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On fuzzy C^*-algebras

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Abstract

In this paper, we show that there is no fuzzy C^*-algebra norm according to the definitions contained in [1, 2, 3]. Based on our claim, some parts of [Theorems 2 and 3, 2], [Corollaries 1 and 2, 2] and [Theorem 4, 3] are incorrect.

Keywords: Fuzzy normed space, Fuzzy normed algebra, Fuzzy normed $*-$algebra, Fuzzy C^*-algebra norm
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1 Introduction

We are well aware that the definition of an induced fuzzy C^*-algebra was introduced by Park et al. at the first time [4]. As far as we know, the definition of a fuzzy C^*-algebra norm was introduced by Ansari et al. at the first time [1]. This definition followed by Lo′lo′ et al. [2]. Also some results according to this definition were presented by Movahednia and De la Sen [3]. This paper aims to inform the authors and the readers to avoid following an incorrect definition.

Definition 1.1 ([1, 2, 3, 4]). Let X be a real vector space. A function $N : X \times \mathbb{R} \longrightarrow [0, 1]$ is called a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$

1. $N(x, t) = 0$ for all $t \leq 0$,
2. $N(x, t) = 1$ for all $t > 0$ if and only if $x = 0$,
3. $N(cx, t) = N(x, \frac{t}{|c|})$ for all $c \neq 0$,
4. $N(x + y, s + t) \geq \min\,(N(x, s), N(y, t))$,
5. $N(x, \cdot)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t\longrightarrow\infty} N(x, t) = 1$,
6. For $x \neq 0, N(x, \cdot)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a fuzzy normed vector space.

Definition 1.2 ([1, 2, 3, 4]). Let (X, N) be a fuzzy normed vector space.

1. A sequence $\{x_n\}$ in X is said to be convergent if there exists an $x \in X$ such that

$$\lim_{n\longrightarrow\infty} N(x_n - x, t) = 1$$

for all $t > 0$.

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2. A sequence $\{x_n\}$ in X is called Cauchy if for each $\epsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n > n_0$ and all $p \geq 1$, we have $N(x_{n+p} - x_n, t) > 1 - \epsilon$. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.

Definition 1.3 ([1, 2, 3, 4]). Let X be an algebra and (X, N) a fuzzy normed space. The fuzzy normed space (X, N) is called a fuzzy normed algebra if

$$N(xx', st) \geq N(x, s)N(x', t)$$

for all $x, x' \in X, s, t \in \mathbb{R}^+$. A complete fuzzy normed algebra is called a fuzzy Banach algebra.

Definition 1.4 ([1, 2, 3]). Let A be an algebra over $\mathbb{C}$. An involution on A is a mapping

$$\begin{aligned} * : & A \longrightarrow A \\ & a \longrightarrow a^* \end{aligned}$$

such that

1. $(\alpha a + \beta b)^* = \bar{\alpha} a^* + \bar{\beta} b^*$ for all $a, b \in A$ and $\alpha, \beta \in \mathbb{C}$.
2. $(ab)^* = b^* a^*$ for all $a, b \in A$.
3. $(a^*)^* = a$ for all $a \in A$.

A complex algebra with an involution is a $*-$algebra.

Definition 1.5 ([1, 2, 3, 4]). Let A be a $*-$algebra and (A, N) a fuzzy normed algebra. The fuzzy normed algebra (A, N) is called a fuzzy normed $*$-algebra if,

$$N(a^*, t) = N(a, t)$$

for all $a \in A$ and for all $t \in \mathbb{R}^+$. A complete fuzzy normed $*-$algebra is called a fuzzy Banach $*-$algebra.

Definition 1.6 ([1, 2, 3]). Let (A, N) be a fuzzy Banach $*-$algebra. Then (A, N) is called a fuzzy C^*-algebra if,

$$N(a^* a, st) = N(a^*, s)N(a, t)$$

for all $a \in A$ and for all $s, t \in \mathbb{R}^+$.

Theorem 1.7. There is no a fuzzy C^*-algebra norm according to Definition1.6.

Proof . We suppose by contradiction that N is a fuzzy C^*-algebra norm on A in the sense Definition 1.6. Let $0 \neq a \in A$. Since $\lim_{t \longrightarrow \infty} N(a^* a, t) = 1$, there exists an $M > 0$ such that $N(a^* a, t) > \frac{1}{2}$ for all $t \geq M$. So by Definition 1.6, for all $n \in \mathbb{N}$ we have, $\frac{1}{2} < N(a^* a, M) = N(a^* a, \frac{M}{n} \cdot n) = N(a^*, \frac{M}{n})N(a, n)$. Hence

$$\frac{N(a^* a, M)}{N(a, n)} = N(a^*, \frac{M}{n}) = N(a, \frac{M}{n})$$

for all $n \in \mathbb{N}$. By parts (6) and (5) of Definition 1.1, $N(a, \cdot)$ is continuous at $t = 0$ and $\lim_{n \longrightarrow \infty} N(a, n) = 1$. So we can conclude that $N(a^* a, M) = \lim_{n \longrightarrow \infty} \frac{N(a^* a, M)}{N(a, n)} = \lim_{n \longrightarrow \infty} N(a, \frac{M}{n}) = 0$. Hence $N(a^* a, M) = 0$, that is a contradiction. $\square$

Example 1.8. Let $(A, *, ||\cdot||)$ be a C^*-algebra. Define $N : A \times \mathbb{R} \longrightarrow [0, 1]$ by

$$N(a, t) = \begin{cases} 0, & t \leq 0 \\ \frac{t}{t + ||a||}, & t > 0. \end{cases}$$

Then (A, N) is a fuzzy Banach $*-$algebra but $N(a^* a, 2 \times 3) \neq N(a^*, 2)N(a, 3)$.

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