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Hyers-Ulam stability of a quadratic-additive functional equation in non-Archimedean fuzzy φ-2-normed spaces

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Abstract

In this work, we introduce the following quadratic-additive functional equation

$$\psi\left(\sum_{a=1}^{n} v_a\right) + \sum_{a=1}^{n} \psi\left(-v_a + \sum_{b=1;a\neq b}^{n} v_b\right)$$
$$= (n-3)\sum_{1\leq a<b\leq n} \psi\left(v_a+v_b\right) - \left(n^2-5n+2\right)\sum_{a=1}^{n}\left[\frac{\psi(v_a)+\phi(-v_a)}{2}\right] - \left(n^2-5n+4\right)\sum_{a=1}^{n}\left[\frac{\psi(v_a)-\phi(-v_a)}{2}\right]$$

where n is a nonnegative integer in $\mathbb{N}-\{0,1,2\}$, and we prove the Hyers-Ulam stability of the quadratic-additive functional equation in non-Archimedean fuzzy φ-2-normed space by utilizing two different techniques.

Keywords: Hyers-Ulam stability; non-Archimedean φ-2-normed space; quadratic-additive functional equation; fixed point method
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1 Introduction

Ulam [25] raised the question: When is it true that the solution of an equation contrasting marginally from a given one, should of need be near the given equation solution? At the starting, the partial answer (on account of Cauchy's condition in Banach spaces) to Ulam's inquiry was given by Hyers [11]. The paper of Rassias [23] has essentially affected the advancement of what we currently call the Hyers-Ulam-Rassias stability. From that point forward a numerous stability issues for different functional equations have been explored in [1, 2, 5, 6, 9, 12, 13, 14, 17, 26]. As of late, the stability issues for different kind of functional equations were examined in [16, 19]. Separately, while the possibility of intuitionistic fuzzy normed space was presented in [22] and further concentrated in [18, 20] to manage some summability issues.

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The functional equations $f(a+b)=f(a)+f(b)$ and $f(a+b)+f(a-b)=2f(a)+2f(b)$ is called the additive and quadratic functional equations, respectively. Especially, every solution of the additive and quadratic functional equations are said to be an additive mapping and a quadratic mapping, respectively.

In 1897, Hensel [10] presented a normed space that doesn't have the Archimedean property. During the most recent thirty years, the hypothesis of non-Archimedean spaces has acquired the interest of physicists for their research specifically in issues coming from quantum physical science, p-adic strings and superstrings [15]. Albeit numerous outcomes in the traditional normed space hypothesis has a non-Archimedean counterpart, their confirmations are basically unique and require a completely new sort of instinct [7, 8, 21, 22, 27].

In this present work, we introduce the following quadratic-additive functional equation

$$\begin{aligned}\psi\left(\sum_{a=1}^{n} v_a\right)+\sum_{a=1}^{n}\psi\left(-v_a+\sum_{b=1;a\neq b}^{n} v_b\right) &= (n-3)\sum_{1\le a<b\le n}\psi\left(v_a+v_b\right)-\left(n^2-5n+2\right)\sum_{a=1}^{n}\left[\frac{\psi(v_a)+\phi(-v_a)}{2}\right]\\ &\quad -\left(n^2-5n+4\right)\sum_{a=1}^{n}\left[\frac{\psi(v_a)-\phi(-v_a)}{2}\right], \qquad (1.1)\end{aligned}$$

where n is a nonnegative integer in $\mathbb{N}-\{0,1,2\}$. We prove the Hyers-Ulam stability of the quadratic-additive functional equation in non-Archimedean fuzzy φ-2-normed space by using two different techniques. It is easy to see that the mappings $\psi(v)=av^2+bv$ is a solution of the functional equation (1.1).

2 Preliminaries

In this section, we recall some usual definitions, terminology and notions to achieve our main results.

Definition 2.1. [10] By a non-Archimedean field, we denote a field $\mathbb{K}$ equipped with a valuation $|\cdot|:\mathbb{K}\to[0,\infty)$ fulfills $|p|=0\Leftrightarrow p=0$, $|pq|=|p||q|$, and $|p+q|\le\max\{|p|,|q|\}$ for every $p,q\in\mathbb{K}$. Clearly, $|1|=|-1|=1$ and $|m|\le 1$ for every $m\in\mathbb{N}$.

Let V be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean nontrivial valuation $|\cdot|$. A mapping $\|\cdot\|:V\to\mathbb{R}$ is called as a non-Archimedean norm (valuation) if it holds the upcoming conditions:

(i) $\|v\|=0\Leftrightarrow v=0$;
(ii) $\|pv\|=|p|\|v\|$, $\quad(v\in V, p\in\mathbb{K})$;
(iii) $\|v_1+v_2\|\le\max\{\|v_1\|,\|v_2\|\}$ $\quad(v_1,v_2\in V)$ (called as ultrametric).

The pair $(V,\|\cdot\|)$ is known as a non-Archimedean normed space.

We know that

$$\|v_n-v_m\|\le\max\{\|v_{j+1}-v_j\|; m\le j\le n-1\}\quad(n\ge m).$$

A sequence $\{v_n\}$ is called Cauchy if $\{v_{n+1}-v_n\}\to 0$ in a non-Archimedean normed space V. By a complete non-Archimedean normed space, we mention one in which every Cauchy sequence is convergent.

Definition 2.2. [24] A t-norm $\Diamond$ is a function $\Diamond:[0,1]\times[0,1]\to[0,1]$ which is commutative, associative, non decreasing and satisfies $\lambda\Diamond 1=\lambda$ for every $\lambda\in[0,1]$.

Definition 2.3. [7, 8, 16, 18, 19, 20] Let V be a real linear space with dimension greater than 1 and $F:V^2\times[0,\infty)\to[0,1]$ fulfilling the upcoming conditions: For every $v_1,v_2,v_3\in V$ and $p,q\in[0,\infty)$,

(NAF1) $F(v_1,v_2,0)=0$;

(NAF2) $F(v_1,v_2,p)=1$, for all $p>0$ iff v_1,v_2 are linear dependent;

(NAF3) $F(v_1,v_2,p)=N(v_2,v_1,p)$ for every $v_1,v_2\in V$, and $p>0$;

(NAF4) $F(v_1+v_2,v_3,\max(p,q))\ge\min(F(v_1,v_3,p)\Diamond F(v_2,v_3,q))$;

(NAF5) $F(v_1,v_2,\cdot):[0,\infty)\to[0,1]$ is left continuous.

(NAF6) $F(\alpha v_1,v_2,p)=F\left(v_1,v_2,\frac{p}{\varphi(\alpha)}\right)$, $\quad\alpha\in\mathbb{R}$.

The triple $(V,F,\Diamond)$ will be called a non-Archimedean fuzzy φ-2-normed space.

Definition 2.4. [7, 8, 16, 18, 19, 20] Let $(V, F, \diamondsuit)$ be a non-Archimedean fuzzy φ-2-normed space and $\{v_n\}$ in V. Then $\{v_n\}$ is called convergent if there is $v \in V$ satisfying

$$\lim_{n\to\infty} F(v_n - v, w, \epsilon) = 1, \quad w \in V,\ \epsilon > 0.$$

In this case, v is the limit of v_n. We denote it by

$$F - \lim_{n\to\infty} v_n = v.$$

Definition 2.5. A sequence $\{x_n\} \in V$ is said to be Cauchy if for a given $\delta > 0$, there is an $N \in \mathbb{N}$ such that $F(v_{n+q} - v_n, w, \epsilon) < 1 - \delta$ for all $w \in V$, $q > 0$, $\varepsilon > 0$ and $n > N$.

Every convergent sequence in a non-Archimedean fuzzy φ-2-normed space is a Cauchy sequence.

Definition 2.6. A non-Archimedean fuzzy φ-2-normed space is said to be a non-Archimedean fuzzy φ-2-Banach space if every Cauchy sequence is convergent.

Theorem 2.7. [3, 4] If (V, d) is a complete generalized metric space and a mapping $\Phi : V \to V$ is strictly contractive with L (Lipschitz constant). Then, for each given element $v \in V$, either

(B1) $d(\Phi^n v, \Phi^{n+1} v) = \infty$ for every $v \geq 0$,

or

(B2) there is $n_0 \in \mathbb{N}$ satisfies
  (i) $d(\Phi^n v, \Phi^{n+1} v) < \infty$ for every $n \geq n_0$;
  (ii) the sequence $\{\Phi^n v\}$ is convergent to a fixed point w^* of Φ;
  (iii) w^* is the unique fixed point of Φ in the set $\Delta = \{w \in V : d(\Phi^{n_0} v, w) < \infty\}$;
  (iv) $d(w^*, w) \leq \frac{1}{1-L} d(w, \Phi w)$ for every $w \in \Delta$.

3 Solution of the functional equation (1.1)

In this section, let V and W are two real vector spaces.

Theorem 3.1. If an odd mapping $\psi : V \to W$ fulfills the functional equation (1.1), then ψ is additive.

Proof . Since $\psi(-v) = -\psi(v), \quad v \in V$, the functional equation (1.1) reduces as

$$\psi\left(\sum_{a=1}^{n} v_a\right) + \sum_{a=1}^{n} \psi\left(-v_a + \sum_{b=1; a\neq b}^{n} v_b\right) = (n-3) \sum_{1\leq a<b\leq n} \psi(v_a + v_b) - \left(n^2 - 5n + 4\right) \sum_{a=1}^{n} \psi(v_a) \tag{3.1}$$

for all $v_1, v_2, \cdots, v_n \in V$. Now, replacing $(v_1, v_2, \cdots, v_n)$ by $(0, 0 \cdots, 0)$ in (3.1), we get $\psi(0) = 0$. Letting $(v_1, v_2, \cdots, v_n) = (v, v, 0, \cdots, 0)$ in (3.1), we get

$$\psi(2v) = 2\psi(v), \quad v \in V.$$

So for a nonnegative integer n, we get

$$\psi(2^n v) = 2^n \psi(v), \quad v \in V.$$

Finally, replacing $(v_1, v_2, \cdots, v_n)$ by $(v_1, v_2, 0, \cdots, 0)$ in (3.1), we have

$$\psi(v_1 + v_2) = \psi(v_1) + \psi(v_2), \quad v_1, v_2 \in V.$$

Hence the mapping ψ is additive. □

Theorem 3.2. If an even mapping $\psi : V \to W$ fulfills the functional equation (1.1), then ψ is quadratic.

Proof . Since $\psi(-v) = \psi(v), \quad v \in V$, the functional equation (1.1) reduces as

$$\psi\left(\sum_{a=1}^{n} v_a\right) + \sum_{a=1}^{n} \psi\left(-v_a + \sum_{b=1;a\neq b}^{n} v_b\right) = (n-3) \sum_{1\leq a<b\leq n} \psi\left(v_a + v_b\right) - \left(n^2 - 5n + 2\right) \sum_{a=1}^{n} \psi(v_a) \tag{3.2}$$

for all $v_1, v_2, \cdots, v_n \in V$. Now, letting $(v_1, v_2, \cdots, v_n) = (0, 0, \cdots, 0)$ in (3.2), we obtain $\psi(0) = 0$. Letting $(v_1, v_2, \cdots, v_n) = (v, v, 0, \cdots, 0)$ in (3.2), we obtain

$$\psi(2v) = 2^2\psi(v), \quad v \in V.$$

So for a nonnegative integer n, we have

$$\psi(2^n v) = 2^{2n}\psi(v), \quad v \in V.$$

Finally, replacing $(v_1, v_2, \cdots, v_n)$ by $(v_1, v_2, 0, \cdots, 0)$ in (3.2), we obtain

$$\psi(v_1 + v_2) + \psi(v_1 - v_2) = 2\psi(v_1) + 2\psi(v_2), \quad v_1, v_2 \in V.$$

Hence the mapping ψ is quadratic. $\square$

Theorem 3.3. A mapping $\psi : V \to W$ fulfills $\psi(0) = 0$ and (1.1) for all $v_1, v_2, \cdots, v_n \in V$ if and only if there exist a mapping $Q : V \times V \to W$, which is symmetric bi-additive, and a mapping $A : V \to W$, which is additive, such that $\psi(v) = Q(v, v) + A(v)$ for all $v \in V$.

Proof . Let ψ with $\psi(0) = 0$ fulfills the functional equation (1.1). We divide ψ into the odd part and even part as

$$\psi_o(v) = \frac{\psi(v) - \psi(-v)}{2}, \quad \psi_e(v) = \frac{\psi(v) + \psi(-v)}{2}, \quad v \in V,$$

respectively. Clearly, $\psi(v) = \psi_e(v) + \psi_o(v), \quad v \in V$. It is easy to prove that ψ_o and ψ_e fulfill the functional equation (1.1). Hence by Theorems 3.1 and 3.2, we conclude that ψ_o and ψ_e are additive and quadratic, respectively. So there exist a symmetric bi-additive mapping $Q : V \times V \to W$ which satisfies $\psi_e(v) = Q(v, v)$ and an additive mapping $A : V \to W$ which satisfies $\psi_o(v) = A(v), \quad v \in V$. Hence $\psi(v) = Q(v, v) + A(v)$ for all $v \in V$.

Conversely, suppose that there exist a mapping $Q : V \times V \to W$ which is symmetric bi-additive and a mapping $A : V \to W$ which is additive such that $\psi(v) = Q(v, v) + A(v)$ for all $v \in V$. Easily, we can show that the mappings $v \mapsto Q(v, v)$ and $A : V \to W$ fulfill the functional equation (1.1). Thus the mapping $\psi : V \to W$ fulfills the functionaql equation (1.1). $\square$

For notational accessibility, we define a mapping $\psi : V \to W$ by

$$\begin{aligned} D\psi(v_1, v_2, \cdots, v_n) &= \psi\left(\sum_{a=1}^{n} v_a\right) + \sum_{a=1}^{n} \psi\left(-v_a + \sum_{b=1;a\neq b}^{n} v_b\right) - (n-3) \sum_{1\leq a<b\leq n} \psi\left(v_a + v_b\right) \\ &\quad + \left(n^2 - 5n + 2\right) \sum_{a=1}^{n} \left[\frac{\psi(v_a) + \phi(-v_a)}{2}\right] + \left(n^2 - 5n + 4\right) \sum_{a=1}^{n} \left[\frac{\psi(v_a) - \phi(-v_a)}{2}\right] \end{aligned}$$

for all $v_1, v_2, \cdots, v_n \in V$.

4 Hyers-Ulam stability in non-Archimedean fuzzy φ-2-normed spaces

In the upcoming subsections, assume that V , $(W, F, \diamondsuit)$ and $(Z, F', \diamondsuit)$ are a lear vector space, real non-Archimedean fuzzy φ-2-Banach space and real non-Archimedean fuzzy φ-2-normed space, respectively.

4.1 Stability results for the even case: Direct method

In this subsection, we investigate the Hyers-Ulam stability of (1.1) for the even case by utilizing direct method.

Theorem 4.1. Let $j \in \{-1, 1\}$ be fixed and $\chi : V^n \to Z$ be a mapping such that for some α with $0 < \left(\frac{\varphi(\alpha)}{\varphi(2^2)}\right)^j < 1$,

$$F^{'}\left(\chi\left(2^j v, 2^j v, 0, \cdots, 0\right), w, \epsilon\right) \geq F^{'}\left([\varphi(\alpha)]^j \chi\left(v, v, 0, \cdots, 0\right), w, \epsilon\right) \tag{4.1}$$

for all $v, w \in V$ and all $\epsilon > 0$, and

$$\lim_{s\to\infty} F^{'}\left(\chi\left(2^{js}v_1, 2^{js}v_2, \cdots, 2^{js}v_n\right), w, \left[\varphi\left(2^{2s}\right)\right]^j \epsilon\right) = 1 \tag{4.2}$$

for all $v_1, v_2, \cdots, v_n, w \in V$ and all $\epsilon > 0$. Suppose that an even mapping $\psi : V \to W$ fulfills

$$F\left(D\psi(v_1, v_2, \cdots, v_n), w, \epsilon\right) \geq F^{'}\left(\chi(v_1, v_2, \cdots, v_n), w, \epsilon\right), \tag{4.3}$$

for all $v_1, v_2, \cdots, v_n, w \in V$ and all $\epsilon > 0$. Then the limit

$$Q_2(v) = F - \lim_{s\to\infty} \frac{\psi(2^{js}v)}{2^{2js}} \tag{4.4}$$

exists for each $v \in V$ and $Q_2 : V \to W$ is a unique quadratic mapping fulfilling (1.1) and

$$F\left(\psi(v) - Q_2(v), w, \epsilon\right) \geq F^{'}\left(\chi(v, v, 0, \cdots, 0), w, 2\epsilon\left|\varphi(2^2) - \varphi(\alpha)\right|\right) \tag{4.5}$$

for all $v, w \in V$ and all $\epsilon > 0$.

Proof . Consider $j = 1$. Replacing $(v_1, v_2, \cdots, v_n)$ by $(v, v, 0, \cdots, 0)$ in (4.3), we get

$$F\left(2\psi(2v) - 8\psi(v), w, \epsilon\right) \geq F^{'}\left(\chi(v, v, 0, \cdots, 0), w, \epsilon\right) \tag{4.6}$$

for all $v, w \in V$ and all $\epsilon > 0$. From (4.6), we obtain

$$F\left(\frac{\psi(2v)}{2^2} - \psi(v), w, \frac{\epsilon}{2\varphi(2^2)}\right) \geq F^{'}\left(\chi(v, v, 0, \cdots, 0), w, \epsilon\right) \tag{4.7}$$

for all $v, w \in V$ and all $\epsilon > 0$. Replacing v by $2^s v$ in(4.7), we get

$$F\left(\frac{\psi(2^{s+1}v)}{2^{2(s+1)}} - \frac{\psi(2^s v)}{2^{2s}}, w, \frac{\epsilon}{2\varphi\left(2^{2(s+1)}\right)}\right) \quad \geq \quad F^{'}\left(\chi(2^s v, 2^s v, 0, \cdots, 0), w, \epsilon\right) \tag{4.8}$$

for all $v, w \in V$ and all $\epsilon > 0$. Utilizing (4.1), (NAF6) in (4.8), we get

$$F\left(\frac{\psi(2^{s+1}v)}{2^{2(s+1)}} - \frac{\psi(2^s v)}{2^{2s}}, w, \frac{\epsilon}{2\varphi(2^{2(s+1)})}\right) \quad \geq \quad F^{'}\left(\chi(v, v, 0, \cdots, 0), w, \frac{\epsilon}{\varphi(\alpha^s)}\right) \tag{4.9}$$

for all $v, w \in V$ and all $\epsilon > 0$. Replacing ϵ by $\varphi(\alpha^s)\epsilon$ in (4.9), we obtain

$$F\left(\frac{\psi(2^{s+1}v)}{2^{2(s+1)}} - \frac{\psi(2^s v)}{2^{2s}}, w, \frac{\varphi(\alpha^s)\epsilon}{2\varphi(2^{2(s+1)})}\right) \geq F^{'}\left(\chi(v, v, 0, \cdots, 0), w, \epsilon\right) \tag{4.10}$$

for all $v, w \in V$ and all $\epsilon > 0$. Since

$$\frac{\psi(2^s v)}{2^{2s}} - \psi(v) = \sum_{i=0}^{s-1} \frac{\psi(2^{i+1}v)}{2^{2(i+1)}} - \frac{\psi(2^i v)}{2^{2i}} \tag{4.11}$$

for all $v \in V$, by (4.10) and (4.11), we have

$$F\left(\frac{\psi(2^s v)}{2^{2s}} - \psi(v), w, \sum_{i=0}^{s-1} \frac{\varphi(\alpha^i)\epsilon}{2\varphi(2^{2(i+1)})}\right) \geq \min \cup_{i=0}^{s-1} \Big\{ F\left(\frac{\psi(2^{i+1}v)}{2^{2(i+1)}} - \frac{\psi(2^i v)}{2^{2i}}, w, \frac{\varphi(\alpha^i)\epsilon}{2\varphi(2^{2(i+1)})}\right) \Big\} \geq F^{'}\left(\chi(v, v, 0, \cdots, 0), w, \epsilon\right) \tag{4.12}$$

for all $v, w \in V$ and all $\epsilon > 0$. Replacing v by $2^k v$ in (4.12) and utilizing (4.1) and (NAF6), we obtain

$$F\left(\frac{\psi(2^{s+k}v)}{2^{2(s+k)}} - \frac{\psi(2^k v)}{2^{2(k)}}, w, \sum_{i=k}^{s+k-1} \frac{\varphi(\alpha^i)\epsilon}{2\varphi(2^{2(i+1)})}\right) \geq F^{'}\left(\chi(v, v, 0, \cdots, 0), w, \epsilon\right) \tag{4.13}$$

for all $v, w \in V$, $\epsilon > 0$ and all $s, k \geq 0$. Replacing ϵ by $\frac{\epsilon}{\sum_{i=k}^{s+k-1} \frac{\varphi(\alpha^i)}{2\varphi(2^{2(i+1)})}}$ in (4.13), we get

$$F\left(\frac{\psi(2^{s+k}v)}{2^{2(s+k)}} - \frac{\psi(2^k v)}{2^{2(k)}}, w, \epsilon\right) \geq F^{'}\left(\chi(v, v, 0, \cdots, 0), w, \frac{\epsilon}{\sum_{i=k}^{s+k-1} \frac{\varphi(\alpha^i)}{2\varphi(2^{2(i+1)})}}\right) \tag{4.14}$$

for all $v, w \in V$, $\epsilon > 0$ and all $s, k \geq 0$. Since $0 < \varphi(\alpha) < \varphi(2^2)$ and $\sum_{i=0}^{\infty} \left(\frac{\varphi(\alpha)}{\varphi(2^2)}\right) < \infty$, the Cauchy criterion for convergence implies that $\left\{\frac{\psi(2^s v)}{2^{2s}}\right\}$ is a Cauchy sequence in $(W, F^{'}, \diamondsuit)$. Since $(W, F^{'}, \diamondsuit)$ is a non-Archimedean fuzzy φ-2-Banach space, this sequence converges to some point $Q_2(v) \in W$. So we can define $Q_2 : V \to W$ by

$$Q_2(v) = F - \lim_{s\to\infty} \frac{\psi(2^s v)}{2^{2s}}$$

for all $v \in V$. Letting $k = 0$ in (4.14), we get

$$F\left(\frac{\psi(2^s v)}{2^{2s}} - \psi(v), w, \epsilon\right) \geq F^{'}\left(\chi(v, v, 0, \cdots, 0), w, \frac{\epsilon}{\sum_{i=0}^{s-1} \frac{\varphi(\alpha^i)}{2\varphi(2^{2(i+1)})}}\right) \tag{4.15}$$

for all $v, w \in V$ and all $\epsilon > 0$. Taking $s \to \infty$ in (4.15) and utilizing (NAF5), we get

$$F\left(\psi(v) - Q_2(v), w, \epsilon\right) \geq F^{'}\left(\chi(v, v, 0, \cdots, 0), w, 2\epsilon\left(\varphi(2^2) - \varphi(\alpha)\right)\right) \tag{4.16}$$

for all $v, w \in V$ and all $\epsilon > 0$. To prove that Q_2 satisfies (1.1), replacing $(v_1, v_2, \cdots, v_n)$ by $(2^s v_1, 2^s v_2, \cdots, 2^s v_n)$ in (4.3), we obtain

$$F\left(\frac{1}{2^{2s}} D\psi(2^s v_1, 2^s v_2, \cdots, 2^s v_n), w, \epsilon\right) \geq F^{'}\left(\chi(2^s v_1, 2^s v_2, \cdots, 2^s v_n), w, \varphi(2^{2s})\epsilon\right) \tag{4.17}$$

for all $v_1, v_2, \cdots, v_n, w \in V$ and all $\epsilon > 0$. Now,

$$F\Big(Q_2\left(\sum_{a=1}^{n} v_a\right) + \sum_{a=1}^{n} Q_2\left(-v_a + \sum_{b=1; a\neq b}^{n} v_b\right) - (n-3) \sum_{1\leq a<b\leq n} Q_2\left(v_a + v_b\right) + \left(n^2 - 5n + 2\right) \sum_{a=1}^{n} Q_2(v_a), w, \epsilon\Big)$$

$$\geq \min\Big\{ F\Big(Q_2\left(\sum_{a=1}^{n} v_a\right) - \frac{1}{2^2}\psi\left(\sum_{a=1}^{n} 2v_a\right), w, \frac{\epsilon}{5}\Big), F\Big(\sum_{a=1}^{n} Q_2\left(-v_a + \sum_{b=1; a\neq b}^{n} v_b\right)$$

$$-\frac{1}{2^2}\sum_{a=1}^{n} \psi\left(2\left(-v_a + \sum_{b=1; a\neq b}^{n} v_b\right)\right), w, \frac{\epsilon}{5}\Big), F\Big(-(n-3) \sum_{1\leq a<b\leq n} Q_2\left(v_a + v_b\right)$$

$$
\begin{aligned}
&+\frac{1}{2^2}(n-3)\sum_{1\le a<b\le n}\psi\left(2\left(v_a+v_b\right)\right),w,\frac{\epsilon}{5}\Big),F\Big(\left(n^2-5n+2\right)\sum_{a=1}^{n}Q_2(v_a)\\
&-\frac{1}{2^2}\left(n^2-5n+2\right)\sum_{a=1}^{n}\psi(2v_a),w,\frac{\epsilon}{5}\Big),F\Big(\frac{1}{2^2}\psi\left(\sum_{a=1}^{n}2v_a\right)+\frac{1}{2^2}\sum_{a=1}^{n}\psi\left(2\left(-v_a+\sum_{b=1;a\neq b}^{n}v_b\right)\right)\\
&-\frac{1}{2^2}(n-3)\sum_{1\le a<b\le n}\psi\left(2\left(v_a+v_b\right)\right)+\frac{1}{2^2}\left(n^2-5n+2\right)\sum_{a=1}^{n}\psi(2v_a),w,\frac{\epsilon}{5}\Big)\Big\} \qquad (4.18)
\end{aligned}
$$

for all $v_1, v_2, \cdots, v_n, w \in V$ and all $\epsilon > 0$. Using (4.17) and (NAF5) in (4.18), we get

$$
\begin{aligned}
F\left(DQ_2(v_1,v_2,\cdots,v_n),w,\epsilon\right) &\ge \min\Big\{1,1,1,1,F'\left(\chi\left(2^sv_1,2^sv_2,\cdots,2^sv_n\right),w,\varphi(2^{2s})\epsilon\right)\Big\}\\
&\ge F'\left(\chi\left(2^sv_1,2^sv_2,\cdots,2^sv_n\right),w,\varphi(2^{2s})\epsilon\right) \qquad (4.19)
\end{aligned}
$$

for all $v_1, v_2, \cdots, v_n, w \in V$ and all $\epsilon > 0$. Letting $s \to \infty$ in (4.19) and using (4.2), we obtain that

$$
F\left(DQ_2(v_1,v_2,\cdots,v_n),w,\epsilon\right)=1
$$

for all $v_1, v_2, \cdots, v_n, w \in V$ and all $\epsilon > 0$. Hence Q_2 fulfills the functional equation (1.1).

Next, we will prove that $Q_2(v)$ is the unique mapping, and consider another mapping $Q_2'(v)$ fulfilling (4.4) and (4.5). Then

$$
\begin{aligned}
F\left(Q_2(v)-Q_2'(v),w,\epsilon\right) &= F\left(\frac{Q_2(2^sv)}{2^{2s}}-\frac{Q_2'(2^sv)}{2^{2s}},w,\epsilon\right)\\
&\ge \min\Big\{F\Big(\frac{Q_2(2^sv)}{2^{2s}}-\frac{\psi(2^sv)}{2^{2s}},w,\frac{\epsilon}{2}\Big),F\Big(\frac{Q_2'(2^sv)}{2^{2s}}-\frac{\psi(2^sv)}{2^{2s}},w,\frac{\epsilon}{2}\Big)\Big\}\\
&\ge F'\left(\chi\left(2^sv,2^sv,0,\cdots,0\right),w,\frac{\epsilon\varphi(2^{2s})\left(\varphi(2^2)-\varphi(\alpha)\right)}{2}\right)\\
&\ge F'\left(\chi\left(v,v,0,\cdots,0\right),w,\frac{\epsilon\varphi(2^{2s})\left(\varphi(2^2)-\varphi(\alpha)\right)}{2\varphi(\alpha^s)}\right)
\end{aligned}
$$

for all $v, w \in V$ and all $\epsilon > 0$. Since $\lim_{s\to\infty}\frac{\epsilon\varphi(2^{2s})\left(\varphi(2^2)-\varphi(\alpha)\right)}{2\varphi(\alpha^s)}=\infty$, we obtain

$$
\lim_{s\to\infty}F'\left(\chi\left(v,v,0,\cdots,0\right),w,\frac{\epsilon\varphi(2^{2s})\left(\varphi(2^2)-\varphi(\alpha)\right)}{2\varphi(\alpha^s)}\right)=1.
$$

Thus $F(Q_2(v)-Q_2'(v),w,\epsilon)=1$ for all $v, w \in V$ and all $\epsilon > 0$. Hence $Q_2(v)=Q_2'(v)$. Therefore, $Q_2(v)$ is unique. In a similar manner, we can prove the other part of the proof for the case $j=-1$. This ends the proof of the theorem. □

Corollary 4.2. If $\psi : V \to W$ is an even mapping fulfilling the inequality

$$
F\left(D\psi(v_1,v_2,\cdots,v_n),w,\epsilon\right)\ge F'\left(c\sum_{i=1}^{n}\|v_i\|^{\gamma},w,\epsilon\right) \qquad (4.20)
$$

for all $v_1, v_2, \cdots, v_n, w, \in V$ and all $\epsilon > 0$, where c and γ are real constants with $c > 0$ and $\gamma \in (0,2)\cup(2,+\infty)$, then there is a unique quadratic mapping $Q_2 : V \to W$ fulfilling

$$
F\left(\psi(v)-Q_2(v),w,\epsilon\right)\ge F'\left(c\|v\|^{\gamma},w,\epsilon\left|\varphi(2^2)-\varphi(2^{\gamma})\right|\right)
$$

for all $v, w \in V$ and all $\epsilon > 0$.

Corollary 4.3. If $\psi : V \to W$ an even mapping fulfilling the inequality

$$F\left(D\psi(v_1, v_2, \cdots, v_n), w, \epsilon\right) \geq F^{'}\left(c\Big(\sum_{i=1}^{n} \|v_i\|^{n\gamma} + \prod_{i=1}^{n} \|v_i\|^{\gamma}\Big), w, \epsilon\right), \tag{4.21}$$

for all $v_1, v_2, \cdots, v_n, w, \in V$ and all $\epsilon > 0$, where c and γ are real constants with $c > 0$ and $n\gamma \in (0, 2) \cup (2, +\infty)$, then there is a unique quadratic mapping $Q_2 : V \to W$ fulfilling

$$F\left(\psi(v) - Q_2(v), w, \epsilon\right) \geq F^{'}\left(c\|v\|^{n\gamma}, w, \epsilon\left|\varphi(2^2) - \varphi(2^{n\gamma})\right|\right)$$

for all $v, w \in V$ and all $\epsilon > 0$.

4.2 Stability results for the even case: Fixed point method

In this subsection, we scrutinize the Hyers-Ulam stability of (1.1) for the even case by utilizing alternative fixed point theorem.

Theorem 4.4. Assume that an even mapping $\psi : V \to W$, for which there is a mapping $\chi : V^n \to Z$ with

$$\lim_{s\to\infty} F^{'}\left(\chi(\delta_i^s v_1, \delta_i^s v_2, \cdots, \delta_i^s v_n), w, \epsilon\varphi(\delta_i^{2s})\right) = 1, \tag{4.22}$$

where $\delta_i = 2$ if $i = 0$ and $\delta_i = \frac{1}{2}$ if $i = 1$, fulfills

$$F\left(D\psi(v_1, v_2, \cdots, v_n), w, \epsilon\right) \geq F^{'}\left(\chi(v_1, v_2, \cdots, v_n), w, \epsilon\right) \tag{4.23}$$

for all $v_1, v_2, \cdots, v_n, w \in V$ and all $\epsilon > 0$. If there exists L fulfilling

$$v \to \rho(v) = \frac{1}{2}\chi\left(\frac{v}{2}, \frac{v}{2}, 0, \cdots, 0\right),$$

which has the property

$$F^{'}\left(\frac{1}{\delta_i^2}\rho(\delta_i v), w, \epsilon\right) \geq F^{'}\left(L\rho(v), w, \epsilon\right), \quad v, w \in V, \ \epsilon > 0, \tag{4.24}$$

then there is a quadratic mapping $Q_2 : V \to W$ fulfilling (1.1) and

$$F\left(\psi(v) - Q_2(v), w, \epsilon\right) \geq F^{'}\left(\frac{L^{1-i}}{1-L}\rho(v), w, \epsilon\right) \tag{4.25}$$

for all $v, w \in V$ and all $\epsilon > 0$.

Proof . Suppose the set $\Lambda = \{q/q : V \to W, q(0) = 0\}$ and define the generalized metric on Λ,

$$d(p, q) = \inf\left\{\theta \in (0, \infty)/F(p(v) - q(v), w, \epsilon) \geq F^{'}(\theta\rho(v), w, \epsilon), \ v, w \in V, \epsilon > 0\right\}.$$

Clearly, (Λ, d) is complete. Define $T : \Lambda \to \Lambda$ by $Tp(v) = \frac{1}{\delta_i^2}p(\delta_i v)$ for all$v \in V$. One can show that $d(Tp, Tq) \leq Ld(p, q)$ for all $p, q \in \Lambda$, i.e., T is a strictly contractive mapping on Λ with Lipschitz constant $L = \delta_i^2$. Setting $(v_1, v_2, \cdots, v_n) = (v, v, 0, \cdots, 0)$ in (4.23), we have

$$F\left(2\psi(2v) - 8\psi(v), w, \epsilon\right) \geq F^{'}\left(\chi(v, v, 0, \cdots, 0), w, \epsilon\right). \tag{4.26}$$

Utilizing (4.24) for $i = 0$ in (4.35), it becomes to

$$\begin{aligned} F\left(\frac{\psi(2v)}{2^2} - \psi(v), w, \frac{\epsilon}{2\varphi(2^2)}\right) &\geq F^{'}\left(\frac{1}{2}\frac{\chi(v, v, 0, \cdots, 0)}{2^2}, w, \epsilon\right) \\ &\geq F^{'}\left(L\rho(v), w, \epsilon\right). \end{aligned} \tag{4.27}$$

That is,

$$d(\psi, T\psi) \leq L = L^1 < \infty.$$

Again, replacing v by $\frac{v}{2}$ in (4.35), we get

$$F\left(\psi(v) - 2^2\psi\Big(\frac{v}{2}\Big), w, \frac{\epsilon}{2}\right) \geq F'\left(\frac{1}{2}\chi\left(\frac{v}{2}, \frac{v}{2}, 0, \cdots, 0\right), w, \epsilon\right). \tag{4.28}$$

Utilizing (4.24) for $i = 1$ in (4.28), it becomes to

$$F\left(\psi(v) - 2^2\psi\Big(\frac{v}{2}\Big), w, \frac{\epsilon}{2}\right) \geq F'\left(\rho(v), w, \epsilon\right).$$

That is,

$$d(\psi, T\psi) \leq 1 = L^{1-i} < \infty.$$

In both cases, we have

$$d(\psi, T\psi) \leq L^{1-i}.$$

Thus (B2)(i) holds. By (B2)(ii), it arises that there is a fixed point Q_2 of T in Λ fulfilling

$$Q_2(v) = F - \lim_{s\to\infty} \frac{\psi(\delta_i v)}{\delta_i^2}.$$

Next, we will show that Q_2 fulfills (1.1). Replacing $(v_1, v_2, \cdots, v_n)$ by $(\delta_i^s v_1, \delta_i^s v_2, \cdots, \delta_i^s v_n)$ in (4.23), we obtain

$$F\left(\frac{1}{\delta_i^{2s}} D\psi(\delta_i^s v_1, \delta_i^s v_2, \cdots, \delta_i^s v_n), w, \epsilon) \geq F'\left(\chi(\delta_i^s v_1, \delta_i^s v_2, \cdots, \delta_i^s v_n\right), w, \varphi(\delta_i^{2s})\epsilon\right) \tag{4.29}$$

for all $v_1, v_2, \cdots, v_n, w \in V$ and all $\epsilon > 0$. Taking $s \to \infty$ in (4.29) and utilizing the definition of $Q_2(v)$, we obtain that Q_2 fulfills (1.1). Thus the mapping Q_2 is quadratic.

By (B2)(iii), Q_2 is the only one fixed point of T in $\Delta = \Big\{\psi \in \Lambda : d(\psi, Q_2) < \infty\Big\}$, i.e., Q_2 is the only one mapping fulfilling

$$F\left(\psi(v) - Q_2(v), w, \epsilon\right) \geq F'\left(\theta\rho(v), w, \epsilon\right)$$

for all $v, w \in V$ and all $\epsilon > 0$, $\theta > 0$. Again by (B2)(iv), we obtain

$$d\left(\psi, Q_2\right) \leq \frac{1}{1-L} d(\psi, T\psi) \Rightarrow d(\psi, Q_2) \leq \frac{L^{1-i}}{1-L}.$$

This yields

$$F\left(\psi(v) - Q_2(v), w, \epsilon\right) \geq F'\left(\frac{L^{1-i}}{1-L}\rho(v), w, \epsilon\right), \quad v, w \in V, \quad \epsilon > 0.$$

Hence the proof of the theorem is now completed. □

Corollary 4.5. If an even mapping $\psi : V \to W$ fulfils

$$F\left(D\psi\left(v_1, v_2, \cdots, v_n\right), w, \epsilon\right) \geq \begin{cases} F'\left(c\sum_{i=1}^n \|v_i\|^\gamma, w, \epsilon\right) \\ F'\left(c\left(\sum_{i=1}^n \|v_i\|^{n\gamma} + \prod_{i=1}^n \|v_i\|^\gamma\right), w, \epsilon\right) \end{cases} \tag{4.30}$$

for all $v_1, v_2, \cdots, v_n, w \in V$ and all $\epsilon > 0$, where c and γ are constants with $c > 0$, then there is only one quadratic mapping $Q_2 : V \to W$ fulfilling

$$F\left(\psi(v) - Q_2(v), w, \epsilon\right) \geq \begin{cases} F'\left(c\|v\|^\gamma, w, \epsilon\left|\varphi(2^2) - \varphi(2^\gamma)\right|\right); & \gamma \neq 2 \\ F'\left(c\|v\|^{n\gamma}, w, \epsilon\left|\varphi(2^2) - \varphi(2^{n\gamma})\right|\right); & \gamma \neq \frac{2}{n} \end{cases}$$

for all $v, w \in V$ and all $\epsilon > 0$.

Proof . Set

$$\chi(v_1, v_2, \cdots, v_n) = \begin{cases} c\sum_{i=1}^{n} \|v_i\|^{\gamma}, \\ c\left(\sum_{i=1}^{n} \|v_i\|^{n\gamma} + \prod_{i=1}^{n} \|v_i\|^{\gamma}\right) \end{cases}$$

for all $v_1, v_2, \cdots, v_n \in V$. Now,

$$\begin{aligned} F^{'}\left(\chi(\delta_i^s v_1, \delta_i^s v_2, \cdots, \delta_i^s v_n), w, \delta_i^{2s}\epsilon\right) &= \begin{cases} F^{'}\left(c\sum_{i=1}^{n} \|\delta_i^s v_i\|^{\gamma}, w, \epsilon\right) \\ F^{'}\left(c\left(\sum_{i=1}^{n} \|\delta_i^s v_i\|^{n\gamma} + \prod_{i=1}^{n} \|\delta_i^s v_i\|^{\gamma}\right), w, \epsilon\right) \end{cases} \\ &= \begin{cases} 1, \text{ if } (i=0 \text{ and } \gamma<2) \text{ or } (i=1 \text{ and } \gamma>2), \\ 1, \text{ if } (i=0 \text{ and } \gamma n<2) \text{ or } (i=1 \text{ and } \gamma n>2). \end{cases} \end{aligned}$$

Let $\rho(v) = \frac{1}{2}\chi\left(\frac{v}{2}, \frac{v}{2}, 0, \cdots, 0\right)$. Next, we have

$$F^{'}\left(\frac{1}{\delta_i^2}\rho(\delta_i v), w, \epsilon\right) = \begin{cases} F^{'}\left(\delta_i^{\gamma-2}\rho(v), w, \epsilon\right) \\ F^{'}\left(\delta_i^{n\gamma-2}\rho(v), w, \epsilon\right) \end{cases}$$

and

$$F^{'}(\rho(v), w, \epsilon) = F^{'}\left(\chi\left(\frac{v}{2}, \frac{v}{2}, 0, \cdots, 0\right), w, 2\epsilon\right) = \begin{cases} F^{'}\left(\frac{2c}{2^{\gamma}}\|v\|^{\gamma}, w, 2\epsilon\right), \\ F^{'}\left(\frac{2c}{2^{n\gamma}}\|v\|^{n\gamma}, w, 2\epsilon\right). \end{cases}$$

Hence the inequality (4.24) holds either $L = 2^{\gamma-2}$ for $\gamma < 2$ if $i = 0$, $L = 2^{2-\gamma}$ for $\gamma > 2$ if $i = 1$, $L = 2^{n\gamma-2}$ for $\gamma < \frac{2}{n}$ if $i = 0$ and $L = 2^{2-n\gamma}$ for $\gamma > \frac{2}{n}$ if $i = 1$. From (4.25), we obtain our results. □

4.3 Stability results for the odd case: Direct method

In this subsection, we scrutinize the Hyers-Ulam stability of (1.1) for the odd case by utilizing direct method.

Theorem 4.6. Let $j \in \{-1, 1\}$ be fixed and $\chi : V^n \to Z$ be a mapping such that for some α with $0 < \left(\frac{\varphi(\alpha)}{\varphi(2)}\right)^j < 1$, (4.1) and

$$\lim_{s\to\infty} F^{'}\left(\chi\left(2^{js}v_1, 2^{js}v_2, \cdots, 2^{js}v_n\right), w, \left[\varphi\left(2^s\right)\right]^j \epsilon\right) = 1$$

for all $v_1, v_2, \cdots, v_n, w \in V$ and all $\epsilon > 0$. If an odd mapping $\psi : V \to W$ fulfills (4.3), then the limit

$$A_1(v) = F - \lim_{s\to\infty} \frac{\psi(2^{js}v)}{2^{js}}$$

exists for each $v \in V$ and the mapping $A_1 : V \to W$ is a unique additive mapping satisfying (1.1) and

$$F\left(\psi(v) - A_1(v), w, \epsilon\right) \geq F^{'}\left(\chi(v, v, 0, \cdots, 0), w, 2\epsilon\left|\varphi(2) - \varphi(\alpha)\right|\right)$$

for all $v, w \in V$ and all $\epsilon > 0$.

Proof . Consider $j = 1$. Replacing $(v_1, v_2, \cdots, v_n)$ by $(v, v, 0, \cdots, 0)$ in (4.3), we get

$$F\left(2\psi(2v) - 4\psi(v), w, \epsilon\right) \geq F^{'}\left(\chi(v, v, 0, \cdots, 0), w, \epsilon\right) \tag{4.31}$$

for all $v, w \in V$ and all $\epsilon > 0$. From (4.31), we obtain

$$F\left(\frac{\psi(2v)}{2} - \psi(v), w, \frac{\epsilon}{2\varphi(2)}\right) \geq F^{'}\left(\chi(v, v, 0, \cdots, 0), w, \epsilon\right) \tag{4.32}$$

for all $v, w \in V$ and all $\epsilon > 0$. Replacing v by $2^s v$ in (4.32), we get

$$F\left(\frac{\psi(2^{s+1}v)}{2^{(s+1)}} - \frac{\psi(2^s v)}{2^s}, w, \frac{\epsilon}{2\varphi\left(2^{(s+1)}\right)}\right) \geq F'\left(\chi(2^s v, 2^s v, 0, \cdots, 0), w, \epsilon\right) \tag{4.33}$$

for all $v, w \in V$ and all $\epsilon > 0$. Utilizing (4.1), (NAF6) in (4.33), we get

$$F\left(\frac{\psi(2^{s+1}v)}{2^{(s+1)}} - \frac{\psi(2^s v)}{2^s}, w, \frac{\epsilon}{2\varphi(2^{(s+1)})}\right) \geq F'\left(\chi(v, v, 0, \cdots, 0), w, \frac{\epsilon}{\varphi(\alpha^s)}\right) \tag{4.34}$$

for all $v, w \in V$ and all $\epsilon > 0$. Replacing ϵ by $\varphi(\alpha^s)\epsilon$ in (4.34), we obtain

$$F\left(\frac{\psi(2^{s+1}v)}{2^{(s+1)}} - \frac{\psi(2^s v)}{2^s}, w, \frac{\varphi(\alpha^s)\epsilon}{2\varphi(2^{(s+1)})}\right) \geq F'\left(\chi(v, v, 0, \cdots, 0), w, \epsilon\right)$$

for all $v, w \in V$ and all $\epsilon > 0$. The remaining proof is the same as in the proof of Theorem 4.1. □

Corollary 4.7. If $\psi : V \to W$ is an odd mapping which fulfills (4.20) for all $v_1, v_2, \cdots, v_n, w, \in V$ and all $\epsilon > 0$, where c and γ are real constants with $c > 0$ and $\gamma \in (0, 1) \cup (1, +\infty)$, then there is a unique additive mapping $A_1 : V \to W$, which fulfills

$$F\left(\psi(v) - A_1(v), w, \epsilon\right) \geq F'\left(c\|v\|^\gamma, w, \epsilon\left|\varphi(2) - \varphi(2^\gamma)\right|\right)$$

for all $v, w \in V$ and all $\epsilon > 0$.

Corollary 4.8. If $\psi : V \to W$ is an odd mapping which fulfils (4.21) for every $v_1, v_2, \cdots, v_n, w, \in V$ and all $\epsilon > 0$, where c and γ are the real constants with $c > 0$ and $n\gamma \in (0, 1) \cup (1, +\infty)$, then there is an additive mapping $A_1 : V \to W$ is unique which fulfils

$$F\left(\psi(v) - A_1(v), w, \epsilon\right) \geq F'\left(c\|v\|^{n\gamma}, w, \epsilon\left|\varphi(2) - \varphi(2^{n\gamma})\right|\right)$$

for all $v, w \in V$ and all $\epsilon > 0$.

4.4 Stability results for the odd case: Fixed point method

In this subsection, we investigate the Hyers-Ulam stability of (1.1) for the odd case by utilizing alternative fixed point theorem.

Theorem 4.9. Assume that an odd mapping $\psi : V \to W$, for which there is a mapping $\chi : V^n \to Z$ with

$$\lim_{s\to\infty} F'\left(\chi(\delta_i^s v_1, \delta_i^s v_2, \cdots, \delta_i^s v_n), w, \epsilon\varphi(\delta_i^s)\right) = 1$$

where $\delta_i = 2$ if $i = 0$ and $\delta_i = \frac{1}{2}$ if $i = 1$, fulfills (4.23). If there exists L satisfying

$$v \to \rho(v) = \frac{1}{2}\chi\left(\frac{v}{2}, \frac{v}{2}, 0, \cdots, 0\right),$$

which has the property

$$F'\left(\frac{1}{\delta_i}\rho(\delta_i v), w, \epsilon\right) \geq F'\left(L\rho(v), w, \epsilon\right)$$

for all $v, w \in V$ and all $\epsilon > 0$, then there is a unique additive mapping $A_1 : V \to W$ fulfilling (1.1) and

$$F\left(\psi(v) - A_1(v), w, \epsilon\right) \geq F'\left(\frac{L^{1-i}}{1-L}\rho(v), w, \epsilon\right)$$

for all $v, w \in V$ and all $\epsilon > 0$.

Proof . Suppose the set $\Lambda = \{q/q : V \to W, q(0) = 0\}$ and define the generalized metric on Λ,

$$d(p,q) = \inf\left\{\theta \in (0,\infty)/F(p(v)-q(v),w,\epsilon) \geq F^{'}(\theta\rho(v),w,\epsilon),\ v,w \in V, \epsilon > 0\right\}.$$

Clearly, (Λ, d) is complete. Define $T : \Lambda \to \Lambda$ by $Tp(v) = \frac{1}{\delta_i}p(\delta_i v)$ for all $v \in V$. One can show that $d(Tp,Tq) \leq Ld(p,q)$ for all $p,q \in \Lambda$, i.e., T is a strictly contractive mapping on Λ with Lipschitz constant $L = \delta_i$. Setting $(v_1, v_2, \cdots, v_n) = (v, v, 0, \cdots, 0)$ in (4.23), we obtain

$$F\left(2\psi(2v) - 4\psi(v), w, \epsilon\right) \geq F^{'}\left(\chi(v, v, 0, \cdots, 0), w, \epsilon\right), \quad v, w \in V,\ \epsilon > 0.$$

The remaining proof is the same as in the proof of Theorem 4.4. □

Corollary 4.10. If an odd mapping $\psi : V \to W$ fulfills (4.30) for all $v_1, v_2, \cdots, v_n, w \in V$ and all $\epsilon > 0$, where c and γ are constants with $c > 0$, then there is only one additive mapping $A_1 : V \to W$ fulfilling

$$F\left(\psi(v) - A_1(v), w, \epsilon\right) \geq \begin{cases} F^{'}\left(c\|v\|^{\gamma}, w, \epsilon\left|\varphi(2) - \varphi(2^{\gamma})\right|\right); & \gamma \neq 1 \\ F^{'}\left(c\|v\|^{n\gamma}, w, \epsilon\left|\varphi(2) - \varphi(2^{n\gamma})\right|\right); & \gamma \neq \frac{1}{n} \end{cases}$$

for all $v, w \in V$ and all $\epsilon > 0$.

5 Conclusion

In this work, we have introduced the new quadratic-additive functional equation (1.1). We investigated our needed results of Hyers-Ulam stability of finite variable quadratic-additive functional equation (1.1) for both even and odd cases by utilizing the direct and fixed point methods in non-Archimedean fuzzy φ-2-normed spaces.

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