph|+|\beth-\aleph|) \\ &= \lambda S(\beth,\beth,\aleph) \\ &= \lambda S(\beth,\beth,\aleph)+\frac{\eta S(\beth,\beth,f(\beth))S(f(\aleph),f(\aleph),\aleph)}{1+S(\beth,\beth,\aleph)}. \end{aligned}$$

All of Theorem 3.1's requirements are then fulfilled with $\lambda = \frac{1}{6}, \eta = 0$ and $\lambda + \sqrt{2}\eta < 1$. As a result, there is only one solution to the linear system of equations. □

5 Conclusions

All fixed point theorems in tricomplex valued S-metric spaces can be regarded as generalizations of fixed point theorems in tricomplex valued metric spaces, complex valued metric spaces, and S-metric spaces. Therefore, studies of fixed point results in tricomplex valued S-metric spaces are significant.

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