Int. J. Nonlinear Anal. Appl. **2** (2011) No.2, 44–50
ISSN: 2008-6822 (electronic)
http://www.ijnaa.com

ON THE FIXED POINT OF ORDER 2

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ABSTRACT. This paper deals with a new type of fixed point, i.e; "fixed point of order 2" which is introduced in a metric space and some results are achieved.

1. Introduction

In 1922, Banach proved the following famous fixed point theorem [1]. Let (X, d) be a complete metric space, $T : X \to X$ be a contraction, there exists a unique fixed point $x_0 \in X$ of T. This theorem, called the Banach contraction principle is a forceful tool in nonlinear analysis. This principle has many applications and is extended by several authors: Caristi [2], Edelstein [4], Ekeland [5, 6], Khan[9], Meir and Keeler [12], Nadler [13] and others. These theorems are also extended; see [3, 7, 8, 10, 15, 16, 17, 18] and others.
Many expressions and generalizations of Banach fixed point theorem were derived in recent years. The results presented in this paper extend properly the Banach contraction principle.
As we have experience with zero's of a map of order 2, we want to introduce a fixed point of order 2 for a map. Our idea goes back to special case in R, which if a real map on R has a fixed point of order 2 means that this map is tangent to axis $y = x$. Therefore, the derivative of map (if exists) is equal to 1 at this point.

2. Main Results

Definition 2.1. Suppose that (X, d) is a metric space, $T : X \to X$ is a function and $x_0 \in X$ is a fixed point for T. We call x_0 is a fixed point of order 2 if it is not alone point and the following satisfies:

$$\lim_{x \to x_0} \frac{d(Tx, x_0)}{d(x, x_0)} = 1.$$

We remember the following definitions. We will show that for the case (a) there is not fixed point of order 2 but in two other cases there is fixed point of order 2.

Definition 2.2. Suppose that (X, d) is a metric space, $T : X \to X$ is a function.

Date: Received: December 2010; Revised: May 2011.
2000 *Mathematics Subject Classification.* 54H25, 55M20.
Key words and phrases. Common fixed point, fixed point, non-expansive map, contractive map.
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- (a) T is a contraction, if there exist $k \in [0,1)$ such that $d(Tx,Ty) \leq kd(x,y)$ for all x,y in X.
- (b) T is a contractive mapping, if $d(Tx,Ty) < d(x,y)$ for all x,y in X which $x \neq y$.
- (c) T is a non-expansive mapping, if $d(Tx,Ty) \leq d(x,y)$ for all x,y in X.

In the following we consider first some properties for fixed point of order 2.

Proposition 2.3. *If $x_0 \in X$ is a fixed point of order 2 for T on X. Then T is continuous at x_0.*

Proof.
$\lim_{n\to\infty} d(Tx,x_0)) = \lim_{x\to x_0} \frac{d(Tx,x_0)}{d(x,x_0)} d(x,x_0) = \lim_{x\to x_0} \frac{d(Tx,x_0)}{d(x,x_0)} \lim_{x\to x_0} d(x,x_0) = 0.$

Proposition 2.4. *Let (X,d) be a metric spaces and $T: X \to X$ be a function such that $x_0 \in X$ is a fixed point for f, not alone point for X and a alone point for $T(X)$. Then x_0 is not fixed point of order 2 for T.*

Proof. According to assume the x_0 is a alone point for $T(X)$. There is a neighborhood of x_0, like $N(x_0)$ such that $N(x_0) \bigcap T(X) = \{x_0\}$ and each $x \in N(x_0)$ implies that $d(Tx,x_0) = 0$. Therefore, $\lim_{x\to x_0} \frac{d(Tx,x_0)}{d(x,x_0)} = 0$, i.e; x_0 is not a fixed point of order 2 for T.

Proposition 2.5. *Suppose that $x_0 \in X$ be a fixed point for $T_i : X \to X$ which $i = 1,...,n$ $(n \in N)$ and also $\lim_{x\to x_0} \frac{d(T_i x,x_0)}{d(x,x_0)} = \lambda_i$. Then x_0 is a fixed point of order 2 for $T_1T_2...T_n$ if and only $\lambda_1\lambda_2...\lambda_n = 1$.*

Proof. T_i is continuous at x_0 for all $i = 1,...,n$, by a simple change of variable, that
$$\lim_{x\to x_0} \frac{d(T_k(T_{k+1}...T_n x), x_0)}{d(T_{k+1}...T_n x, x_0)} = \lim_{t\to x_0} \frac{d(T_k t, x_0)}{d(t,x_0)}$$
and the last limit is equal with λ_k for $k = 1,...,n$. Hence,
$$\begin{aligned}\lim_{x\to x_0} \frac{d(T_1T_2...T_n x, x_0)}{d(x,x_0)} &= \lim_{x\to x_0} \frac{d(T_1(T_2...T_n)x, x_0)}{d(T_2...T_n, x_0)} \frac{d(T_2(T_3...T_n), x_0)}{d(T_3...T_n, x_0)} \cdots \frac{d(T_n x, x_0)}{d(x,x_0)} \\ &= \lambda_1\lambda_2...\lambda_n.\end{aligned}$$

Proposition 2.6. *Let $x_0 \in X$ be a fixed point for $T_i : X \to X$ for $i = 1,...,n$ and $n \in N$.*

- *(a) If x_0 is fixed point order 2 for all T_i, then x_0 is fixed point of order 2 for $T_1T_2...T_n$.*
- *(b) If x_0 is fixed point order 2 for T_1T_2 and T_2, then x_0 is fixed point of order 2 for T_1.*

Proof. (a) By proposition 2.3.
(b) x_0 is fixed point order 2 for T_1T_2 and T_2. Thus, $\lim_{x\to x_0} \frac{d(T_1T_2x,x_0)}{d(x,x_0)} = 1$, $\lim_{x\to x_0} \frac{d(T_2x,x_0)}{d(x,x_0)} = 1$. Since T is continuous at x_0 for $t = T_2x$.
$$1 = \frac{\lim_{x\to x_0} \frac{d(T_1T_2x,x_0)}{d(x,x_0)}}{\lim_{x\to x_0} \frac{d(T_2x,x_0)}{d(x,x_0)}} = \lim_{x\to x_0} \frac{d(T_1T_2x,x_0)}{d(T_2x,x_0)} = \lim_{t\to x_0} \frac{d(T_1t,x_0)}{d(t,x_0)}.$$

Proposition 2.7. *Suppose that x_0 is not alone point and is a fixed point for $T_i : X \to X$ for $i = 1, ..., n$ and $n \in N$.*

- *(a) If T_i be a contractive mapping or non-expansive mapping for all $i = 1, ..., n$ and $\lim_{x\to x_0} \frac{d(T_i x, x_0)}{d(x, x_0)} = \lambda_i$. Then $x_0 \in X$ is a fixed point of order 2 for $T_1T_2...T_n$ if and only if x_0 is a fixed point of order 2 for all T_i.*
- *(b) If $lim_{x\to x_0} \frac{d(T_1 x, x_0)}{d(x, x_0)} = \lambda$ then x_0 is a fixed point of order 2 for T_1 if and only if x_0 be a fixed point of order 2 for T_1^n, where n is arbitrary positive integer.*
- *(c) If T_1 be a contractive mapping or non-expansive mapping, then x_0 is a fixed point of order 2 for T_1 if and only if there exist $n \in N$ such that x_0 be a fixed point of order 2 for T_1^n.*

Proof. (a) Let T_i be a contractive mapping for all $i = 1, ..., n$. If x_0 is a fixed point of order 2 for all T_i then by proposition 2.5, x_0 is a fixed point of order 2 for $T_1T_2...T_n$. Now assume that x_0 is a fixed point of order 2 for $T_1T_2...T_n$, then by proposition 2.4, $1 = \lim_{x\to x_0} \frac{d(T_1T_2...T_n x, x_0)}{d(x, x_0)} = \lambda_1\lambda_2...\lambda_n$. But all T_i are contractive mappings so $\frac{d(T_1 x, x_0)}{d(x, x_0)} < 1$ which implies that $\lambda_i \leq 1$ for all $i = 1, ..., n$. Hence, $\lambda_1 = \lambda_2 = ... = \lambda_n = 1$. Proof for non-expansive is similar.
(b) By proposition 2.4, $lim_{x\to x_0} \frac{d(T_1^n x, x_0)}{d(x, x_0)} = \lambda^n$. Then $\lambda^n = 1$ if and only if $\lambda = 1$ because $\lambda \geq 0$.
(c) Let T_1 be a contractive mapping and there exists $n \in N$ such that x_0 is a fixed point of order 2 for T_1^n. T_1 is a contractive mapping, so

$$d(T_1^n x, x_0) < ... < d(T_1 x, x_0) < d(x, x_0)$$

$$1 = lim_{x\to x_0} \frac{d(T_1^n x, x_0)}{d(x, x_0)} \leq lim_{x\to x_0} \frac{d(T_1 x, x_0)}{d(x, x_0)} \leq 1.$$

Therefore, $lim_{x\to x_0} \frac{d(T_1 x, x_0)}{d(x, x_0)} = 1$.

Proposition 2.8. *Suppose that (X, d) is a metric space, $T : X \to X$ is a function and $x_0 \in X$ is a fixed point for T. If T is a contraction then x_0 is not a fixed point of order 2 for T.*

Proof. Since T is a contractive mapping so there exists $\alpha \in [0, 1)$ such that $d(Tx, Ty) \leq \alpha d(x, y)$ for all $x, y \in X$. Therefore, $\frac{d(Tx, x_0)}{d(x, x_0)} \leq \alpha < 1$ and x_0 can not be a fixed point of order 2 for T.

Proposition 2.9. *Suppose that $x_0 \in X$ be a fixed point of order 2 for $T : X \to X$, where T is one-to-one and g is left inverse of T. Then x_0 is also a fixed point of order 2 for g.*

Proof. It is clear that x_0 is a fixed point for g. On the other hand, since T is continuous at x_0 for $t = Tx$ so

$$\begin{aligned} 1 = \lim_{x\to x_0} \frac{d(Tx, x_0)}{d(x, x_0)} &= \lim_{x\to x_0} \frac{d(g(T(Tx), x_0)}{d(gT(x), x_0)} \\ &= \lim_{t\to x_0} \frac{d(gT(t), x_0)}{d(g(t), x_0)} \\ &= \lim_{t\to x_0} \frac{d(t, x_0)}{d(g(t), x_0)} = \lim_{t\to x_0} \frac{1}{\frac{d(g(t),x_0)}{d(t,x_0)}}. \end{aligned}$$

Therefore, $\lim_{t\to x_0} \frac{d(g(t),x_0)}{d(t,x_0)} = 1$.
In the following we give another condition for the fixed point of order 2.

Proposition 2.10. *Suppose that x_0 is not alone point and is a fixed point for $T : X \to X$.*

- *(a) If $\lim_{x\to x_0} \frac{d(Tx,x)}{d(x,x_0)} = 0$ then x_0 is a fixed point of order 2 for T.*
- *(b) If $\lim_{x\to x_0} \frac{d(Tx,x)}{d(Tx,x_0)} = 0$ then x_0 is a fixed point of order 2 for T.*

Proof.
(a) Using $| \, d(Tx, x_0) - d(x, x_0) \, | \le d(Tx, x)$,

$$1 - \frac{d(Tx, x)}{d(x, x_0)} \le \frac{d(Tx, x_0)}{d(x, x_0)} \le 1 + \frac{d(Tx, x)}{d(x, x_0)},$$

$\lim_{x\to x_0} \frac{d(Tx,x_0)}{d(x,x_0)} = 1$.
(b) Using $| \, d(Tx, x_0) - d(x, x_0) \, | \le d(Tx, x)$,

$$1 - \frac{d(Tx, x)}{d(Tx, x_0))} \le \frac{d(x, x_0)}{d(Tx, x_0)} \le 1 + \frac{d(Tx, x)}{d(Tx, x_0)}.$$

This shows that $\lim_{x\to x_0} \frac{d(x,x_0)}{d(Tx,x_0)} = 1$. Therefore, $\lim_{x\to x_0} \frac{d(Tx,x_0)}{d(x,x_0)} = 1$.

Proposition 2.11. *Suppose that x_0 is a fixed point for $T : X \to X$ and $\varphi : X \to R^+$ is a real valued function.*

- *(a) If x_0 be a fixed point of order 2 for T then $\lim_{x\to x_0} \frac{d(Tx,x)}{d(x,x_0)} \le 2$.*
- *(b) If $d(Tx, x) \le \varphi(x) - \varphi(Tx) \le d(x, x_0)$ for all x in X then x_0 is a fixed point of order 2 for T if and only if $\lim_{x\to x_0} \frac{d(Tx,x)}{d(x,x_0)} = 0$.*

Proof.
(a) From the inequality $d(Tx, x) \le d(Tx, x_0) + d(x, x_0)$,

$$\frac{d(Tx, x)}{d(x, x_0)} \le \frac{d(Tx, x_0)}{d(x, x_0)} + 1.$$

Therefore, $\lim_{x\to x_0}\frac{d(Tx,x)}{d(x,x_0)}\leq 2$.
(b) From inequality $d(Tx,x)\leq\varphi(x)-\varphi(Tx)\leq d(x,x_0)$,

$$\begin{aligned} d(x,Tx)+d(Tx,T^2x)+...+d(T^{n-1}x,T^nx) &\leq \sum_{i=1}^{n}\varphi(T^{i-1}x)-\varphi(T^ix)\\ &= \varphi(x)-\varphi(T^nx)\end{aligned}$$

and

$$\begin{aligned}\frac{d(T^{n-1}x,T^nx)}{d(x,x_0)} &= \frac{d(T^{n-1}x,T^nx)}{d(T^{n-1}x,T^{n-2}x_0)}\frac{d(T^{n-1}x,T^{n-2}x_0)}{d(T^{n-2}x,T^{n-3}x_0)},...,\frac{d(T^2x,Tx_0)}{d(Tx,x_0)}\frac{d(Tx,x_0)}{d(x,x_0)}\\ &= \frac{d(T^{n-1}x,T^nx)}{d(T^{n-1}x,x_0)}\frac{d(T^{n-1}x,x_0)}{d(T^{n-2}x,x_0)}...,\frac{d(T^2x,x_0)}{d(Tx,x_0)}\frac{d(Tx,x_0)}{d(x,x_0)},\end{aligned}$$

since $\lim_{x\to x_0}\frac{d(T^{n-1}x,T^nx)}{d(T^{n-1}x,x_0)}=\lim_{x\to x_0}\frac{d(Tx,x)}{d(x,x_0)}$ and $\lim_{x\to x_0}\frac{d(T^{n-k}x,x_0)}{d(T^{n-k-1}x,x_0)}=1$ which $k=1,2,...,n-1$, so $\lim_{x\to x_0}\frac{d(T^{n-1}x,T^nx)}{d(x,x_0)}=\lim_{x\to x_0}\frac{d(Tx,x)}{d(x,x_0)}$. From inequality $d(Tx,x)\leq\varphi(x)-\varphi(Tx)$. It is clear that $\varphi(T^nx)$ is strict decreasing.

$$\begin{aligned} n\lim_{x\to x_0}\frac{d(Tx,x)}{d(x,x_0)} &\leq \lim_{x\to x_0}\frac{\varphi(x)-\varphi(T^nx)}{d(x,x_0)}\\ &\leq \lim_{x\to x_0}\frac{\varphi(x)-\varphi(T^nx)}{\varphi(x)-\varphi(Tx)}\\ &\leq \lim_{x\to x_0}\frac{\varphi(x)-\varphi(T^nx)}{\varphi(x)-\varphi(T^nx)}=1.\end{aligned}$$

Hence, $\lim_{x\to x_0}\frac{d(Tx,x)}{d(x,x_0)}\leq\frac{1}{n}$. Since n is arbitrary positive integer, $\lim_{x\to x_0}\frac{d(Tx,x)}{d(x,x_0)}=0$. In the following we prove common fixed point of order 2.

Proposition 2.12. *Suppose that (X,d) is a metric space, $f,g:X\to X$ are two function and $x_0\in X$ is a fixed point for f such that f,g satisfies*

$$\begin{cases} d(f(x),g(x))\leq d(f(x),x)\leq\varphi(x)-\varphi(f(x))\leq d(x,x_0)\\ \\ \lim_{x\to x_0}\frac{d(f(x),x_0)}{d(x,x_0)}\geq 1\end{cases}.$$

for all x in X. Then x_0 is a common fixed point of order 2 for f,g.

Proof. First we show that x_0 is a fixed point of order 2 for f. From inequality $d(f(x),x)\leq\varphi(x)-\varphi(f(x))\leq d(x,x_0)$ we have

$$\begin{aligned} d(x,f(x))+d(f(x),f^2(x))+...+d(f^{n-1}(x),f^n(x)) &\leq \sum_{i=1}^{n}\varphi(f^{i-1}(x))-\varphi(f^i(x))\\ &= \varphi(x)-\varphi(f^n(x))\end{aligned}$$

and

$$\begin{aligned}\frac{d(f^{n-1}(x),f^n(x))}{d(x,x_0)} &= \frac{d(f^{n-1}(x),f^n(x))}{d(f^{n-1}(x),f^{n-2}(x_0()}\frac{d(f^{n-1}(x),f^{n-2}(x_0))}{d(f^{n-2}(x),f^{n-3}(x_0))}\cdots\frac{d(f^2(x),f(x_0))}{d(f(x),x_0)}\frac{d(f(x),x_0)}{d(x,x_0)}\\ &= \frac{d(f^{n-1}(x),f^n(x))}{d(f^{n-1}(x),x_0)}\frac{d(f^{n-1}(x),x_0)}{d(f^{n-2}(x),x_0)}\cdots\frac{d(f^2(x),x_0)}{d(f(x),x_0)}\frac{d(f(x),x_0)}{d(x,x_0)}.\end{aligned}$$

Since $\lim_{x\to x_0}\frac{d(f^{n-1}(x),f^n(x))}{d(f^{n-1}(x),x_0)}=\lim_{x\to x_0}\frac{d(f(x),x)}{d(x,x_0)}$ and $\lim_{x\to x_0}\frac{d(f^{n-k}(x),x_0)}{d(f^{n-k-1}(x),x_0)}=\lambda$ which $k=1,2,...,n-1$, we see that

$$\lim_{x\to x_0}\frac{d(f(x),x)}{d(x,x_0)}(1+\lambda+\lambda^2+...+\lambda^n)\le \lim\frac{\varphi(x)-\varphi(f^n(x))}{d(x,x_0)}.$$

But $\varphi(f^n(x))$ is strict decreasing so,

$$\begin{aligned}\lim_{x\to x_0}\frac{d(f(x),x)}{d(x,x_0)}(1+\lambda+\lambda^2+...+\lambda^n) &\le \lim_{x\to x_0}\frac{\varphi(x)-\varphi(f^n(x))}{d(x,x_0)}\\ &\le \lim_{x\to x_0}\frac{\varphi(x)-\varphi(f^n(x))}{\varphi(x)-\varphi(f(x))}\\ &\le \lim_{x\to x_0}\frac{\varphi(x)-\varphi(f^n(x))}{\varphi(x)-\varphi(f^n(x))}=1,\end{aligned}$$

and also $\lim_{x\to x_0}\frac{d(f(x),x)}{d(x,x_0)}\le\frac{1}{(1+\lambda+\lambda^2+...+\lambda^n)}$, but $\lambda\ge 1$ and n is arbitrary positive integer. Then $\lim_{x\to x_0}\frac{d(f(x),x)}{d(x,x_0)}=0$ and proposition 2.9 implies that x_0 is a fixed point of order 2 for f. Now, we show that x_0 is a fixed point of order 2 for g. It is clear that x_0 is a fixed point for g, because $d(f(x),g(x))\le d(f(x),x)$ and x_0 is fixed point for f. From inequality $d(f(x),g(x))\le d(f(x),x)$ and triangle inequality,

$$0\le\frac{d(g(x),x)}{d(x,x_0)}\le\frac{d(g(x),f(x))}{d(x,x_0)}+\frac{d(f(x),x)}{d(x,x_0)}\le 2\frac{d(f(x),x)}{d(x,x_0)}.$$

Therefore, $\lim_{x\to x_0}\frac{d(g(x),x)}{d(x,x_0)}=0$ and x_0 is a fixed point of order 2 for g.

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