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ON THE NATURE OF SOLUTIONS OF THE DIFFERENCE EQUATION $X_{N+1} = X_N X_{N-3} - 1$

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ABSTRACT. We investigate the long-term behavior of solutions of the difference equation

$$x_{n+1} = x_n x_{n-3} - 1\,, \quad n = 0\,, 1\,, \ldots\,,$$

where the initial conditions $x_{-3}\,, x_{-2}\,, x_{-1}\,, x_0$ are real numbers. In particular, we look at the periodicity and asymptotic periodicity of solutions, as well as the existence of unbounded solutions.

1. Introduction and preliminaries

Recently there has been a surge of interest in studying nonlinear difference equations which do not stem from differential equations (see, for example, [1]–[37] and the references therein). Usual properties which have been studied are the boundedness character ([8, 14, 16], [29]–[31], [34, 36, 37]), the periodicity [8, 14], asymptotic periodicity ([17]–[20], [22]), local and global stability ([1, 8, 14, 16, 17], [29]–[35]), as well as the existence of specific solutions such as those that are monotonic or nontrivial ([2, 3, 5, 9, 10, 16, 17], [23]–[28]).

In this paper we study solutions of the difference equation

$$x_{n+1} = x_n x_{n-3} - 1\,, \quad n = 0\,, 1\,, \ldots\,. \tag{1.1}$$

Equation (1.1) belongs to the class of equations of the form

$$x_{n+1} = x_{n-l} x_{n-k} - 1\,, \quad n = 0\,, 1\,, \ldots\,, \tag{1.2}$$

with particular choices of k and l, where $k\,, l \in \{0\,, 1\,, \ldots\}$. The relatively simple appearance of Eq.(1.2) is deceiving in that its behavior changes significantly for different choices of k and l. The cases (1) $l = 0$ and $k = 1$, (2) $l = 1$ and $k = 2$, and (3) $l = 0$ and $k = 2$ have recently been investigated in papers [10, 11, 12]. This paper can be regarded as a continuation of our systematic investigation of Eq.(1.2).

Note that Eq.(1.2) has two equilibria:

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$$\bar{x}_1 = \frac{1-\sqrt{5}}{2} \quad \text{and} \quad \bar{x}_2 = \frac{1+\sqrt{5}}{2}$$

(i.e., the Golden Number and its conjugate). For the sake of comparison in the sequel, we summarize the important results for the three different choices of k and l in the general equation $x_{n+1} = x_{n-l}x_{n-k} - 1$.

(1) $l = 0\,, k = 1$: The negative equilibrium $\bar{x}_1$ is locally asymptotically stable, and the positive equilibrium $\bar{x}_2$ is unstable and hyperbolic. There are no eventually constant solutions, and there do not exist periodic solutions with prime period two. There exist exactly three prime period-three solutions with the following sets of initial values:
   (i) $x_{-1} = x_0 = -1$;
   (ii) $x_{-1} = -1\,, x_0 = 0$;
   (iii) $x_{-1} = 0\,, x_0 = -1$.

   The interval $(-1\,,0)$ is invariant, and it is conjectured that every solution in $(-1\,,0)$ converges to the negative equilibrium $\bar{x}_1$ contained in $(-1\,,0)$. A solution $\{x_n\}_{n=-1}^{\infty}$ is unbounded if, for example,
   (i) $x_{-1}\,, x_0 > \bar{x}_2$, in which case the solution is strictly increasing to $+\infty$; or
   (ii) $x_{-1}\,, x_0 < -1$, in which case the subsequences $\{x_{3n}\}_{n=0}^{\infty}$, $\{x_{3n+2}\}_{n=0}^{\infty}$ are strictly decreasing to $-\infty$ and the subsequence $\{x_{3n+1}\}_{n=0}^{\infty}$ is strictly increasing to $+\infty$.

(2) $l = 1\,, k = 2$: The negative equilibrium $\bar{x}_1$ is nonhyperbolic (its stability nature is unknown) and the positive equilibrium $\bar{x}_2$ is unstable. There are no eventually constant solutions, and there do not exist periodic solutions with prime period two, three, or four. There exist infinitely many period-five solutions each of which has its set of initial values satisfy one of the following:
   (i) $x_{-2} = a\,, x_{-1} = b\,, x_0 = \dfrac{a+1}{ab-1}$, $a\,, b \in \mathbf{R}$, $ab \neq -1$;
   (ii) $x_{-2} = -1 = x_{-1} = -1\,, x_0 = c$, $c \in \mathbf{R}$;
   (iii) $x_{-2} = a\,, x_{-1} = 0\,, x_0 = a - 1$, $a \in \mathbf{R}$.

   The interval $(-1\,,0)$ is invariant, and every solution in $(-1\,,0)$ converges to a period five solution in $(-1\,,0)$. A solution $\{x_n\}_{n=-1}^{\infty}$ is unbounded if, for example,
   (i) $x_{-2}\,, x_{-1}\,, x_0 > \bar{x}_2$, in which case each of the subsequences $\{x_{2n-1}\}_{n=0}^{\infty}$, $\{x_{2n}\}_{n=0}^{\infty}$ is strictly decreasing to $-\infty$; or
   (ii) $x_{-2} = a\,, x_{-1} = b\,, x_0 > \dfrac{a+1}{ab-1}$, $a\,, b > 0$ and $ab > 1$, in which case the solution tends to $+\infty$.

(3) $l = 0\,, k = 2$: The negative equilibrium $\bar{x}_1$ and the positive equilibrium $\bar{x}_2$ are both unstable and hyperbolic. There are no eventually constant solutions, and there do not exist periodic solutions with prime period three or four. There exist exactly two prime period-two solutions with the following sets of initial values:
   (i) $x_{-2} = 0\,, x_{-1} = -1\,, x_0 = 0$;
   (ii) $x_{-2} = -1\,, x_{-1} = 0\,, x_0 = -1$.

   The interval $(-1\,,0)$ is invariant, and under certain conditions, a solution in $(-1\,,0)$ converges to the two-cycle $\{-1\,,0\}$. There also exist solutions which converge to the negative equilibrium $\bar{x}_1$ in $(-1\,,0)$. A solution $\{x_n\}_{n=-1}^{\infty}$ is unbounded if, for example, $x_{-1}\,,x_0 > \bar{x}_2$, in which case the solution is strictly increasing to $+\infty$.

2. Periodic solutions

In this section we prove some results regarding periodicity of solutions of Eq.(1.1). Note that when $l = 1$ and $k = 2$ in the general equation

$$x_{n+1} = x_{n-l}x_{n-k} - 1\,,$$

the results obtained are essentially the same as those below, i.e., there are no prime period-two, three, or four solutions and there exist infinitely many period-five solutions (see [11]).

Theorem 2.1. *Eq.(1.1) has no eventually constant solutions.*

Proof. Suppose that $\{x_n\}_{n=-3}^{\infty}$ is an eventually constant solution, $\bar{x}$, of Eq.(1.1), which cannot be zero. Then there is an integer $N \geq -2$ such that $x_N = x_{N+1} = x_{N+2} = x_{N+3} = \cdots = \bar{x}$. In this case, Eq.(1.1) gives $x_{N+3} = x_{N+2}x_{N-1} - 1$, which implies that $\bar{x} = \bar{x}x_{N-1} - 1$, and thus we have that

$$x_{N-1} = \frac{\bar{x}+1}{\bar{x}} = \bar{x}\,.$$

Repeating this procedure, we obtain $x_n = \bar{x}$ for $n \geq -3$, a contradiction. Hence, there are no eventually constant solutions. □

Theorem 2.2. *Eq.(1.1) has no prime period-two solutions.*

Proof. Suppose there exists a prime period-two solution, $\{x_n\}_{n=-3}^{\infty}$, of Eq.(1.1), with initial values $x_{-3} = a\,, x_{-2} = b\,, x_{-1} = a\,, x_0 = b$, where $a\,,b \in \mathbf{R}$ and $a \neq b$. Then $x_1 = ab - 1 = a$ and $x_2 = ab - 1 = b$.
It follows that $a = b$, a contradiction. □

Theorem 2.3. *Eq.(1.1) has no prime period-three solutions.*

Proof. Suppose there exists a prime period-three solution, $\{x_n\}_{n=-3}^{\infty}$, of Eq.(1.1), with initial values $x_{-3} = a\,, x_{-2} = b\,, x_{-1} = c\,, x_0 = a$, where $a\,,b\,,c \in \mathbf{R}$. Then, upon computation of $x_1\,,x_2\,,x_3$, we obtain the three equations

$$a^2 - 1 = b\,, \tag{2.1}$$

$$b^2 - 1 = c\,, \tag{2.2}$$

and

$$c^2 - 1 = a\,. \tag{2.3}$$

From Eqs.(2.1)-(2.3), we have that a, b, and c all satisfy the equation $x^8 - 4x^6 + 4x^4 - x - 1 = 0$, which is equivalent to $(x^2 - x - 1)(x^6 + x^5 - 2x^4 - x^3 + x^2 + 1) = 0$. Suppose that a satisfies $x^2 - x - 1 = 0$. Then either $a = \bar{x}_1$ or $a = \bar{x}_2$. By Eqs.(2.1)-(2.3), it follows that either

$$a = b = c = \bar{x}_1 \quad \text{or} \quad a = b = c = \bar{x}_2\,.$$

Hence, $\{x_n\}_{n=-3}^{\infty}$ is the trivial solution, which is impossible. Consequently, we must have that a satisfies the equation $x^6+x^5-2x^4-x^3+x^2+1 = 0$. By similar arguments as above, we must also have that b and c satisfy $x^6 + x^5 - 2x^4 - x^3 + x^2 + 1 = 0$.

Now, let

$$P(x) = x^6 + x^5 - 2x^4 - x^3 + x^2 + 1\,.$$

By Descartes' rule, P has
(1) either two positive zeros or none;
(2) either two negative zeros or none.
We claim that if P has two positive zeros, then both must be less than one. For, if we let

$$f(x) = x^6 + x^5 + x^2 + 1 \quad \text{and} \quad g(x) = 2x^4 + x^3\,,$$

then $f(1) > g(1)$ and, for $x \geq 1$,

$$f'(x) = 6x^5 + 5x^4 + 2x > 6x^3 + 5x^3 = 11x^3 = 8x^3 + 3x^3 \geq 8x^3 + 3x^2 = g'(x)\,.$$

Thus, $a\,, b\,, c \in (-\infty\,, 0) \cup (0\,, 1)$. Suppose $a \in (0\,, 1)$. Then, by Eqs.(2.1) and (2.2), $c \in (-1\,, 0)$. This, together with Eq.(2.3), implies that $a \in (-1\,, 0)$, a contradiction. Therefore, $a \in (-\infty\,, 0)$. Similarly, $b\,, c \in (-\infty\,, 0)$. However, there cannot be greater than two negative zeros of P. Thus, $a = b$, $a = c$, or $b = c$. From Eqs.(2.1)-(2.3), we then have

$$a = b = c = \bar{x}_1 \quad \text{or} \quad a = b = c = \bar{x}_2\,,$$

and therefore $\{x_n\}_{n=-3}^{\infty}$ is the trivial solution, which again is impossible.

We conclude that $a\,, b\,, c$, as the initial values of the *prime* period-three solution $\{x_n\}_{n=-3}^{\infty}$ of Eq.(1.1), do not satisfy the equation $x^8 - 4x^6 + 4x^4 - x - 1 = 0$. This gives us a contradiction, and, hence, Eq.(1.1) has no prime period-three solutions. The proof is complete. ☐

Theorem 2.4. *Eq.(1.1) has no prime period-four solutions.*

Proof. Let $\{x_n\}_{n=-3}^{\infty}$ be a period-four solution of Eq.(1.1), with initial values $x_{-3} = a\,, x_{-2} = b\,, x_{-1} = c\,, x_0 = d$, where $a\,, b\,, c\,, d \in \mathbf{R}$. We consider the following four cases and show that $\{x_n\}_{n=-3}^{\infty}$ must be trivial.

Case 1.: Suppose that $a = 0$. Then $x_1 = ad - 1 = -1 = a$, which is a contradiction.

Case 2.: Suppose that $a = 1$. Then $x_1 = a$ and so $x_2 = ab - 1 = -b - 1 = b$, from which it follows that $-1 = 0$, an impossibility.

Case 3.: Suppose that $b = 1$. Then $x_1 = a$, $x_2 = b$, and so $x_3 = bc-1 = c-1 = c$, which implies that $-1 = 0$. This is not possible.

Case 4.: Suppose that $c = 1$. Then $x_1 = a$, $x_2 = b$, $x_3 = c$, and so $x_4 = cd - 1 = d - 1 = d$, which once again implies that $-1 = 0$. This is not possible.

Therefore, $a \neq 0$, $a \neq 1$, $b \neq 1$, and $c \neq 1$, and from $x_1 = ad - 1 = a$, $x_2 = ab - 1 = b$, $x_3 = bc - 1 = c$, and $x_4 = cd - 1 = d$, we have that

$$d = \frac{a+1}{a}\,, \quad b = \frac{1}{a-1}\,, \quad c = \frac{1}{b-1}\,, \quad \text{and} \quad d = \frac{1}{c-1}\,, \tag{2.4}$$

respectively. We can then write

$$d = \frac{1}{c-1} = \frac{1}{\frac{1}{b-1} - 1} = \frac{b-1}{1-(b-1)} = \frac{b-1}{2-b} = \frac{\frac{1}{a-1} - 1}{2 - \frac{1}{a-1}} = \frac{1-(a-1)}{2(a-1)-1} = \frac{2-a}{2a-3}\,.$$

This, together with the fact that $d = \frac{a+1}{a}$, gives us $\frac{a+1}{a} = \frac{2-a}{2a-3}$, or, equivalently, $a^2 - a - 1 = 0$. Thus, $a = \bar{x}_1$ or $\bar{x}_2$, and so from (2.4)

$$a = b = c = d = \bar{x}_1 \quad \text{or} \quad a = b = c = d = \bar{x}_2\,.$$

Hence, $\{x_n\}_{n=-3}^{\infty}$ is the trivial solution, and we are done. □

We now present the main result of this section, which will play an important role in the sequel.

Theorem 2.5. *There exist prime period-five solutions of Eq.(1.1). Furthermore, if $\{x_n\}_{n=-3}^{\infty}$ is a prime period-five solution of Eq.(1.1), then its set of initial values satisfies one of the following:*

(i) $x_{-3} = 0\,, x_{-2} = -1\,, x_{-1} = -d-1\,, x_0 = d$ *(so $x_1 = -1$), $d \in \mathbf{R}$;*
(ii) $x_{-3} = -1\,, x_{-2} = 0\,, x_{-1} = -1\,, x_0 = d$ *(so $x_1 = -d-1$), $d \in \mathbf{R}$;*
(iii) $x_{-3} = a\,, x_{-2} = b\,, x_{-1} = \frac{b+1}{a}\,, x_0 = \frac{a+b+1}{ab}$ *(so $x_1 = \frac{a+1}{b}$), $a\,, b \in \mathbf{R}$ and $ab \neq 0$.*

Proof. Clearly, any solution $\{x_n\}_{n=-3}^{\infty}$ of Eq.(1.1) whose set of initial values satisfies

(i) $x_{-3} = 0\,, x_{-2} = -1\,, x_{-1} = -d-1\,, x_0 = d$ (so $x_1 = -1$), $d \in \mathbf{R}$, or
(ii) $x_{-3} = -1\,, x_{-2} = 0\,, x_{-1} = -1\,, x_0 = d$ (so $x_1 = -d-1$), $d \in \mathbf{R}$, or
(iii) $x_{-3} = a\,, x_{-2} = b\,, x_{-1} = \frac{b+1}{a}\,, x_0 = \frac{a+b+1}{ab}$ (so $x_1 = \frac{a+1}{b}$), $a\,, b \in \mathbf{R}$ and $ab \neq 0$,

is periodic with prime period five. Hence there exist infinitely many prime period-five solutions of Eq.(1.1).

Now, let $\{x_n\}_{n=-3}^{\infty}$ be a prime period-five solution of Eq.(1.1) with initial values $x_{-3} = a\,, x_{-2} = b\,, x_{-1} = c\,, x_0 = d$, where $a\,, b\,, c\,, d \in \mathbf{R}$. Then

$$\begin{aligned}
x_1 &= ad - 1 \equiv e\,,\\
x_2 &= be - 1 = (ad-1)b - 1 = a\,,\\
x_3 &= ac - 1 = b\,,\\
x_4 &= bd - 1 = c\,,\\
x_5 &= ce - 1 = d\,,\\
x_6 &= ad - 1 = e\,.
\end{aligned}$$

From this we obtain the following:

$$a = abd - b - 1\,, \tag{2.5}$$

$$a = be - 1\,, \tag{2.6}$$

$$b = ac - 1\,, \tag{2.7}$$

and

$$c = bd - 1\,. \tag{2.8}$$

We consider three cases:

Case 1.: Suppose $a = 0$. Then $x_{-3} = a = 0$. From Eq.(2.7) we have $x_{-2} = b = ac - 1 = -1$. Also, by Eq.(2.8) we have $x_{-1} = c = bd - 1 = -d - 1$, with $x_0 = d$. Note that $x_1 = ad - 1 = -1$. Simple calculations show that these five values repeat all over again. This gives the solution in Part (i).

Case 2.: Suppose $b = 0$. Then $x_{-3} = a$ and $x_{-2} = b = 0$. From Eq.(2.8) we have $x_{-1} = c = bd - 1 = -1$. Thus Eq.(2.8) gives us $a = -1$. Also $x_0 = d$. Then we have $x_1 = ad - 1 = -d - 1$. Simple calculations show that these five values repeat all over again. This gives the solution in Part (ii).

Case 3.: Suppose $a \neq 0$ and $b \neq 0$. Then we have the solution in Part (iii), where

$$x_{-3} = a\,;$$

$$x_{-2} = b\,;$$

$$x_{-1} = c = \frac{b+1}{a} \quad \text{from Eq.(2.7)};$$

$$x_0 = d = \frac{a+b+1}{ab} \quad \text{from Eq.(2.5)};$$

$$x_1 = e = \frac{a+1}{b} \quad \text{from Eq.(2.6)}.$$

This completes the proof. □

Theorem 2.6. *There exist solutions of Eq.(1.1) that are eventually periodic with prime period five.*

Proof. Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq.(1.1) with initial values $x_{-3} = a\,, x_{-2} = b\,, x_{-1} = c\,, x_0 = 0$, where $a\,, b\,, c \in \mathbf{R}$. Then we calculate

$$x_1 = -1\,, x_2 = -b - 1\,, x_3 = -bc - c - 1\,, x_4 = -1\,, x_5 = 0\,, x_6 = -1\,,$$
$$x_7 = bc + c\,, x_8 = -bc - b - 1\,.$$

From Theorem 2.5, it follows that $\{x_n\}_{n=4}^{\infty}$ is periodic with period five. □

Remark 2.7. There also exist eventually prime period-five solutions of Eq.(1.1) with $x_0 \neq 0$. For example, consider the solution $\{x_n\}_{n=-3}^{\infty}$ with initial values $x_{-3} = 1\,, x_{-2} = 1\,, x_{-1} = 1\,, x_0 = 1$.

Then

$$\{x_n\}_{n=-3}^{\infty} = 1\,,1\,,1\,,1\,,0\,,-1\,,-2\,,-3\,,-1\,,0\,,-1\,,2\,,\ldots\,,$$

and the last five values repeat.

Also, the solution $\{x_n\}_{n=-3}^{\infty}$ with initial values $x_{-3} = -1\,, x_{-2} = b\,, x_{-1} = c\,, x_0 = -1\,, b$, with $c \in \mathbf{R}$, is eventually periodic with period five, where

$$\{x_n\}_{n=-3}^{\infty} = -1\,,b\,,c\,,-1\,,0\,,-1\,,-c-1\,,c\,,-1\,,0\,,-1\,,-c-1\,,\ldots\,,$$

and the last five values repeat.

3. Local stability

Here we study the local stability of the equilibrium points $\bar{x}_1 = \dfrac{1-\sqrt{5}}{2}$ and $\bar{x}_2 = \dfrac{1+\sqrt{5}}{2}$.

Theorem 3.1. *The positive equilibrium of Eq.(1.1), $\bar{x}_2$, is unstable.*

Proof. The linearized equation associated with the equilibrium $\bar{x}_2$ is

$$x_{n+1} - \bar{x}_2 x_n - \bar{x}_2 x_{n-3} = 0\,.$$

Its characteristic polynomial is

$$P_{\bar{x}_2}(\lambda) = \lambda^4 - \bar{x}_2\lambda^3 - \bar{x}_2\,.$$

Since

$$P_{\bar{x}_2}(1) = 1 - 2\bar{x}_2 = -\sqrt{5} < 0 \quad \text{and} \quad \lim_{\lambda\to+\infty} P_{\bar{x}_2}(\lambda) = +\infty\,,$$

it follows that there is a $\lambda_0 > 1$ such that $P_{\bar{x}_2}(\lambda_0) = 0$, from which the result follows for the equilibrium $\bar{x}_2$. □

Remark 3.2. The negative equilibrium of Eq.(1.1), $\bar{x}_1$, turns out to be nonhyperbolic:

The linearized equation associated with the equilibrium $\bar{x}_1$ is

$$x_{n+1} - \bar{x}_1 x_n - \bar{x}_1 x_{n-3} = 0\,,$$

and so its characteristic polynomial is

$$P_{\bar{x}_1}(\lambda) = \lambda^4 - \bar{x}_1\lambda^3 - \bar{x}_1\,.$$

We then factor the characteristic polynomial as

$$P_{\bar{x}_1}(\lambda) = \left(x^2 - x - \frac{1-\sqrt{5}}{2}\right)\left(x^2 + \frac{1+\sqrt{5}}{2}x + 1\right)\,,$$

and determine its zeroes, which are

$$\lambda_1\,,\bar{\lambda}_1 = \frac{1}{2} \pm i\frac{1}{2}\sqrt{2\sqrt{5}-3}\,, \quad \lambda_2\,,\bar{\lambda}_2 = \frac{-1-\sqrt{5}}{4} \pm i\frac{1}{4}\sqrt{10-2\sqrt{5}}\,.$$

Then, $|\lambda_1| = |\bar{\lambda}_1| < 1$ and $|\lambda_1| = |\bar{\lambda}_1| = 1$, and so $\bar{x}_1$ is nonhyperbolic.

Open Problem 3.3. Determine the sability nature of the negative equilibrium of Eq.(1.1), $\bar{x}_1$.

4. The case $x_{-3}, x_{-2}, x_{-1}, x_0 \in (-1, 0)$

This section considers the solutions of Eq.(1.1) with initial conditions in the interval $(-1, 0)$. We first show that the interval $(-1, 0)$ is invariant, in the general case $x_{n+1} = x_{n-l}x_{n-k} - 1$.

Theorem 4.1. *Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of Eq.(1.2). If $x_{-k}, x_{-k+1}, \ldots x_{-1}, x_0 \in (-1, 0)$, then $x_n \in (-1, 0)$ for all $n \geq -k$.*

Proof. Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of Eq.(1.2) and suppose that $x_{-k}, x_{-k+1}, \ldots x_{-1}, x_0 \in (-1, 0)$. Then $x_1 = x_0 x_{-k} - 1 \in (-1, 0)$. From Eq.(1.2) and by induction, we obtain the result. □

Note that there exist solutions, not all of whose initial conditions are in the interval $(-1, 0)$, that eventually enter and remain in the interval $(-1, 0)$. For example, suppose $\{x_n\}_{n=-3}^{\infty}$ is a solution of Eq.(1.1) with $x_{-3} = a, x_{-2} = b, x_{-1} = c, x_0 = d$, where $a, b, c \in (-1, 0]$, $abc = 0$, and $d \in (-1, 0)$. Then it is easy to show that there exists $N \geq 2$ such that $x_N, x_{N+1}, x_{N+2} \in (-1, 0)$. By Theorem 4.1, $x_n \in (-1, 0)$ for all $n \geq N$. Another example is when either $a \in (-1, 0)$, $b, c, d \in [-1, 0)$, and any of b, c, d is -1 or $d \in (-1, 0)$, $a, b, c \in [-1, 0)$, and any of a, b, c is -1.

We now present some lemmas that are needed for a final result and conjecture on the asymptotic periodicity of solutions in the invariant interval $(-1, 0)$.

Lemma 4.2. *There exist infinitely many prime period-five solutions in the interval $(-1, 0)$ and they are all of the form*

$$\{x_n\}_{n=-3}^{\infty} = a, b, \frac{b+1}{a}, \frac{a+b+1}{ab}, \frac{a+1}{b}, a, b, \ldots,$$

where $a, b \in (-1, 0)$ and $a + b + 1 < 0$.

Proof. From Theorem 2.5, there exist prime period-five solutions and they are of the form

(1) $\{x_n\}_{n=-3}^{\infty} = 0, -1, -d-1, d, -1, 0, -1, \ldots$; or
(2) $\{x_n\}_{n=-3}^{\infty} = -1, 0, -1, d, -d-1, -1, 0, \ldots$; or
(3) $\{x_n\}_{n=-3}^{\infty} = a, b, \dfrac{b+1}{a}, \dfrac{a+b+1}{ab}, \dfrac{a+1}{b} a, b, \ldots$.

Clearly, the former two forms cannot exist in the *open* interval $(-1, 0)$ for any value of d. On the other hand, one can easily show that if $a, b \in (-1, 0)$ and $a+b+1 < 0$, then $\dfrac{b+1}{a}, \dfrac{a+b+1}{ab}, \dfrac{a+1}{b} \in (-1, 0)$, and the proof is complete. □

Of course, if there is (as we consider below) convergence of solutions in the interval $(-1, 0)$ to period-five solutions, these period-five solutions can conceivably include solutions of the form $\{x_n\}_{n=-3}^{\infty} = 0, -1, -d-1, d, -1, 0, -1, \ldots$ or $\{x_n\}_{n=-3}^{\infty} = -1, 0, -1, d, -d-1, -1, 0, \ldots$.

Lemma 4.3. *Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq.(1.1). Then $\{x_n\}_{n=-3}^{\infty}$ satisfies the identity*

$$x_{n+10} - x_{n+5} = x_{n+5}(x_{n+9}x_{n+2} - x_{n+7}x_{n+4}), \quad n \geq 0. \tag{4.1}$$

Proof. We obtain the identity (4.1) through several backwards iterations of the right-hand side of $x_{n+10} = x_{n+9}x_{n+6} - 1$:

$$\begin{aligned} x_{n+10} &= x_{n+9}x_{n+6} - 1 \\ &= x_{n+9}(x_{n+5}x_{n+2} - 1) - 1 \\ &= x_{n+5}x_{n+9}x_{n+2} - (x_{n+8}x_{n+5} - 1) - 1 \\ &= x_{n+5}x_{n+9}x_{n+2} - (x_{n+7}x_{n+4} - 1)x_{n+5} \\ &= x_{n+5}x_{n+9}x_{n+2} - x_{n+5}x_{n+7}x_{n+4} + x_{n+5}, \end{aligned}$$

from which the result follows. □

Observe that from the identity (4.1), we obtain

$$x_{n+10} - x_{n+5} = x_{n+5}[x_{n+2}(x_{n+9} - x_{n+4}) - x_{n+4}(x_{n+7} - x_{n+2})].$$

Then, taking absolute values of both sides, we obtain

$$|x_{n+10} - x_{n+5}| \leq |x_{n+5}||x_{n+2}||x_{n+9} - x_{n+4}| + |x_{n+5}||x_{n+4}||x_{n+7} - x_{n+2}|,$$

from which we can write

$$|x_{n+10} - x_{n+5}| \leq 2 \cdot \max\{|x_{n+5}||x_{n+2}||x_{n+9} - x_{n+4}|, |x_{n+5}||x_{n+4}||x_{n+7} - x_{n+2}|\}.$$

Again from (4.1), we have

$$|x_{n+9} - x_{n+4}| \leq 2 \cdot \max\{|x_{n+4}||x_{n+1}||x_{n+8} - x_{n+3}|, |x_{n+4}||x_{n+3}||x_{n+6} - x_{n+1}|\};$$

$$|x_{n+7} - x_{n+2}| \leq 2 \cdot \max\{|x_{n+2}||x_{n-1}||x_{n+6} - x_{n+1}|, |x_{n+2}||x_{n+1}||x_{n+4} - x_{n-1}|\}.$$

Putting everything together, we end up with

$$\begin{aligned} |x_{n+10} - x_{n+5}| \leq\ & 2^2 \cdot \max\{|x_{n+5}||x_{n+2}||x_{n+4}||x_{n+1}||x_{n+8} - x_{n+3}|, \\ & |x_{n+5}||x_{n+2}||x_{n+4}||x_{n+3}||x_{n+6} - x_{n+1}|, \\ & |x_{n+5}||x_{n+4}||x_{n+2}||x_{n-1}||x_{n+6} - x_{n+1}|, \\ & |x_{n+5}||x_{n+4}||x_{n+2}||x_{n+1}||x_{n+4} - x_{n-1}|\}. \end{aligned} \tag{4.2}$$

We continue this backwards iterative process up to that iterative step when one of the arguments of the maximum function is found to contain any one of the following differences in absolute value: $|x_2 - x_{-3}|$, $|x_3 - x_{-2}|$, $|x_4 - x_{-1}|$. This will mean that the number of factors in each argument will be the same in the final inequality. In actuality, it will turn out that if computations are performed as above, the argument that will be the first to contain one of these differences in absolute value, $|x_2 - x_{-3}|$, $|x_3 - x_{-2}|$, or $|x_4 - x_{-1}|$, will be the last argument in the list of arguments of the maximum function. At this point the iterative process will stop.

Remark 4.4. We look at a generalization of the inequality (4.2) and introduce some terminology, for the sake of convenience, together with some results that are necessary for our final result below.

Generalization/Extension of (4.2) up through the final m*th iterative step, where* $m \in \{0\,,1\,,\ldots\}$ *and for some* $q \in \{5\,,6\,,\ldots\}$*:*

$$\begin{aligned}|x_q - x_{q-5}| \quad \leq \quad & 2^m \cdot \max\left\{\left|x_{p_1^{(1)}}\right| \cdots \left|x_{p_{2m}^{(1)}}\right| \left|x_{q^{(1)}} - x_{q^{(1)}-5}\right|\ ,\ldots\ ,\right. \\ & \left.\left|x_{p_1^{(m)}}\right| \cdots \left|x_{p_{2m}^{(m)}}\right| \left|x_{q^{(m)}} - x_{q^{(m)}-5}\right|\right\}\ . \end{aligned} \tag{4.3}$$

Terminology:

(1) We refer to $\left|x_{q^{(i)}} - x_{q^{(i)}-5}\right|$, $i = 1\,,2\,,\ldots\,,m$, as the ith *absolute difference* in the ith argument. Without loss of generality, we assume that

$$\left|x_{q^{(m)}} - x_{q^{(m)}-5}\right| = |x_2 - x_{-3}|\,, |x_3 - x_{-2}|\,,\ \text{or}\ |x_4 - x_{-1}|\,.$$

(2) We refer to $\left|x_{p_1^{(i)}}\right|\,,\left|x_{p_2^{(i)}}\right|\,,\ldots\,,\left|x_{p_{2m-1}^{(i)}}\right|\,,\left|x_{p_{2m}^{(i)}}\right|$, $i = 1\,,2\,,\ldots\,,m$, as the ith collection of $2m$ *factors* (or the ith collection of m pairs of *factors*) in the ith argument.

(3) We refer to the multiplicative factor 2^m of the maximum function as the *multiplier* of the maximum function. We emphasize here that the exponent of the multiplier corresponds to the number of pairs of factors in each argument.

Now, if $\{x_n\}_{n=-3}^{\infty}$ is a solution of Eq.(1.1) in the interval $(-1\,,0)$, then the absolute difference and every factor in each argument of the maximum function in Eq.(4.3) is less than one, by Theorem 4.1. We can then make the following technical observation.

Observation:

Fix $j \in \{-3\,,-2\,,-1\,,0\,,1\}$ and consider the set of terms $S_j = \{|x_{5n+j}|\ :\ n = 0\,,1\,,\ldots\}$, i.e., consider every fifth term in the solution $\{x_n\}_{n=-3}^{\infty}$ starting with x_{-3}, x_{-2}, x_{-1}, x_0, or x_1. Then one can show that, with the collection of factors in each argument of the maximum function in Eq.(4.3), every consecutive block of eight factors will contain at least one element from S_j. So, if we make the assumption that, say,

$$|x_{5n+j}| < \frac{1}{2^{21}}\,,\qquad \text{for all } n \geq 0\,,$$

then the product of factors in each argument of the maximum function in Eq.(4.3) is such that

$$\left|x_{p_1^{(i)}}\right|\left|x_{p_2^{(i)}}\right| \cdots \left|x_{p_{2m-1}^{(i)}}\right|\left|x_{p_{2m}^{(i)}}\right| < \frac{1}{2^m} \cdot \frac{1}{2^{2m}}\,.$$

Thus, from Eq.(4.3),

$$|x_q - x_{q-5}| < \frac{1}{2^{2m}} \cdot \max\left\{\left|x_{q^{(1)}} - x_{q^{(1)}-5}\right|\ ,\ldots\ ,\left|x_{q^{(m)}} - x_{q^{(m)}-5}\right|\right\} < \frac{1}{2^{2m}}\,,$$

where

$$\max\left\{\left|x_{q^{(1)}}-x_{q^{(1)}-5}\right|\,,\ldots,\left|x_{q^{(m)}}-x_{q^{(m)}-5}\right|\right\}<1\,,$$

by Theorem 4.1.

Our final result states that if every fifth term in a solution of Eq.(1.1) in the interval $(-1\,,0)$ is "small enough" in magnitude, then the solution will converge to a period-five solution in the interval $[-1\,,0]$.

Theorem 4.5. *Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq.(1.1) in the interval $(-1\,,0)$. Suppose that for some $j_0\in\{-3\,,-2\,,-1\,,0\,,1\}$, $N\in\{0\,,1\,,\ldots\}$, and $r\in\{21\,,22\,,\ldots\}$,*

$$|x_{5(N+n)+j_0}|<\frac{1}{2^r}\,,\quad \textit{for all } n\geq 0\,.$$

Then $\{x_n\}_{n=-3}^{\infty}$ converges to a period-five solution in the interval $[-1\,,0]$.

Proof. Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq.(1.1) in the interval $(-1\,,0)$. Suppose that for some $j_0\in\{-3\,,-2\,,-1\,,0\,,1\}$, $N\in\{0\,,1\,,\ldots\}$, and $r\in\{21\,,22\,,\ldots\}$,

$$|x_{5(N+n)+j_0}|<\frac{1}{2^r}\,,\quad \text{for all } n\geq 0\,.$$

Without loss of generality, we assume that $N=0$. Based on Remark 4.4 and the discussion prior to it, we make the following computations, with details omitted. However, we note that all inequalities follow from the fact that there exists the subsequence $\{x_{5n+j_0}\}_{n=0}^{\infty}$, all of whose terms are bounded in magnitude by $\dfrac{1}{2^r}$ and whose terms in absolute value occur at least once every eight factors in each argument of the maximum function, described in Remark 4.4 and the discussion prior to it.

(1) $|x_{10}-x_5|<1=\left(\frac{1}{2^2}\right)^0$, $|x_{15}-x_{10}|<\frac{1}{2^8}<\left(\frac{1}{2^2}\right)^1$, $|x_{20}-x_{15}|<\frac{1}{2^{12}}<\left(\frac{1}{2^2}\right)^2$, $|x_{25}-x_{20}|<\frac{1}{2^{14}}<\left(\frac{1}{2^2}\right)^3$, and $|x_{30}-x_{25}|<\frac{1}{2^{18}}<\left(\frac{1}{2^2}\right)^4$.

(2) $|x_{11}-x_6|<1=\left(\frac{1}{2^2}\right)^0$, $|x_{16}-x_{11}|<\frac{1}{2^8}<\left(\frac{1}{2^2}\right)^1$, $|x_{21}-x_{16}|<\frac{1}{2^{12}}<\left(\frac{1}{2^2}\right)^2$, $|x_{26}-x_{21}|<\frac{1}{2^{16}}<\left(\frac{1}{2^2}\right)^3$, and $|x_{31}-x_{26}|<\frac{1}{2^{18}}<\left(\frac{1}{2^2}\right)^4$.

(3) $|x_{12}-x_7|<1=\left(\frac{1}{2^2}\right)^0$, $|x_{17}-x_{12}|<\frac{1}{2^{10}}<\left(\frac{1}{2^2}\right)^1$, $|x_{22}-x_{17}|<\frac{1}{2^{12}}<\left(\frac{1}{2^2}\right)^2$, $|x_{27}-x_{22}|<\frac{1}{2^{16}}<\left(\frac{1}{2^2}\right)^3$, and $|x_{32}-x_{27}|<\frac{1}{2^{20}}<\left(\frac{1}{2^2}\right)^4$.

(4) $|x_{13}-x_8|<1=\left(\frac{1}{2^2}\right)^0$, $|x_{18}-x_{13}|<\frac{1}{2^{10}}<\left(\frac{1}{2^2}\right)^1$, $|x_{23}-x_{18}|<\frac{1}{2^{14}}<\left(\frac{1}{2^2}\right)^2$, $|x_{28}-x_{23}|<\frac{1}{2^{16}}<\left(\frac{1}{2^2}\right)^3$, and $|x_{33}-x_{28}|<\frac{1}{2^{20}}<\left(\frac{1}{2^2}\right)^4$.

(5) $|x_{14}-x_9| < \frac{1}{2^8} < \left(\frac{1}{2^2}\right)^0$, $|x_{19}-x_{14}| < \frac{1}{2^{10}} < \left(\frac{1}{2^2}\right)^1$, $|x_{24}-x_{19}| < \frac{1}{2^{14}} < \left(\frac{1}{2^2}\right)^2$, $|x_{29}-x_{24}| < \frac{1}{2^{18}} < \left(\frac{1}{2^2}\right)^3$, and $|x_{34}-x_{29}| < \frac{1}{2^{20}} < \left(\frac{1}{2^2}\right)^4$.

We conclude by induction that for $n \geq 1$ and $i \in \{0\,,1\,,2\,,3\,,4\}$,

$$|x_{5(n+1)+i} - x_{5n+i}| < \left(\frac{1}{2^2}\right)^{n-1} \;;$$

and so, since $\lim_{K\to\infty} \sum_{k=K}^{\infty} \left(\frac{1}{2^2}\right)^k = 0$, given $0 < \epsilon < 1$, there exists $N = N(\epsilon) \geq 2$ with

$$\sum_{k=N-2}^{\infty} \left(\frac{1}{2^2}\right)^k < \epsilon$$

such that

$$|x_{5(n+1)+i} - x_{5n+i}| < \epsilon \quad \text{for all } n\,,m \geq N\,.$$

Hence, for $i \in \{0\,,1\,,2\,,3\,,4\}$, the subsequence $\{x_{5n+i}\}_{n=0}^{\infty} \subset (-1\,,0)$ is Cauchy. Then, for $i \in \{0\,,1\,,2\,,3\,,4\}$ there exists $L_i \in [-1\,,0]$ such that

$$\lim_{n\to\infty} x_{5n+i} = L_i\,.$$

We thus have that $\{x_n\}_{n=-3}^{\infty}$ converges to a period-five solution in the interval $[-1\,,0]$, and the proof is complete. □

As can be seen above, if we wish to use the identity given in Lemma 4.3 to prove convergence of a solution $\{x_n\}_{n=-3}^{\infty}$ of Eq.(1.1) in the interval $(-1\,,0)$ to a period-five solution, we need to assume that the following condition holds: Starting with the term x_{-3}, x_{-2}, x_{-1}, x_0, or x_1, every fifth term of $\{x_n\}_{n=-3}^{\infty}$ is "small enough" in magnitude. We believe that such a condition is not *necessary* for asymptotic periodicity and that

Conjecture 4.6. Every solution of Eq.(1.1) in the interval $(-1\,,0)$ converges to a period-five solution in the interval $[-1\,,0]$.

5. Unbounded solutions of Eq.(1.1)

In this section we state a result that gives a set of initial conditions for which there exists an unbounded solution.

Theorem 5.1. *Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq.(1.1) such that*

$$\min\{|x_{-3}|\,,|x_{-2}|\,,|x_{-1}|\,,|x_0|\} > \bar{x}_2 = \frac{1+\sqrt{5}}{2}\,.$$

Then

$$\bar{x}_2 < |x_0| < |x_1| < |x_2| < \cdots < |x_n| < \cdots\,.$$

Proof. From the hypothesis, we have that $|x_{-3}|-1 > \bar{x}_2-1$, and so $|x_0|\,(|x_{-3}-1) > \bar{x}_2\,(\bar{x}_2-1) = 1$. Therefore, $|x_0||x_{-3}| - |x_0| > 1$, which in turn implies that $|x_0||x_{-3}| - 1 > |x_0|$. We also have $|x_1| = |x_0 x_{-3} - 1| \geq |x_0||x_{-3}| - 1$. Combining these last two inequalities, we then have that $\bar{x}_2 < |x_0| < |x_1|$. By induction, the result follows. □

Corollary 5.2. *Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq.(1.1) such that*

$$\min\{x_{-3}\,, x_{-2}\,, x_{-1}\,, x_0\} > \bar{x}_2 = \frac{1+\sqrt{5}}{2}\,.$$

Then the solution tends to $+\infty$.

Proof. Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq.(1.1) such that

$$\min\{x_{-3}\,, x_{-2}\,, x_{-1}\,, x_0\} > \bar{x}_2 = \frac{1+\sqrt{5}}{2}\,,$$

By Theorem 5.1,

$$\bar{x}_2 < x_0 < x_1 < x_2 < \cdots < x_n < \cdots\,.$$

Now, assume to the contrary that the solution does not tend to $+\infty$. Since the sequence of terms of the solution is increasing and bounded, then it must converge. However, Eq.(1.1) has only two equilibria, and they are both less than x_0. We have a contradiction, and so this solution must tend to infinity. □

Stević [38] provides a generalization of Theorem 5.1 and Corollary 5.2 for Eq.(1.2, given that the condition $\gcd(k\,,l) = 1$ holds. We present modified versions of his results, which do not require that the condition $\gcd(k\,,l) = 1$ hold in their proofs.

For the following lemma, we need some notation and terminology for continued fractions (see [13]). Consider the following continued fraction representation for a real number β:

$$\beta = b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cdots + \cfrac{1}{b_n + \cfrac{1}{\beta_{n+1}}}}}\,.$$

The numbers $b_1\,, b_2\,, b_3\,, \ldots\,, b_n$ are called the *partial denominators* of β, and β_{n+1} is the $(n+1)$st *remainder*. A continued fraction that is *periodic* is one whose sequence of partial denominators, $\{b_n\}_{n=1}^{\infty}$, is periodic. Shorthand notation for this continued fraction of β is

$$\beta = b_1 + \frac{1}{b_2+}\,\frac{1}{b_3+}\,\cdots\,\frac{1}{b_n+}\,\frac{1}{\beta_{n+1}}$$

or

$$\beta = [b_1\,, b_2\,, b_3\,, \ldots\,, b_n\,, \beta_{n+1}]\,;$$

and the expression

$$[b_1\,, b_2\,, b_3\,, \ldots\,, b_n]\,,$$

represents a number called the *nth convergent* of β. The process of generating the continued fraction representation for β is referred to as the *continued fraction algorithm* (and if the process *terminates*, the continued fraction represents a rational number).

Lemma 5.3. *Consider Eq.(1.2), where $k\,, l \in \{0\,, 1\,, \ldots\}$, $k > l$. Then there do not exist period-$(l+1)$ solutions of Eq.(1.2) all of whose terms are strictly greater than* $\bar{x}_2 = \dfrac{1+\sqrt{5}}{2}$.

Proof. For the sake of contradiction, assume that we have a period-$(l+1)$ solution, $\{x_n\}_{n=-k}^{\infty} = \{a_i\}_{i=1}^{\infty}$ such that $a_i > \bar{x}_2$ for all $i = 1\,, 2\,, \ldots$. We begin with the following relations, which follow from the assumption that $\{x_n\}_{n=-k}^{\infty}$ is periodic with period $l+1$ and that $k > l$.

(1) Since $k > l$, we have the relation

$$k+1 = u(l+1) + (r-1)\,, \quad u \in \mathbf{N}\,, r-1 \in \{0\,, 1\,, \ldots\,, l\}\,. \tag{5.1}$$

Observe that r is such that $x_1 = a_r$. Equivalently, we have

$$k+2 = u(l+1) + r\,, \quad u \in \mathbf{N}\,, r \in \{1\,, 2\,, \ldots\,, l+1\}\,.$$

(2) Given that $\{x_n\}_{n=-k}^{\infty}$ is periodic with period $l+1$ and that $k > l$, we have the relation

$$s \equiv (k+1) - l \pmod{l+1}\,, \tag{5.2}$$

where s is such that $x_{-l} = a_s$. Then, from Eq.(5.2), there exists some positive integer v such that

$$s = (k+1) - l - v(l+1)\,,$$

which implies that

$$s = (l+2) - (l+1) - v(l+1)\,,$$

which, in turn, implies that

$$k+2 = (v+1)(l+1) + s\,. \tag{5.3}$$

Given Eqs.(5.1) and (5.3), we then have, by uniqueness in application of Euclid's algorithm, $(v+1 = u$ and$)$ $s = r$. Therefore, $x_{-l} = a_r = x_1$.

We now consider the following two cases.

Case 1.: $k+1$ is a multiple of $l+1$.

Note that if $k+1$ is a multiple of $l+1$, then from Eq.(5.1) and the assumtion that $\{x_n\}_{n=-k}^{\infty} = \{a_i\}_{i=1}^{\infty}$ is periodic with period-$(l+1)$, we see that $r = 1$, and thus

$$a_1 = x_1 = x_{-l}x_{-k} - 1 = a_1 a_1 - 1\,,$$

which implies that

$$a_1 = x_1 = x_{-k} = x_{-l} = \bar{x}_1 \text{ or } \bar{x}_2\,.$$

But then, we do not have $a_i > \bar{x}_2$ for all $i = 1, 2, \ldots$ (where $\bar{x}_2 > \bar{x}_1$). This case gives us a contradiction and so is not possible.

Case 2.: $k+1$ is not a multiple of $l+1$.

Since $k+1$ is not a multiple of $l+1$, we then have that $r \neq 1$ and

$$a_r = x_1 = x_{-l}x_{-k} - 1 = a_r a_1 - 1\,,$$

which implies that

$$a_1 = 1 + \frac{1}{a_r}\,. \tag{5.4}$$

We also then have

$$a_{r+1} = x_2 = x_{-l+1}x_{-k+1} - 1 = a_{r+1}a_2 - 1$$

and

$$a_{r+2} = x_3 = x_{-l+2}x_{-k+2} - 1 = a_{r+2}a_3 - 1\,.$$

By induction, we have

$$a_{2r-1} = a_{r+(r-1)} = x_r = x_{-l+(r-1)}x_{-k+(r-1)} = a_{2r-1}a_r - 1\,,$$

which implies that

$$a_r = 1 + \frac{1}{a_{2r-1}}\,. \tag{5.5}$$

By Eqs.(5.4) and (5.5), we can express a_1 as the continued fraction

$$a_1 = 1 + \cfrac{1}{1 + \cfrac{1}{a_{2r-1}}}\,. \tag{5.6}$$

Next

$$a_{3r-2} = a_{r+(2r-2)} = x_{2r-1} = x_{-l+(2r-2)}x_{-k+(2r-2)} = a_{3r-2}a_{2r-1} - 1\,,$$

which implies that

$$a_{2r-1} = 1 + \frac{1}{a_{3r-2}}\,. \tag{5.7}$$

By Eqs.(5.6) and (5.7), we can express a_1 as the continued fraction

$$a_1 = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{a_{3r-2}}}}\,. \tag{5.8}$$

Then, by induction, we obtain the continued fractions, for $m = 1, 2, \ldots$,

$$a_1 = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots \cfrac{1}{a_{(m+1)r-m}}}}},$$

or

$$a_1 = [b_1\,, b_2\,, \ldots b_{m+1}\,, a_{(m+1)r-m}] = [1\,, 1\,, \ldots\,, 1\,, a_{(m+1)r-m}]\,,$$

where $b_j = 1$ for $j = 1\,, 2\,, \ldots\,, m+1$. We claim that there exists m_0 such that

$$(m_0 + 1)r - m_0 \equiv 1 \pmod{l+1}\,,$$

i.e., there exists some m_0 such that $a_{(m_0+1)r-m_0} = a_1$, with $\{a_i\}_{n=1}^{\infty}$ periodic with period-$(l+1)$), for we have that

$$(m_0 + 1)r - m_0 \equiv 1 \pmod{l+1}\,,$$

if and only if

$$(m_0 + 1)r - (m_0 + 1) \equiv 0 \pmod{l+1}\,,$$

if and only if

$$(m_0 + 1)(r - 1) = t(l + 1)\,, \quad \text{for some } t \in \mathbf{N}\,, \tag{5.9}$$

and certainly Eq.(5.9) holds for $m_0 = l$ and $t = r - 1$. Then

$$a_1 = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots \cfrac{1}{a_1}}}}, \tag{5.10}$$

or

$$a_1 = [b_1\,, b_2\,, \ldots b_{m_0+1}\,, a_1] = [1\,, 1\,, \ldots\,, 1\,, a_1]\,.$$

where $b_j = 1$ for $j = 1\,, 2\,, \ldots\,, m_0+1$. However, given Eq.(5.10) with remainder a_1, we can write a_1 as the infinite periodic continued fraction

$$a_1 = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}, \tag{5.11}$$

or

$$[1\,, 1\,, \ldots] = [\bar{1}]\,.$$

Since the continued fraction is infinite and periodic with partial denominaters all ones, Eq.(5.11) can equivalently be written as

$$a_1 = 1 + \frac{1}{a_1}\,.$$

Solving for a_1, we obtain

$$a_1 = \bar{x}_1 \quad \text{or} \quad \bar{x}_2 \,.$$

In either case, we do not have that $a_1 > \bar{x}_2$ for all $i = 1\,,2\,,\ldots$. Again we have a contradiction.

Therefore, there do not exist any period-$(l+1)$ solutions, all of whose terms are strictly greater than $\bar{x}_2$, and the proof is complete. □

Theorem 5.4. *Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of Eq.(1.2), where $k\,,l \in \{0\,,1\,,\ldots\}$ and $k > l$, and such that*

$$\min\{|x_{-k}|\,,|x_{-k+1}|\,,\ldots\,,|x_0|\} > \bar{x}_2 = \frac{1+\sqrt{5}}{2}\,.$$

Then there are exactly $l+1$ disjoint subsequences,

$$\left\{|x_{(l+1)n+j}|\right\}_{n=0}^{\infty}\,, \quad j = -l\,,-l+1\,,\ldots\,,-1\,,0\,,$$

of the sequence $\{|x_n|\}_{n=-k}^{\infty}$, each of which is strictly increasing from above $\bar{x}_2$.

Proof. From the hypothesis, we have that $|x_{-k}|-1 > \bar{x}_2-1$, and so $|x_{-l}|\,(|x_{-k}|-1) > \bar{x}_2\,(\bar{x}_2-1) = 1$. Therefore, $|x_{-l}||x_{-k}| - |x_{-l}| > 1$, which in turn implies that $|x_{-l}||x_{-k}| - 1 > |x_{-l}|$. We also have $|x_1| = |x_{-l}x_{-k} - 1| \geq ||x_{-l}||x_{-k}| - 1| = |x_{-l}||x_{-k}| - 1$. Combining these last two inequalities, we then have that $\bar{x}_2 < |x_{-l}| < |x_1|$. Continuing this process, we then have the following:

$$\begin{array}{lll}
|x_1| > |x_{1-(l+1)}| = |x_{-l}| & \text{or} & |x_{1\cdot(l+1)+(-l)}| > |x_{0\cdot(l+1)+(-l)}| \\
|x_2| > |x_{2-(l+1)}| = |x_{-l+1}| & \text{or} & |x_{1\cdot(l+1)+(-l+1)}| > |x_{0\cdot(l+1)+(-l+1)}| \\
 & \vdots & \\
|x_l| > |x_{l-(l+1)}| = |x_{-1}| & \text{or} & |x_{1\cdot(l+1)+(-1)}| > |x_{0\cdot(l+1)+(-1)}| \\
|x_{l+1}| > |x_{(l+1)-(l+1)}| = |x_0| & \text{or} & |x_{1\cdot(l+1)+0}| > |x_{0\cdot(l+1)+0}| \\
|x_{l+2}| > |x_{(l+2)-(l+1)}| = |x_1| > |x_{-l}| & \text{or} & |x_{2\cdot(l+1)+(-l)}| > |x_{1\cdot(l+1)+(-l)}| > |x_{0\cdot(l+1)+(-l)}| \\
|x_{l+3}| > |x_{(l+3)-(l+1)}| = |x_2| > |x_{-l+1}| & \text{or} & |x_{2\cdot(l+1)+(-l+1)}| > |x_{1\cdot(l+1)+(-l+1)}| > |x_{0\cdot(l+1)+(-l+1)}| \\
 & \vdots &
\end{array}$$

Clearly, by induction, we see that there exist $l+1$ strictly increasing subsequences

$$\left\{|x_{n(l+1)+j}|\right\}_{n=0}^{\infty}\,, \quad j = -l\,,-l+1\,,\ldots\,,-1\,,0\,,$$

all of whose terms are greater than $\bar{x}_2$. Furthermore, these $l+1$ subsequences are disjoint. Otherwise, there exist $u\,,v \in \{0\,,1\,,\ldots\}$ and $r\,,s \in \{-l\,,-l+1\,,\ldots\,,-1\,,0\}$, with $u \neq v$ or $r \neq s$, such that $x_{u(l+1)+r} = x_{v(l+1)+s}$. Then $u(l+1)+r = v(l+1)+s$, and so

$$|(u-v)(l+1)| = |r-s|\,.$$

But $|r-s| \leq l$, which gives us a contradiction. The proof is complete. □

Corollary 5.5. *Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of Eq.(1.2), where $k \in \{0\,,1\,,\ldots\}$ and $k > l$, and such that*

$$\min\{x_{-k}, x_{-k+1}, \ldots, x_0\} > \bar{x}_2 = \frac{1+\sqrt{5}}{2}.$$

Then the solution tends to $+\infty$.

Proof. In order to prove that the solution, $\{x_n\}_{n=-k}^{\infty}$, of Eq.(1.2) tends to $+\infty$, we assume to the contrary, that it is bounded, i.e., there does not exist a subsequence $\{x_{n_i}\}_{i=0}^{\infty}$ such that $x_{n_i} \uparrow \infty$. Then, given the hypothesis

$$\min\{x_{-k}, x_{-k+1}, \ldots, x_0\} > \bar{x}_2 = \frac{1+\sqrt{5}}{2},$$

we have, by assumption and Theorem 5.4, that each of the following $l+1$ disjoint subsequences

$$\left\{x_{n(l+1)+j}\right\}_{n=0}^{\infty}, \quad j = -l, -l+1, \ldots, -1, 0,$$

of the solution is bounded (otherwise, by definition, $\{x_n\}_{n=-k}^{\infty}$ is unbounded) and strictly increasing from above $\bar{x}_2 = \dfrac{1+\sqrt{5}}{2}$, and so must converge. Therefore, the solution converges to a period-$(l+1)$ solution of Eq.(1.2), all of whose terms are greater than $\bar{x}_2$. But, by Lemma 5.3, there do not exist period-$(l+1)$ solutions of Eq.(1.2) all of whose terms are strictly greater than $\bar{x}_2$. Hence we obtain a contradiction, and the proof is complete. □

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