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General 3D-Jensen ρ-functional equation and ternary Hom-Jordan derivation

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Abstract

In this paper, we introduce the concept of ternary Hom-Jordan derivation and solve the new 3D-Jensen ρ-functional equations in the sense of ternary Banach algebras. Moreover, we prove its Hyers-Ulam stability using the fixed point method.

Keywords: Ternary Hom-Jordan Derivation, 3D-Jensen ρ-functional equations, Fixed point method
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1 Introduction and preliminaries

The Hyers-Ulam stability problem which arises from Ulam's question says that for two given fixed functions φ and ψ, the functional equation $\mathcal{F}_1(G) = \mathcal{F}_2(G)$ is stable if for a function g for which $d(\mathcal{F}_1(g), \mathcal{F}_2(g)) \le \varphi$ holds, there is a function h such that $\mathcal{F}_1(h) = \mathcal{F}_2(h)$ and $d(g, h) \le \psi$ [7, 9, 18, 20]. In 1941 [9], Hyers solved the approximately additive mappings on the setting of Banach spaces. First, Th. M. Rassias [18] and Aoki [1] and then a number of authors extended this result by considering the unbounded Cauchy differences in different spaces. For example see [6, 8, 11, 12, 14]. F. Skof in 1983 [19], proved the stability problem of quadratic functional equation between normed and Banach spaces.

A ternary Banach algebra $\mathfrak{A}$ with $\|.\|$ is a complex Banach algebra equipped with a ternary product $(a, b, c) \to [a, b, c]$ of $\mathfrak{A}^3$ into $\mathfrak{A}$. This product is $\mathbb{C}$-linear in the outer variable, conjugate $\mathbb{C}-$linear in the middle variable associative in the sense that $[a, b, [c, v, u]] = [a, [b, c, v], u] = [[a, b, c], v, u]$ and satisfies $\|[a, b, c]\| \le \|a\|.\|b\|.\|c\|$ and $\|[a, a, a]\| = \|a\|^3$ (see [21]). Ternary structures and their extensions, known as n-ary algebras have many applications in mathematical physics and photonics, such as the quark model and Nambu mechanics [4, 5, 10, 13, 16]. Today, many physical systems can be modeled as a linear system. The principle of additivity has various applications in physics especially in calculating the internal energy in thermodynamic and also the meaning of the superposition principle. Throughout this paper, $\mathfrak{A}$ is a ternary Banach algebra.

Definition 1.1. A mapping $h : \mathfrak{A} \to \mathfrak{A}$ is called a ternary homomorphism, if h is a $\mathbb{C}$-linear and

$$h([x_1, x_2, x_3]) = [h(x_1), h(x_2), h(x_3)] \quad \forall\ x_1, x_2, x_3 \in \mathfrak{A}.$$

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Definition 1.2. Let $h : \mathfrak{A} \to \mathfrak{A}$ be a ternary homomorphism. A $\mathbb{C}$-linear $D : \mathfrak{A} \to \mathfrak{A}$ is called a ternary hom-derivation if D satisfies

$$D([x_1, x_2, x_3]) = [D(x_1), h(x_2), h(x_3)] + [h(x_1), D(x_2), h(x_3)] + [h(x_1), h(x_2), D(x_3)]$$

for all $x_1, x_2, x_3 \in \mathfrak{A}$.

Consider the generalized functional equation

$$\begin{aligned} &f(\frac{x+y}{2}+z)+f(\frac{x+z}{2}+y)+f(\frac{y+z}{2}+x)-2f(x)-2f(y)-2f(z) \\ &=\rho(f(x+y+z)+f(x)-f(x+z)-f(x+y)), \end{aligned} \tag{1.1}$$

where $\rho \neq 0, \pm 1$ is a complex number. In this paper, we solve (1.1) and show that a function which satisfies (1.1) is additive. We also prove its Hyers-Ulam stability by using the fixed point method. To do this, we use the Diaz-Margolis fixed point theorem [15].

Theorem 1.3. [15] Let $(\mathfrak{A}, d)$ be a complete generalized metric space and let $\Gamma : \mathfrak{A} \to \mathfrak{B}$ be a strictly contractive mapping with Lipschitz constant $0 < L < 1$. Then for each given element $x \in \mathfrak{A}$, either

$$d(\Gamma^i(x), \Gamma^{i+1}(x)) = \infty$$

for all nonnegative integers i or there exists a positive integer i_0 such that

(1) $d(\Gamma^i(x), \Gamma^{i+1}(x)) < \infty, \quad \forall i \geq i_0$;

(2) the sequence $\{F^i(x)\}$ converges to a unique fixed point y^* of Γ in the set $\mathfrak{B} = \{y \in \mathfrak{A} \mid d(\Gamma^{i_0}x, y) < \infty\}$;

(3) $d(y, y^*) \leq \frac{1}{1-L} d(y, \Gamma(y))$ for all $y \in \mathfrak{B}$.

2 Main results

Throughout the section, let $\mathbb{T}^1_{1/n_0}$ be the set of all complex numbers $e^{i\theta}$, where $0 \leq \theta \leq \frac{2\pi}{n_0}$. To prove the main theorems, we need the following lemmas. Firstly, in the next lemma, we prove that f is a additive mapping.

Lemma 2.1. Let $\mathfrak{A}$ and $\mathfrak{B}$ are two vector spaces. Let mapping $f : \mathfrak{A} \to \mathfrak{B}$ satisfies

$$\begin{aligned} &f(\frac{x+y}{2}+z)+f(\frac{x+z}{2}+y)+f(\frac{y+z}{2}+x)-2f(x)-2f(y)-2f(z) \\ &=\rho(f(x+y+z)+f(x)-f(x+z)-f(x+y)), \end{aligned} \tag{2.1}$$

for all $x, y, z \in \mathfrak{A}$. Then $f : \mathfrak{A} \to \mathfrak{B}$ is a additive.

Proof . First of all, let $x = y = z = 0$ in (2.1), we get $f(0) = 0$. Putting $y = z = 0$ in (2.1), we have

$$f(\frac{x}{2}) = \frac{1}{2} f(x). \tag{2.2}$$

Again putting $x = -y$, $z = 0$ in (2.1), we have

$$\frac{1}{2} f(y) + \frac{1}{2} f(-y) - 2f(-y) - 2f(y) = 0. \tag{2.3}$$

Now by (2.3) and using (2.2), we get

$$f(-y) = -f(y).$$

Let $z = -y$ in (2.1), we have

$$f(\frac{x-y}{2}) + f(\frac{x+y}{2}) - f(x) = 0, \tag{2.4}$$

replacing x and y by $x + y$ and $x - y$ respectively in (2.4), we have

$$f(x + y) = f(x) + f(y).$$

Hence, f is a additive mapping. □

Lemma 2.2. Let $\mathfrak{A}$ and $\mathfrak{B}$ are two linear spaces. Let mapping $f : \mathfrak{A} \to \mathfrak{B}$ satisfies

$$\begin{aligned} &f(\frac{\lambda x+\lambda y}{2}+\lambda z)+f(\frac{\lambda x+\lambda z}{2}+\lambda y)+f(\frac{\lambda y+\lambda z}{2}+\lambda x)-2\lambda f(x)-2\lambda f(y)-2\lambda f(z) \\ &=\rho(f(\lambda x+\lambda y+\lambda z)+\lambda f(x)-\lambda f(x+z)-\lambda f(x+y)), \end{aligned} \tag{2.5}$$

for all $\lambda \in \mathbb{T}^1_{1/n_0}$ and $x, y, z \in \mathfrak{A}$. Then f is a $\mathbb{C}-$linear.

Proof . By lemma 2.1 f is additive. letting $y = z = 0$ in (2.5), we have $\lambda f(x) = f(\lambda x)$ for all $\lambda \in \mathbb{T}^1_{1/n_0}$ and $x, y, z \in \mathfrak{A}$. By the same reasoning as in proof [[17], Theorem 2.1] the mapping f is $\mathbb{C}-$linear. □

Lemma 2.3. [3] Let $f : \mathfrak{A} \to \mathfrak{A}$ be an linear mapping. As a result, are equivalent the following relations:

$$f([x,x,x])=[f(x),x,x]+[x,f(x),x]+[x,x,f(x)],$$

and

$$\begin{aligned} f([x_1,x_2,x_3]+[x_2,x_3,x_1]+[x_3,x_1,x_2]) &= [f(x_1),x_2,x_3]+[x_1,f(x_2),x_3]+[x_1,x_2,f(x_3)] \\ &+[f(x_2),x_3,x_1]+[x_2,f(x_3),x_1]+[x_2,x_3,f(x_1)] \\ &+[f(x_3),x_1,x_2]+[x_3,f(x_1),x_2]+[x_3,x_1,f(x_2)]. \end{aligned}$$

In the following lemma, we investigate equality ternary hom-Jordan derivation by non-same components.

Lemma 2.4. Let $d : \mathfrak{A} \to \mathfrak{A}$ be an linear mapping. As a result, are equivalent the following relations:

$$d([x,x,x])=[d(x),h(x),h(x)]+[h(x),d(x),h(x)]+[h(x),h(x),d(x)] \tag{2.6}$$

and

$$\begin{aligned} d([x_1,x_2,x_3]+[x_2,x_3,x_1]+[x_3,x_1,x_2]) =&[d(x_1),h(x_2),h(x_3)]+[h(x_1),d(x_2),h(x_3)] \\ &+[h(x_1),h(x_2),d(x_3)]+[d(x_2),h(x_3),h(x_1) \\ &+[h(x_2),d(x_3),h(x_1)]+[h(x_2),h(x_3),d(x_1)] \\ &+[d(x_3),h(x_1),h(x_2)]+[h(x_3),d(x_1),h(x_2)] \\ &+[h(x_3),h(x_1),d(x_2)], \end{aligned} \tag{2.7}$$

where $h : \mathfrak{A} \to \mathfrak{A}$ is a ternary homomorphism.

Proof . In the first equation, we replace x by $x_1 + x_2 + x_3$, then we have

$$\begin{aligned} &d([(x_1+x_2+x_3),(x_1+x_2+x_3),(x_1+x_2+x_3)])=[d(x_1+x_2+x_3),h(x_1+x_2+x_3),h(x_1+x_2+x_3)] \\ &+[h(x_1+x_2+x_3),d(x_1+x_2+x_3),h(x_1+x_2+x_3)]+[h(x_1+x_2+x_3),h(x_1+x_2+x_3),d(x_1+x_2+x_3)], \end{aligned}$$

for all $x_1, x_2, x_3 \in \mathfrak{A}$. We determine as follows

$$\begin{aligned} &d([(x_1+x_2+x_3),(x_1+x_2+x_3),(x_1+x_2+x_3)])=d([x_1,x_1,x_1]+[x_1,x_2,x_1] \\ &+[x_1,x_3,x_1]+[x_2,x_1,x_1]+[x_2,x_2,x_1]+[x_2,x_3,x_1]+[x_3,x_1,x_1]+[x_3,x_2,x_1] \\ &+[x_3,x_3,x_1]+[x_1,x_1,x_2]+[x_1,x_2,x_2]+[x_1,x_3,x_2]+[x_2,x_1,x_2]+[x_2,x_2,x_2] \\ &+[x_2,x_3,x_2]+[x_3,x_1,x_2]+[x_3,x_2,x_2]+[x_3,x_3,x_2]+[x_1,x_1,x_3]+[x_1,x_2,x_3] \\ &+[x_1,x_3,x_3]+[x_2,x_1,x_3]+[x_2,x_2,x_3]+[x_2,x_3,x_3]+[x_3,x_1,x_3]+[x_3,x_2,x_3]+[x_3,x_3,x_3])= \\ &d([x_1,x_1,x_1])+d([x_1,x_2,x_1])+d([x_1,x_3,x_1])+d([x_2,x_1,x_1])+d([x_2,x_2,x_1])+d([x_2,x_3,x_1]) \\ &+d([x_3,x_1,x_1])+d([x_3,x_2,x_1])+d([x_3,x_3,x_1])+d([x_1,x_1,x_2])+d([x_1,x_2,x_2])+d([x_1,x_3,x_2]) \\ &+d([x_2,x_1,x_2])+d([x_2,x_2,x_2])+d([x_2,x_3,x_2])+d([x_3,x_1,x_2])+d([x_3,x_2,x_2])+d([x_3,x_3,x_2]) \\ &+d([x_1,x_1,x_3])+d([x_1,x_2,x_3])+d([x_1,x_3,x_3])+d([x_2,x_1,x_3])+d([x_2,x_2,x_3])+d([x_2,x_3,x_3]) \\ &+d([x_3,x_1,x_3])+d([x_3,x_2,x_3])+d([x_3,x_3,x_3]), \end{aligned}$$

for all $x_1, x_2, x_3 \in \mathfrak{A}$. One the other hand, we have

$$\begin{aligned}
&[d(x_1+x_2+x_3), h(x_1+x_2+x_3), h(x_1+x_2+x_3)] + [h(x_1+x_2+x_3), d(x_1+x_2+x_3),\\
&h(x_1+x_2+x_3)] + [h(x_1+x_2+x_3), h(x_1+x_2+x_3), d(x_1+x_2+x_3)] =\\
&[d(x_1), h(x_1), h(x_1)] + [d(x_1), h(x_1), h(x_2)] + [d(x_1), h(x_1), h(x_3)] + [d(x_1), h(x_2), h(x_1)]\\
&+ [d(x_1), h(x_2), h(x_2)] + [d(x_1), h(x_2), h(x_3)] + [d(x_1), h(x_3), h(x_1)] + [d(x_1), h(x_3), h(x_2)]\\
&+ [d(x_1), h(x_3), h(x_3)] + [d(x_2), h(x_1), h(x_1)] + [d(x_2), h(x_1), h(x_2)] + [d(x_2), h(x_1), h(x_3)]\\
&+ [d(x_2), h(x_2), h(x_1)] + [d(x_2), h(x_2), h(x_2)] + [d(x_2), h(x_2), h(x_3)] + [d(x_2), h(x_3), h(x_1)]\\
&+ [d(x_2), h(x_3), h(x_2)] + [d(x_2), h(x_3), h(x_3)] + [d(x_3), h(x_1), h(x_1)] + [d(x_3), h(x_1), h(x_2)]\\
&+ [d(x_3), h(x_1), h(x_3)] + [d(x_3), h(x_2), h(x_1)] + [d(x_3), h(x_2), h(x_2)] + [d(x_3), h(x_2), h(x_3)]\\
&+ [d(x_3), h(x_3), h(x_1)] + [d(x_3), h(x_3), h(x_2)] + [d(x_3), h(x_3), h(x_3)] + [h(x_1), d(x_1), h(x_1)]\\
&+ [h(x_1), d(x_1), h(x_2)] + [h(x_1), d(x_1), h(x_3)] + [h(x_1), d(x_2), h(x_1)] + [h(x_1), d(x_2), h(x_2)]\\
&+ [h(x_1), d(x_2), h(x_3)] + [h(x_1), d(x_3), h(x_1)] + [h(x_1), d(x_3), h(x_2)] + [h(x_1), d(x_3), h(x_3)]\\
&+ [h(x_2), d(x_1), d(x_1)] + [h(x_2), d(x_1), h(x_2)] + [h(x_2), d(x_1), h(x_3)] + [h(x_2), d(x_2), h(x_1)]\\
&+ [h(x_2), d(x_2), h(x_2)] + [h(x_2), d(x_2), h(x_3)] + [h(x_2), d(x_3), h(x_1)] + [h(x_2), d(x_3), h(x_2)]\\
&+ [h(x_2), d(x_3), h(x_3)] + [h(x_3), d(x_1), h(x_1)] + [h(x_3), d(x_1), h(x_2)] + [h(x_3), d(x_1), h(x_3)]\\
&+ [h(x_3), d(x_2), h(x_1)] + [h(x_3), d(x_2), h(x_2)] + [h(x_3), d(x_2), h(x_3)] + [h(x_3), d(x_3), h(x_1)]\\
&+ [h(x_3), d(x_3), h(x_2)] + [h(x_3), d(x_3), h(x_3)] + [h(x_1), h(x_1), d(x_1)] + [h(x_1), h(x_1), d(x_2)]\\
&+ [h(x_1), h(x_1), d(x_3)] + [h(x_1), h(x_2), d(x_1)] + [h(x_1), h(x_2), d(x_2)] + [h(x_1), h(x_2), d(x_3)]\\
&+ [h(x_1), h(x_3), d(x_1)] + [h(x_1), h(x_3), d(x_2)] + [h(x_1), h(x_3), d(x_3)] + [h(x_2), h(x_1), d(x_1)]\\
&+ [h(x_2), h(x_1), d(x_2)] + [h(x_2), h(x_1), d(x_3)] + [h(x_2), h(x_2), d(x_1)] + [h(x_2), h(x_2), d(x_2)]\\
&+ [h(x_2), h(x_2), d(x_3)] + [h(x_2), h(x_3), d(x_1)] + [h(x_2), h(x_3), d(x_2)] + [h(x_2), h(x_3), d(x_3)]\\
&+ [h(x_3), h(x_1), d(x_1)] + [h(x_3), h(x_1), d(x_2)] + [h(x_3), h(x_1), d(x_3)] + [h(x_3), h(x_2), d(x_1)]\\
&+ [h(x_3), h(x_2), d(x_2)] + [h(x_3), h(x_2), d(x_3)] + [h(x_3), h(x_3), d(x_1)] + [h(x_3), h(x_3), d(x_2)] + [h(x_3), h(x_3), d(x_3)],
\end{aligned}$$

for all $x_1, x_2, x_3 \in \mathfrak{A}$. We have the above two relations

$$\begin{aligned}
&d([x_1, x_2, x_3] + [x_2, x_3, x_1] + [x_3, x_1, x_2]) = [d(x_1), h(x_2), h(x_3)] + [h(x_1), d(x_2), h(x_3)] + [h(x_1), h(x_2), d(x_3)]\\
&+ [d(x_2), h(x_3), h(x_1)] + [h(x_2), d(x_3), h(x_1)] + [h(x_2), h(x_3), d(x_1)]\\
&+ [d(x_3), h(x_1), h(x_2)] + [h(x_3), d(x_1), h(x_2)] + [h(x_3), h(x_1), d(x_2)],
\end{aligned}$$

for all $x_1, x_2, x_3 \in \mathfrak{A}$. Now, for the converse proof, putting $x_1 = x_2 = x_3 = x$ in (2.7), we get

$$d([x, x, x]) = [d(x), h(x), h(x)] + [h(x), d(x), h(x)] + [h(x), h(x), d(x)],$$

for all $x_1, x_2, x_3, x \in \mathfrak{A}$. According to the above proof, we proved that (2.6) and (2.7) are equivalent, which completes this proof. □

In the following, we give Hyers-Ulam stability of 3D-Jensen ρ-functional equations on ternary Banach algebras. Assume that $\varphi, \psi : \mathfrak{A}^3 \to [0, \infty)$ be a function satisfies condition

$$\varphi(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}) \leq \frac{L}{2}\varphi(x, y, z), \quad \forall x, y, z \in \mathfrak{A} \tag{2.8}$$

$$\psi(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}) \leq \frac{L}{2^3}\psi(x, y, z), \quad \forall x, y, z \in \mathfrak{A}, \tag{2.9}$$

some $0 < L < 1$. Therefore $\varphi(0, 0, 0) = 0$. Clearly, by induction one can obtain that

$$2^n\varphi(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}) \leq L^n\varphi(x, y, z), \quad \forall n \in \mathbb{N}, \tag{2.10}$$

$$2^{3n}\psi(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}) \leq L^n\psi(x, y, z), \quad \forall n \in \mathbb{N}. \tag{2.11}$$

Theorem 2.5. Let $f : \mathfrak{A} \to \mathfrak{A}$ be a mapping satisfies

$$\begin{aligned} &\|f(\frac{x+y}{2}+z)+f(\frac{x+z}{2}+y)+f(\frac{y+z}{2}+x)-2f(x)-2f(y)-2f(z)- \\ &\rho(f(x+y+z)+f(x)-f(x+z)-f(x+y))\| \leq \varphi(x,y,z), \end{aligned} \tag{2.12}$$

where φ fulfills (2.8). Then there exists a unique additive $T : \mathfrak{A} \to \mathfrak{A}$, such that

$$\|f(x)-T(x)\| \leq \frac{1}{1-L}\varphi(x,0,0).$$

Proof . Let $x = y = z = 0$ in (2.12), we have $f(0) = 0$ and putting $y = z = 0$ in (2.12), we get

$$\|2f(\frac{x}{2})-f(x)\| \leq \varphi(x,0,0), \tag{2.13}$$

for all $x \in \mathfrak{A}$. Let Ω be the set of all functions $h : \mathfrak{A} \to \mathfrak{A}$ with $h(0) = 0$. Define the mapping $\Lambda : \Omega \to \Omega$ by $\Lambda(h)(x) = 2h(\frac{x}{2})$ and for every $h, k \in \Omega$ and $x \in \mathfrak{A}$ define

$$d(h,k) = \inf\{\beta > 0 : \ \|h(x)-k(x)\| \leq \beta\varphi(x,0,0)\},$$

where $\inf \emptyset = +\infty$. It is easy to show that d is a generalized metric on Ω and (Ω, d) is a complete generalized metric space. It follows from (2.13) that $d(f, \Lambda f) \leq 1$.

By theorem Diaz, there exists a mapping $T : \mathfrak{A} \to \mathfrak{A}$ such that mapping T is the unique fixed point of Λ in the set $\Gamma = \{h \in \Omega : \ d(f,h) < \infty\}$ and $\lim_{n\to\infty} \Lambda^n T(x) = T(x)$. This implies that T is a unique mapping such that there exists a $\beta \in (0,\infty)$ satisfying

$$\|f(x)-T(x)\| \leq \beta\varphi(x,0,0).$$

Also we have $d(f,h) \leq \frac{1}{1-L}$, which implies that

$$\|f(x)-T(x)\| \leq \frac{1}{1-L}\varphi(x,0,0).$$

It follows (2.10) and (2.12) that

$$\begin{aligned} &\|T(\frac{x+y}{2}+z)+T(\frac{x+z}{2}+y)+T(\frac{y+z}{2}+x)-2T(x)-2T(y)-2T(z) \\ &-\rho(T(x+y+z)+T(x)-T(x+z)-T(x+y))\| \\ &= \lim_{n\to\infty} 2^n \|f(\frac{x+y}{2^{n+1}}+\frac{z}{2^n})+f(\frac{x+z}{2^{n+1}}+\frac{y}{2^n})+f(\frac{y+z}{2^{n+1}}+\frac{x}{2^n}) \\ &-2f(\frac{x}{2^n})-2f(\frac{y}{2^n})-2f(\frac{z}{2^n})-\rho(f(\frac{x+y+z}{2^n})+f(\frac{x}{2^n})-f(\frac{x+z}{2^n})-f(\frac{x+y}{2^n}))\| \\ &\leq \lim_{n\to\infty} 2^n \varphi(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}) = 0, \end{aligned}$$

for all $x, y, z \in \mathfrak{A}$. By lemma 2.1 T is additive mapping and the proof is complete. □

Corollary 2.6. Let $r \neq 1$ and θ be nonnegative real numbers, and let $f : \mathfrak{A} \to \mathfrak{A}$ be a mapping satisfying

$$\begin{aligned} &\|f(\frac{x+y}{2}+z)+f(\frac{x+z}{2}+y)+f(\frac{y+z}{2}+x)-2f(x)-2f(y)-2f(z) \\ &-\rho(f(x+y+z)+f(x)-f(x+z)-f(x+y))\| \leq \theta(\|x\|^r+\|y\|^r+\|z\|^r), \end{aligned}$$

for all $x, y, z \in \mathfrak{A}$. Then there exists a unique additive mapping $T : \mathfrak{A} \to \mathfrak{A}$ such that

$$\|f(x)-T(x)\| \leq \frac{2^r\theta}{2^r-2}\|x\|^r \quad \text{for} \quad r < 1,$$

$$\|f(x)-T(x)\| \leq \frac{2^r\theta}{2-2^r}\|x\|^r \quad \text{for} \quad r > 1.$$

Proof . The proof follows from previous theorem by taking

$$\varphi(x,y,z)=\theta(\|x\|^r+\|y\|^r+\|z\|^r).$$

Then we can choose $L=2^{1-r}$ or $L=2^{r-1}$ and we get the desired result. □ For simplicity, denote

$$\Delta_\rho f_\lambda(x,y,z)=f(\frac{\lambda x+\lambda y}{2}+\lambda z)+f(\frac{\lambda x+\lambda z}{2}+\lambda y)+f(\frac{\lambda y+\lambda z}{2}+\lambda x)-2\lambda f(x)-2\lambda f(y)-2\lambda f(z)\\ -\rho(f(\lambda x+\lambda y+\lambda z)+\lambda f(x)-\lambda f(x+z)-\lambda f(x+y)),$$

and

$$\begin{aligned}&\mathcal{D}_h f([x_1,x_2,x_3]+[x_2,x_3,x_1]+[x_3,x_1,x_2]):=\\ &f([x_1,x_2,x_3]+[x_2,x_3,x_1]+[x_3,x_1,x_2])-[f(x_1),h(x_2),h(x_3)]-[h(x_1),f(x_2),h(x_3)]\\ &-[h(x_1),h(x_2),f(x_3)]-[f(x_2),h(x_3),h(x_1)]-[h(x_2),f(x_3),h(x_1)]-[h(x_2),h(x_3),f(x_1)]\\ &-[f(x_3),h(x_1),h(x_2)]-[h(x_3),f(x_1),h(x_2)]-[h(x_3),h(x_1),f(x_2)],\end{aligned}$$

for all $x,y,z,x_1,x_2,x_3\in\mathfrak{A}$. In the following, we prove the Hyers-Ulam stability of ternary Hom-Jordan derivations on ternary Banach algebras for the functional equation (1.1).

Theorem 2.7. Let $f,h:\mathfrak{A}\to\mathfrak{A}$ are two mappings satisfying

$$\|\Delta_\rho f_\lambda(x,y,z)\|\le\varphi(x,y,z), \tag{2.14}$$

$$\|\Delta_\rho h(x,y,z)\|\le\varphi(x,y,z), \tag{2.15}$$

$$\|h([x_1,x_2,x_3])-[h(x_1),h(x_2),h(x_3)]\|\le\psi(x_1,x_2,x_3), \tag{2.16}$$

$$\|\mathcal{D}_h f([x_1,x_2,x_3]+[x_2,x_3,x_1]+[x_3,x_1,x_2])\|\le\psi(x_1,x_2,x_3), \tag{2.17}$$

where φ and ψ satisfying conditions (2.8) and (2.9) for some constant $0<L<1$. Then there exists a unique ternary homomorphism $H:\mathfrak{A}\to\mathfrak{A}$ and unique ternary Hom-Jordan derivation $D:\mathfrak{A}\to\mathfrak{A}$, such that

$$\|h(x)-H(x)\|\le\frac{1}{1-L}\varphi(x,0,0),\quad \|f(x)-D(x)\|\le\frac{1}{1-L}\varphi(x,0,0).$$

Proof . First of all, let $\lambda=1$ in (2.14) and let Ω, d and Λ be those as defined in the proof of theorem 2.5, as a result, there exist unique mappings $H,D:\mathfrak{A}\to\mathfrak{A}$ such that

$$H(x)=\lim_{n\to\infty}2^n h(\frac{x}{2^n}), \tag{2.18}$$

$$D(x)=\lim_{n\to\infty}2^n f(\frac{x}{2^n}), \tag{2.19}$$

and satisfying (2.14), (2.15), (2.16) and (2.17) as desired. By attention to (2.16) and (2.18) we have

$$\begin{aligned}\|H([x_1,x_2,x_3]-[H(x_1),H(x_2),H(x_3)])\|&=\lim_{n\to\infty}2^{3n}\|f([\frac{x_1}{2^n},\frac{x_2}{2^n},\frac{x_3}{2^n}]-[f(\frac{x_1}{2^n}),f(\frac{x_2}{2^n}),f(\frac{x_3}{2^n})])\|\\ &\le\lim_{n\to\infty}2^{3n}\psi(\frac{x_1}{2^n},\frac{x_2}{2^n},\frac{x_3}{2^n})\\ &\le L^n\psi(x_1,x_2,x_3)\\ &=0,\end{aligned}$$

as a result, H is a ternary homomorphism. It follows (2.17) and (2.19), imply that $\mathcal{D}_h D$ is a ternary Hom-Jordan derivation

$$\begin{aligned}\|\mathcal{D}_h D([x_1,x_2,x_3]+[x_2,x_3,x_1]+[x_3,x_1,x_2])\|&=\lim_{n\to\infty}2^{3n}\|\mathcal{D}_h f([\frac{x_1}{2^n},\frac{x_2}{2^n},\frac{x_3}{2^n}]+[\frac{x_2}{2^n},\frac{x_3}{2^n},\frac{x_1}{2^n}]+[\frac{x_3}{2^n},\frac{x_1}{2^n},\frac{x_2}{2^n}])\|\\ &\le\lim_{n\to\infty}2^{3n}\psi(\frac{x_1}{2^n},\frac{x_2}{2^n},\frac{x_3}{2^n})\\ &\le L^n\psi(x_1,x_2,x_3)\\ &=0.\end{aligned}$$

Now, the proof is complete. □

Corollary 2.8. Let $r < 1$ and θ be two elements of $\mathbb{R}^+$. and θ be nonnegative real numbers, and let $f, h : \mathfrak{A} \to \mathfrak{A}$ are two mappings satisfying

$$\|\Delta_\rho f_\lambda(x,y,z)\| \leq \theta(\|x\|^r + \|y\|^r + \|z\|^r)$$
$$\|\Delta_\rho h(x,y,z)\| \leq \theta(\|x\|^r + \|y\|^r + \|z\|^r)$$
$$\|h([x_1,x_2,x_3]) - [h(x_1),h(x_2),h(x_3)]\| \leq \theta(\|x_1\|^r + \|x_2\|^r + \|x_3\|^r),$$
$$\|\mathcal{D}_h f([x_1,x_2,x_3] + [x_2,x_3,x_1] + [x_3,x_1,x_2])\| \leq \theta(\|x_1\|^r + \|x_2\|^r + \|x_3\|^r),$$

for all $x, y, z \in \mathfrak{A}$. Then there exists unique ternary homomorphism H and unique ternary Hom-Jordan derivation D such that

$$\|h(x) - H(x)\| \leq \frac{2^r\theta}{2^r - 2}\|x\|^r,$$
$$\|f(x) - D(x)\| \leq \frac{2^r\theta}{2^r - 2}\|x\|^r.$$

Proof . The proof follows from previous theorem by taking

$$\varphi(x,y,z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r) \qquad \psi(x_1,x_2,x_3) = \theta(\|x_1\|^r + \|x_2\|^r + \|x_3\|^r).$$

Then we can choose $L = 2^{1-r}$ and we get the desired result. □

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