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NEW INEQUALITIES FOR A CLASS OF DIFFERENTIABLE FUNCTIONS

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ABSTRACT. In this paper, we use the Riemann-Liouville fractional integrals to establish some new integral inequalities related to Chebyshev's functional in the case of two differentiable functions.

1. Introduction and Basic Definitions

Let us consider

$$T(f,g) := \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{b-a}\left(\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right) \tag{1.1}$$

where f and g are two integrable functions on $[a,b]$ [4].
The relation (1.1) has evoked the interest of many researchers and several inequalities related to this functional have appeared in the literature, to mention a few, see [1, 2, 6, 7] and the references cited therein.
The main aim of this paper is to establish some new inequalities for (1.1) by using the Riemann-Liouville fractional integrals. We give our results in the case of differentiable functions.
We shall introduce the following spaces which are used throughout this paper.
Let $C([0,\infty[)$ the space of all continuous functions from $[0,\infty[$ into $\mathbb{R}$ and let $L_\infty([0,\infty[)$ the space of essentially bounded functions $f(x)$ on $[0,\infty[$, with the norm

$$||f||_\infty := inf\{C \geq 0, |f(x)| \leq C; for\ almost\ every\ x \in [0,\infty[\}.$$

For the Riemann-Liouville integrals, we give the following definitions and properties.

Definition 1.1. A real valued function $f(t), t \geq 0$ is said to be in the space $C_\mu, \mu \in \mathbb{R}$ if there exists a real number $p > \mu$ such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C([0,\infty[)$.

Definition 1.2. A function $f(t), t \geq 0$ is said to be in the space $C_\mu^n, \mu \in \mathbb{R}$, if $f^{(n)} \in C_\mu$

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Definition 1.3. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a function $f \in C_\mu, (\mu \geq -1)$ is defined as

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau; \quad \alpha > 0, t > 0, \\ J^0 f(t) = f(t), \tag{1.2}$$

where $\Gamma(\alpha) := \int_0^\infty e^{-u} u^{\alpha-1} du$.

For the convenience of establishing the results, we give the semigroup property:

$$J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t), \alpha \geq 0, \beta \geq 0, \tag{1.3}$$

which implies the commutative property

$$J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t). \tag{1.4}$$

For more details, one can consult [8].

2. Main Results

Theorem 2.1. *Let f and g be two differentiable functions on $[0, \infty[$ such that $f', g' \in L_\infty([0, \infty[)$. Then for all $t > 0, \alpha > 0$, we have:*

$$\Big| \frac{t^\alpha}{\Gamma(\alpha+1)} J^\alpha fg(t) - J^\alpha f(t) J^\alpha g(t) \Big| \\ \leq ||f'||_\infty ||g'||_\infty \Big[\frac{t^\alpha}{\Gamma(\alpha+1)} J^\alpha t^2 - (J^\alpha t)^2 \Big]. \tag{2.1}$$

Proof. Let f and g be two functions satisfying the conditions of Theorem 2.1. Define

$$H(\tau, \rho) := (f(\tau) - f(\rho))(g(\tau) - g(\rho)); \ \tau, \rho \in (0, t), t > 0. \tag{2.2}$$

Multiplying (2.2) by $\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}$; $\tau \in (0, t)$ and integrating the resulting identity with respect to τ from 0 to t, we can state that

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} H(\tau, \rho) d\tau \\ = J^\alpha fg(t) - f(\rho) J^\alpha g(t) - g(\rho) J^\alpha f(t) + f(\rho) g(\rho) \frac{t^\alpha}{\Gamma(\alpha+1)}. \tag{2.3}$$

Now, multiplying (2.3) by $\frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}$; $\rho \in (0, t)$ and integrating the resulting identity with respect to ρ over $(0, t)$, we can write

$$\frac{1}{\Gamma^2(\alpha)} \int_0^t \int_0^t (t-\tau)^{\alpha-1} (t-\rho)^{\alpha-1} H(\tau, \rho) d\tau d\rho \\ = 2\Big(\frac{t^\alpha}{\Gamma(\alpha+1)} J^\alpha fg(t) - J^\alpha f(t) J^\alpha g(t) \Big). \tag{2.4}$$

On the other hand, we have

$$H(\tau, \rho) = \int_\tau^\rho \int_\tau^\rho f'(y) g'(z) dy dz. \tag{2.5}$$

Since $f', g' \in L_\infty([0,\infty[)$, then we can write

$$|H(\tau,\rho)| \leq \Big|\int_\tau^\rho f'(y)dy\Big|\Big|\int_\tau^\rho g'(z)dz\Big| \leq ||f'||_\infty ||g'||_\infty (\tau-\rho)^2. \tag{2.6}$$

Consequently,

$$\begin{aligned}&\frac{1}{\Gamma^2(\alpha)}\int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}|H(\tau,\rho)|d\tau d\rho\\ &\leq \tfrac{||f'||_\infty ||g'||_\infty}{\Gamma^2(\alpha)}\int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}(\tau^2-2\tau\rho+\rho^2)d\tau d\rho.\end{aligned} \tag{2.7}$$

Thus, we obtain the following estimate

$$\begin{aligned}&\frac{1}{\Gamma^2(\alpha)}\int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}|H(\tau,\rho)|d\tau d\rho\\ &\leq ||f'||_\infty ||g'||_\infty\Big[\tfrac{t^\alpha}{\Gamma(\alpha+1)}J^\alpha t^2 - 2(J^\alpha t)^2 + \tfrac{t^\alpha}{\Gamma(\alpha+1)}J^\alpha t^2\Big].\end{aligned} \tag{2.8}$$

By the relations (2.4),(2.8) and using the properties of the modulus, we get the desired inequality (2.1). □

Remark 2.2. Applying Theorem 2.1 for $\alpha = 1$, we obtain (Corollary 6.2 of[7] on $[0,t]$):

$$\Big|t\int_0^t f(\tau)g(\tau)d\tau - \Big(\int_0^t f(\tau)d\tau\Big)\Big(\int_0^t g(\tau)d\tau\Big)\Big| \leq t^4/12.$$

Our next result is the following theorem, in which we use two real positive parameters.

Theorem 2.3. *Let f and g be two differentiable functions on $[0,\infty[$ such that $f', g' \in L_\infty([0,\infty[)$. Then for all $t>0, \alpha>0, \beta>0$, we have*

$$\begin{aligned}&\Big|\frac{t^\alpha}{\Gamma(\alpha+1)}J^\beta fg(t) + \frac{t^\beta}{\Gamma(\beta+1)}J^\alpha fg(t) - J^\alpha f(t)J^\beta g(t) - J^\beta f(t)J^\alpha g(t)\Big|\\ &\leq ||f'||_\infty ||g'||_\infty\Big[\tfrac{t^\alpha}{\Gamma(\alpha+1)}J^\beta t^2 - 2(J^\alpha t)(J^\beta t) + \tfrac{t^\beta}{\Gamma(\beta+1)}J^\alpha t^2\Big].\end{aligned} \tag{2.9}$$

Proof. The relation (2.3) implies that

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}H(\tau,\rho)d\tau d\rho\\ &= \tfrac{t^\alpha}{\Gamma(\alpha+1)}J^\beta fg(t) + \tfrac{t^\beta}{\Gamma(\beta+1)}J^\alpha fg(t) - J^\alpha f(t)J^\beta g(t) - J^\beta f(t)J^\alpha g(t).\end{aligned} \tag{2.10}$$

On the other hand, the relation (2.6) implies that

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}|H(\tau,\rho)|d\tau d\rho\\ &\leq \tfrac{||f'||_\infty ||g'||_\infty}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}(\tau-\rho)^2 d\tau d\rho.\end{aligned} \tag{2.11}$$

Using (2.10) and (2.11), we get the inequality (2.9). □

Remark 2.4. Applying Theorem 2.3 for $\alpha = \beta$ we obtain Theorem 2.1.

The following results have some applications in the perturbed quadrature rules (see, for example, [3, 5]).

Theorem 2.5. *Let f and g be two differentiable functions on $[0, \infty[$ with $g'(t) \neq 0, t \in [0, \infty[$. If there exists a constant $M > 0$ such that $\left|\frac{f'(t)}{g'(t)}\right| \leq M$, then for all $\alpha > 0, \beta > 0$, we have*

$$\begin{aligned} &\Big|\frac{t^\alpha}{\Gamma(\alpha+1)} J^\beta fg(t) + \frac{t^\beta}{\Gamma(\beta+1)} J^\alpha fg(t) - J^\alpha f(t) J^\beta g(t) - J^\beta f(t) J^\alpha g(t)\Big| \\ &\leq M\Big[\tfrac{t^\beta}{\Gamma(\beta+1)} J^\alpha g^2(t) + \tfrac{t^\alpha}{\Gamma(\alpha+1)} J^\beta g^2(t) - 2J^\alpha g(t) J^\beta g(t)\Big]. \end{aligned} \tag{2.12}$$

Proof. Let f and g be two functions satisfying the conditions of Theorem 2.5. Then for every $\tau, \rho \in [0, t]; \tau = \rho, t > 0$ there exists a c between τ and ρ so that

$$\frac{f(\tau) - f(\rho)}{g(\tau) - g(\rho)} = \frac{f'(c)}{g'(c)}.$$

Hence for every $\tau, \rho \in [0, t]; t > 0$, we have

$$|f(\tau) - f(\rho)| \leq M|g(\tau) - g(\rho)|. \tag{2.13}$$

This implies that

$$\Big|H(\tau, \rho)\Big| \leq M\Big(g(\tau) - g(\rho)\Big)^2, \tau, \rho \in [0, t]. \tag{2.14}$$

Then, it follows that

$$\begin{aligned} &\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_0^t (t-\tau)^{\alpha-1} (t-\rho)^{\beta-1} |H(\tau, \rho)| d\tau d\rho \\ &\leq \tfrac{M}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_0^t (t-\tau)^{\alpha-1} (t-\rho)^{\beta-1} \Big(g^2(\tau) - 2g(\tau)g(\rho) + g^2(\rho)\Big) d\tau d\rho. \end{aligned} \tag{2.15}$$

Therefore,

$$\begin{aligned} &\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_0^t (t-\tau)^{\alpha-1} (t-\rho)^{\beta-1} |H(\tau, \rho)| d\tau d\rho \\ &\leq M\Big[\tfrac{t^\beta}{\Gamma(\beta+1)} J^\alpha g^2(t) + \tfrac{t^\alpha}{\Gamma(\alpha+1)} J^\beta g^2(t) - 2J^\alpha g(t) J^\beta g(t)\Big]. \end{aligned} \tag{2.16}$$

Theorem 2.5 is thus proved. □

Corollary 2.6. *Let f and g be two differentiable functions on $[0, \infty[$; with $g'(t) \neq 0, t \in [0, \infty[$. If there exists a constant $M > 0$ such that $\left|\frac{f'(t)}{g'(t)}\right| \leq M$, then for all $\alpha > 0$, we have:*

$$\begin{aligned}&\Big|\frac{t^{\alpha}}{\Gamma(\alpha+1)}J^{\alpha}fg(t)-J^{\alpha}f(t)J^{\alpha}g(t)\Big|\\&\leq M\Big[\tfrac{t^{\alpha}}{\Gamma(\alpha+1)}J^{\alpha}g^{2}(t)-(J^{\alpha}g(t))^{2}\Big].\end{aligned}\tag{2.17}$$

Proof. We apply Theorem 2.5 for $\alpha=\beta$. □

Remark 2.7. Applying Theorem 2.5 for $\alpha=\beta=1$, we obtain (Corollary 4.2 of[7] on $[0,t]$):

$$\begin{aligned}&\Big|t\int_0^t f(\tau)g(\tau)d\tau-\Big(\int_0^t f(\tau)d\tau\Big)\Big(\int_0^t g(\tau)d\tau\Big)\Big|\\&M\leq\Big[t\int_0^t g^{2}(\tau)d\tau-\Big(\int_0^t g(\tau)d\tau\Big)^{2}\Big].\end{aligned}\tag{2.18}$$

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