Int. J. Nonlinear Anal. Appl. 16 (2025) 12, 177–186
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.25586.3062

# On the study of nonlinear sequential fractional integro-differential equation with nonseparated boundary conditions

Mariam El Hassnaoui[a], Hssaine Oummi[b,*]

[a] Laboratory of Applied Mathematics and Scientific Calculus, Sultan Moulay Sliman University, Morocco

[b] Department of Applied Mathematics, Faculty of Applied Sciences Ait Melloul. University Ibn Zohr, Morocco

*(Communicated by Zakieh Avazzadeh)*

**Abstract**

The aim of this paper is to study the existing results of nonlinear sequential fractional integro-differential equations with nonseparated boundary conditions. In this work, we consider a nonlinear problem and general boundary conditions. This extension introduces mathematical difficulties which we will overcome by using fixed-point techniques. For this, we rewrite the nonlinear boundary problem as a fixed point one involving two operators. Then, we show that these operators satisfy the conditions of the Krasnoselskii theorem. An example is given to illustrate our result.

Keywords: Fractional differential equation, Krasnoselskii fixed point theorem, nonseparated boundary conditions
2020 MSC: 34A08, 34B18, 34B40

## 1 Introduction

Fractional differential equations have been proved to be important tools in the modeling of many phenomena in various fields of applied sciences and engineering. Such as control theory, signal processing, rheology, fractals, chaotic dynamics, optics, medicine, economics, astrophysics, chemical engineering and so on, (see [1, 2, 4, 9, 12, 14, 15, 18, 22] and the references therein).

Recently, many problems can be modeled by fractional integro–differential equations, for an extensive literature in the study of fractional differential equations, we refer the reader to [11, 23, 25]. In the mathematical context, several interesting results about the fractional integro– differential equations supplemented by many boundary conditions classical, periodic, antiperiodic, nonlocal, ..., see [3, 5, 6, 7, 17, 19]. For example, Baleanu et al. [7] have considered the following fractional integro-differential equation:

$$^{c}D^{\alpha}u(t) + f(t, u(t), \varphi u(t), \psi u(t)) = 0,\ t \in ]0, 1[,$$

where $n-1 < \alpha \leq n$, $n \geq 3$. It should be noted that $\varphi$ and $\psi$ are linear operators. The authors established the existence and uniqueness of positive solutions for the above equation with the integral boundary conditions. The

*Corresponding author
*Email addresses:* hassnaoui.mariam@gmail.com (Mariam El Hassnaoui), hssaine.oummi@gmail.com (Hssaine Oummi)

*Received:* October 2021 *Accepted:* December 2021

essential tools in the proof are the Banach contraction principle and the Krasnoselskii fixed point theorem under the sufficient conditions. An another existence result was obtained in [3], the authors have considered fractional differential transform method to solve linear and nonlinear fractional integro-differential equations of Volterra type. The existence and controllability of this fractional integro-differential system is done for $\alpha \in (0,1)$ with infinite delay [17]. The proofs are based on the theoretical concepts related to the fractional calculus and the measure of noncompactness.

In this paper, we discuss the existence of solutions for the following nonlinear sequential fractional integro–differential equation:

$$^{c}D^{\alpha}(^{c}D^{\beta})u(t) = f(t, u(t), Tu(t), Gu(t)),\ t \in ]0,1[, \tag{1.1}$$

where $1 < \alpha \leq 2$, $1 < \beta \leq 2$ and $^{c}D^{(\cdot)}$ is Caputo's fractional derivative. This fractional integro-differential equations can model many physical and biological phenomena and, in general, they are not linear. In fact, linear problems are only approximations of reality which is rather complex involves nonlinears phenomena [19]. The main purpose of this paper is to extend the results of the above-mentioned works [3, 7, 17]. The difficulties arising from this model compared to previous ones is the fact that we consider the sequential fractional derivative. Moreover, the operators $T$ and $G$ are nonlinear and general boundary conditions, which are the nonseparated boundary ones. This extend introduce many mathematical difficulties. To overcome this task, we use the techniques of nonlinear analysis, precisely the fixed point theory.

The rest of this paper is organized as follow. In section 2, we introduce the functional setting of the problem and fixing the different notations. The position of the problem and data assumptions are presented in section 3. In section 4, we prove the existence results of the the problem (3.1)-(3.2), we first transform our problem in the fixed point one involving two nonlinear operators $\mathbf{A}$ and $\mathbf{B}$, next we apply the Krasnosel'skii's fixed-point theorem to show the existence of the solution. In section 5, we give an example to illustrate our main result.

## 2 Preliminaries

In this section, we recall some basic definitions and properties of the fractional calculus theory and auxiliary lemmas which will be used throughout this paper. For more details, see [10, 16, 18, 20]. Moreover, we state the Krasnoselskii fixed point theorem which is an important tool in our work, see [21, 24]. Let $\mathbf{X} = C([0, 1]; \mathbb{R})$ be the space of all continuous real-valued functions on $[0, 1]$ endowed with the norm

$$\|u\| = \sup_{0\leq t\leq 1} |u(t)|.$$

We recall the definition of the Caputo fractional order derivative.

**Definition 2.1.** The Caputo fractional order derivative of order $\alpha > 0$ with the lower limit zero of a continuous function $u$ is defined by

$$^{c}D^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}u^{(n)}(s)ds,$$

where $0 \leq n-1 < \alpha < n$, $n \in \mathbb{N}$, $t > 0$ and $\Gamma(\cdot)$ is Euler's Gamma function, which is defined by $\Gamma(x) = \displaystyle\int_0^{\infty} t^{x-1}e^{-t}dt$.

Next, we introduce The fractional integral.

**Definition 2.2.** The fractional integral of order $\alpha > 0$ with the lower limit zero of a continuous function $u$ can be defined as

$$I^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds.$$

Now, we present the following property of the fractional integral.

**Lemma 2.3.** Let $\alpha$, $\beta \geq 0$. Then, the following relation holds:

$$I^{\alpha}t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}t^{\alpha+\beta}.$$

The following lemma plays an important role in obtaining our main results.

**Lemma 2.4.** [18] Let $n \in \mathbb{N}$ and $n-1 < \alpha < n$. If $u$ is a continuous function, then we have

$$I^{\alpha}({}^{c}D^{\alpha}u(t)) = u(t) + a_0 + a_1 t + \cdots + a_{n-1}t^{n-1},$$

where $a_i \in \mathbb{R}$, $i = 0, 1, \cdots, n-1$.

As a consequence of Lemma 2.4, we have the following result which is useful in our existence result.

**Lemma 2.5.** Let $h \in C[0,1]$, then the boundary value problem

$$\begin{cases} {}^{c}D^{\alpha}({}^{c}D^{\beta})u(t) = h(t), \ t \in ]0,1[, \\ u(0) = \lambda_1 u(1), \quad u'(0) = \lambda_2 u'(1), \\ {}^{c}D^{\beta}u(0) = \lambda_3^{c} D^{\beta}u(1), \\ {}^{c}D^{\beta+1}u(0) = \lambda_4^{c} D^{\beta+1}u(1), \end{cases} \tag{2.1}$$

has the unique solution given by

$$\begin{aligned} u(t) \quad = \quad & \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}h(s)ds + \frac{\Lambda_1(t)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s))ds + \frac{\Lambda_2(t)}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}h(s)ds \\ & + \frac{\lambda_1}{(1-\lambda_1)\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}h(s)ds + \frac{\lambda_3(t)}{\Gamma(\alpha+\beta-1)}\int_0^1 (t-s)^{\alpha+\beta-2}h(s)ds, \end{aligned}$$

where

$$\begin{aligned} \Lambda_1(t) = & \frac{\lambda_3}{1-\lambda_3}\Big(\frac{t^{\beta}}{\Gamma(\beta+1)} + \frac{\lambda_1}{(1-\lambda_1)\Gamma(\beta+1)} + \frac{\lambda_1\lambda_2}{(1-\lambda_1)(1-\lambda_2)\Gamma(\beta)} + \frac{t\lambda_2}{(1-\lambda_2)\Gamma(\beta)}\Big), \\ \Lambda_2(t) = & \frac{\lambda_4}{1-\lambda_4}\Big(\frac{t^{\beta}\lambda_3}{(1-\lambda_3)\Gamma(\beta+1)} + \frac{t^{\beta+1}}{\Gamma(\beta+2)} + \frac{\lambda_1\lambda_2}{(1-\lambda_1)(1-\lambda_2)\Gamma(\beta+1)} + \frac{\lambda_1}{(1-\lambda_1)\Gamma(\beta+2)} + \frac{\lambda_2 t}{(1-\lambda_2)\Gamma(\beta+1)} \\ & + \frac{\lambda_1\lambda_2\lambda_3}{(1-\lambda_1)(1-\lambda_2)(1-\lambda_3)\Gamma(\beta)} + \frac{\lambda_1\lambda_3}{(1-\lambda_1)(1-\lambda_3)\Gamma(\beta+1)} + \frac{\lambda_2\lambda_3 t}{(1-\lambda_2)(1-\lambda_3)\Gamma(\beta)}\Big), \\ \Lambda_3(t) = & \frac{\lambda_2}{1-\lambda_2}\Big(t + \frac{\lambda_1}{1-\lambda_1}\Big). \end{aligned}$$

**Proof .** According to Lemma 2.4, we have

$${}^{c}D^{\alpha}u(t) = I^{\alpha}h(t) + c_0 + c_1 t, \quad \text{and} \quad {}^{c}D^{\beta+1}u(t) = I^{\alpha-1}h(t) + c_1.$$

Then

$$u(t) = I^{\alpha+\beta}h(t) + I^{\beta}c_0 + I^{\beta}c_1 t + c_2 + c_3 t, \tag{2.2}$$

where $c_0, c_1, c_2, c_3 \in \mathbb{R}$. Using the conditions ${}^{c}D^{\beta+1}u(0) = \lambda_4^{c} D^{\beta+1}u(1)$, ${}^{c}D^{\beta}u(0) = \lambda_3^{c} D^{\beta}u(1)$, $u'(0) = \lambda_2 u'(1)$ and $u(0) = \lambda_1 u(1)$, we get

$$c_0 = \frac{\lambda_3}{(1-\lambda_3)\Gamma(\alpha)}\int_0^t (1-s)^{\alpha-1}h(s)ds + \frac{\lambda_3\lambda_4}{(1-\lambda_3)(1-\lambda_4)\Gamma(\alpha-1)}\int_0^t (1-s)^{\alpha-2}h(s)ds,$$

$$c_1 = \frac{\lambda_4}{(1-\lambda_4)\Gamma(\alpha-1)}\int_0^t (1-s)^{\alpha-2}h(s)ds,$$

$$
\begin{aligned}
c_2 \;=\;& \frac{\lambda_1}{(1-\lambda_1)\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}h(s)ds + \frac{\lambda_1\lambda_2}{(1-\lambda_1)(1-\lambda_2)\Gamma(\alpha+\beta-1)} \times \int_0^1 (1-s)^{\alpha+\beta-2}h(s)ds\\
&+\Big(\frac{\lambda_1\lambda_3}{(1-\lambda_1)(1-\lambda_3)\Gamma(\beta+1)} + \frac{\lambda_1\lambda_2\lambda_3}{(1-\lambda_1)(1-\lambda_2)(1-\lambda_3)\Gamma(\beta)}\Big)\\
&\times\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}h(s)ds + \Big(\frac{\lambda_1\lambda_3\lambda_4}{(1-\lambda_1)(1-\lambda_3)(1-\lambda_4)\Gamma(\beta+1)}\\
&+\frac{\lambda_1\lambda_4}{(1-\lambda_1)(1-\lambda_4)\Gamma(\beta+2)} + \frac{\lambda_1\lambda_2\lambda_4}{(1-\lambda_1)(1-\lambda_2)(1-\lambda_4)\Gamma(\beta+1)}\\
&+\frac{\lambda_1\lambda_2\lambda_3\lambda_4}{(1-\lambda_1)(1-\lambda_2)(1-\lambda_3)(1-\lambda_4)\Gamma(\beta)}\Big)\frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}h(s)ds,
\end{aligned}
$$

and

$$
\begin{aligned}
c_3 =& \frac{\lambda_2}{(1-\lambda_2)\Gamma(\alpha+\beta-1)}\int_0^t (1-s)^{\alpha+\beta-2}h(s)ds + \frac{\lambda_2\lambda_3\lambda_4}{(1-\lambda_2)(1-\lambda_3)(1-\lambda_4)\Gamma(\alpha-1)\Gamma(\beta)}\int_0^t (1-s)^{\alpha-2}h(s)ds\\
&+\frac{\lambda_2\lambda_3}{(1-\lambda_2)(1-\lambda_3)\Gamma(\alpha)\Gamma(\beta)}\int_0^t (1-s)^{\alpha-1}h(s)ds.
\end{aligned}
$$

Substituting the values of $c_0$, $c_1$, $c_2$ and $c_3$ in (2.2), we get the desired results. By direct computing, we can prove that $u(t)$ is the solution of the problem (2.1). □ □

We close this section by recalling the Krasnosel'skii fixed point theorem (see [21], [24]).

**Theorem 2.6.** Let $\mathcal{M}$ be a nonempty closed, bounded and convex subset of a Banach space $X$. Assume that $A : \mathcal{M} \to X$ and $B : \mathcal{M} \to X$ are such that:

1. $A$ is continuous and $A\mathcal{M}$ is a relatively compact subset of $X$.
2. $B$ is a strict contraction.
3. $A\mathcal{M} + B\mathcal{M} \subset \mathcal{M}$.

Then there exists $x \in \mathcal{M}$ such that $Ax + Bx = x$.

## 3 Position of the problem

In this section, we describe the boundary value problem. Precisely, we are concerned with the existence of solution to the nonlinear sequential fractional integro-differential equation

$$
{}^cD^{\alpha}({}^cD^{\beta})u(t) = f(t, u(t), Tu(t), Gu(t)),\; t \in ]0,1[, \tag{3.1}
$$

where $1 < \alpha \leq 2$, $1 < \beta \leq 2$, ${}^cD^{(\cdot)}$ is Caputo's fractional derivative, $f : [0,1]\times\mathbb{R}^3 \longrightarrow \mathbb{R}$ is a given continuous function, and the operators $T$, $G$ are nonlinear given by:

$$
Tu(t) = \int_0^t \lambda(t,s)g(s,u(s))ds, \quad \text{and} \quad Gu(t) = \int_0^t \delta(t,s)g(s,u(s))ds,
$$

where $g : [0,1]\times\mathbb{R} \longrightarrow \mathbb{R}$ is a given continuous function, and $\lambda$, $\delta$ are the functions defined from : $[0,1]\times[0,1] \longrightarrow [0,+\infty)$ verifying

$$
\sup_{0\leq t\leq 1}\int_0^1 \lambda(t,s)ds < \infty \text{ and } \sup_{0\leq t\leq 1}\int_0^1 \delta(t,s)ds < \infty,
$$

In our framework, equation (3.1) is completed with nonseparated boundary conditions, i.e.,

$$
\begin{cases}
u(0) = \lambda_1 u(1), \quad u'(0) = \lambda_2 u'(1),\\
{}^cD^{\beta}u(0) = \lambda_3^c D^{\beta}u(1),\\
{}^cD^{\beta+1}u(0) = \lambda_4^c D^{\beta+1}u(1),
\end{cases} \tag{3.2}
$$

where $\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4 \in \mathbb{R} \setminus \{0,1\}$. In order to establish the existence results of the problem (3.1)-(3.2), we need to introduce the following assumptions, denoted (H):

(H1)- There exists a continuous functions $\theta$ and $\Lambda$ such that ,

$$|f(t,x_1,y_1,z_1) - f(t,x_2,y_2,z_2)| \leq \theta(t)\Big(|x_1-x_2| + |y_1-y_2| + |z_1-z_2|\Big),$$

and

$$|g(t,x_1) - f(t,x_2)| \leq \Lambda(t)|x_1-x_2|,$$

for all $t \in [0,1]$ and $x_i, y_i, z_i \in \mathbb{R}$, $i = 1,2$.

(H2)- $\Big[\frac{\Lambda_1}{\Gamma(\alpha)} + \frac{\Lambda_2}{\Gamma(\alpha-1)} + \frac{|\lambda_1||1-\lambda_1|}{|1-\lambda_1|\Gamma(\alpha+\beta)} + \frac{\Lambda_3}{\Gamma(\alpha+\beta-1)}\Big](1+\lambda_0\Lambda^* + \delta_0\Lambda^*)\theta^* \leq \frac{1}{2}$,

(H3)- $C^* = \frac{\theta^*(1+\Lambda^*\lambda_0+\Lambda^*\delta_0)}{\Gamma(\alpha+\beta)} < 1$, where

$$\theta^* = \|\theta\|,\ \ \Lambda^* = \|\Lambda\|,\ \ \lambda_0 = \sup_{0\leq t\leq 1}\int_0^1 \lambda(t,s)ds \ \ \text{and} \ \ \delta_0 = \sup_{0\leq t\leq 1}\int_0^1 \delta(t,s)ds.$$

## 4 Main results

In this section, we discuss existence results of the nonlinear sequential fractional integro-differential equations with nonseparated boundary conditions (3.1)-(3.2) in Banach spaces. We first transform the problem into a fixed point one involving two operators, say, $\mathbf{A}u+\mathbf{B}u = u$. The proof is based on The Krasnosel'skii's fixed point theorem. For $r > 0$, we denote by $\mathbb{B}_r$ the set

$$\mathbb{B}_r = \big\{u \in X \ : \ \|u\| \leq r\big\}.$$

The following lemma deals with the equivalence of a nonlinear fractional integro-differential equation (3.1)-(3.2).

**Lemma 4.1.** The problem (3.1)-(3.2) may be written in the form

$$\mathbf{A}u(t) + \mathbf{B}u(t) = u(t), \tag{4.1}$$

where where $\mathbf{A}$, $\mathbf{B} : X \longrightarrow X$ are given by

$$\begin{aligned}
\mathbf{A}u(t) \ &= \ \frac{\Lambda_1(t)}{\Gamma(\alpha)}\int_0^1 (t-s)^{\alpha-1} f(s,u(s),Tu(s),Gu(s))ds + \frac{\Lambda_2(t)}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2} f(s,u(s),Tu(s),Gu(s))ds \\
&+ \frac{\lambda_1}{(1-\lambda_1)\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1} f(s,u(s),Tu(s),Gu(s))ds \\
&+ \frac{\Lambda_3(t)}{\Gamma(\alpha+\beta-1)}\int_0^1 (t-s)^{\alpha+\beta-2} f(s,u(s),Tu(s),Gu(s))ds,
\end{aligned}$$

and $\mathbf{B}u(t) = \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1} f(s,u(s),Tu(s),Gu(s))ds$.

We denote by $\Lambda_1$, $\Lambda_2$, $\Lambda_3$ and $f^*$ the following quantities:

$$\begin{aligned}
\Lambda_1 :=& \frac{|\lambda_3|}{|1-\lambda_3|}\Big(\frac{1}{\Gamma(\beta+1)} + \frac{|\lambda_1|}{|1-\lambda_1|\Gamma(\beta+1)} + \frac{|\lambda_1\lambda_2|}{|(1-\lambda_1)(1-\lambda_2)|\Gamma(\beta)} + \frac{|\lambda_2|}{|1-\lambda_2|\Gamma(\beta)}\Big), \\
\Lambda_2 :=& \frac{|\lambda_4|}{|1-\lambda_4|}\Big(\frac{|\lambda_3|}{|1-\lambda_3|\Gamma(\beta+1)} + \frac{1}{\Gamma(\beta+2)} + \frac{|\lambda_1\lambda_2|}{|(1-\lambda_1)(1-\lambda_2)|\Gamma(\beta+1)} + \frac{|\lambda_1|}{|1-\lambda_1|\Gamma(\beta+2)} \\
&+ \frac{|\lambda_1\lambda_2\lambda_3|}{|(1-\lambda_1)(1-\lambda_2)(1-\lambda_3)|\Gamma(\beta)} + \frac{|\lambda_1\lambda_3|}{|(1-\lambda_1)(1-\lambda_3)|\Gamma(\beta+1)} + \frac{|\lambda_2\lambda_3|}{|(1-\lambda_2)(1-\lambda_3)|\Gamma(\beta)} + \frac{|\lambda_2|}{|1-\lambda_2|\Gamma(\beta+1)}\Big), \\
\Lambda_3 :=& \frac{|\lambda_2|}{|1-\lambda_2|}\Big(1 + \frac{|\lambda_1|}{|1-\lambda_1|}\Big),
\end{aligned}$$

and $f^* = \sup_{0\leq t\leq 1}|f(t,0,0,0)|$.

In order to show the existence of the problem (3.1)-(3.2), we use the Krasnosel'skii's fixed point theorem. For this reason, we need to show the following Lemmas. In the first one, we show that the operator $\mathbf{A}+\mathbf{B}$ leaves $\mathbb{B}_r$ invariant.

**Lemma 4.2.** Assume that the hypotheses (H1)-(H2) are satisfied. Then, for all $r>0$ satisfying

$$r \geq 2\Big[\frac{\Lambda_1}{\Gamma(\alpha)} + \frac{\Lambda_2}{\Gamma(\alpha-1)} + \frac{|\lambda_1||1-\lambda_1|}{|1-\lambda_1|\Gamma(\alpha+\beta)} + \frac{\Lambda_3}{\Gamma(\alpha+\beta-1)}\Big](\theta^*\lambda_0 g^* + \theta^*\delta_0 g^* + f^*), \tag{4.2}$$

the operator $\mathbf{A}+\mathbf{B}$ leaves $\mathbb{B}_r$ invariant.

**Proof .** For any $u\in\mathbb{B}_r$ and $t\in(0,1)$, we have

$$\begin{aligned}|\mathbf{A}u(t)| \leq& \frac{\Lambda_1}{\Gamma(\alpha)}\int_0^1 (t-s)^{\alpha-1}|f(s,u(s),Tu(s),Gu(s))|ds + \frac{\Lambda_2}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}|f(s,u(s),Tu(s),Gu(s))|ds\\ &+ \frac{|\lambda_1|}{|1-\lambda_1|\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}|f(s,u(s),Tu(s),Gu(s))|ds\\ &+ \frac{\Lambda_3}{\Gamma(\alpha+\beta-1)}\int_0^1 (t-s)^{\alpha+\beta-2}|f(s,u(s),Tu(s),Gu(s))|ds.\end{aligned}$$

Using the hypothesis (H1), we get

$$\begin{aligned}|\mathbf{A}u(t)| \leq& \frac{\Lambda_1\theta^*}{\Gamma(\alpha)}\int_0^1 (t-s)^{\alpha-1}(|u(s)|+|Tu(s)|+|Gu(s)|)ds + \frac{\Lambda_1}{\Gamma(\alpha)}\int_0^1 (t-s)^{\alpha-1}|f(s,0,0,0)|ds\\ &+ \frac{\Lambda_2\theta^*}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}(|u(s)|+|Tu(s)|+|Gu(s)|)ds + \frac{\Lambda_2}{\Gamma(\alpha-1)}\int_0^1 (t-s)^{\alpha-2}|f(s,0,0,0)|ds\\ &+ \frac{|\lambda_1|\theta^*}{|1-\lambda_1|\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}(|u(s)|+|Tu(s)|+|Gu(s)|)ds\\ &+ \frac{|\lambda_1|}{|1-\lambda_1|\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}|f(s,0,0,0)|ds + \frac{\Lambda_3\theta^*}{\Gamma(\alpha+\beta-1)}\int_0^1 (t-s)^{\alpha+\beta-2}(|u(s)|\\ &+|Tu(s)|+|Gu(s)|)ds + \frac{\Lambda_3}{\Gamma(\alpha+\beta-1)}\int_0^1 (1-s)^{\alpha+\beta-2}|f(s,0,0,0)|ds.\end{aligned}$$

We set $g^* = \sup\limits_{0\leq t\leq 1}|g(t,0)|$, thus

$$\begin{aligned}|\mathbf{A}u(t)| \leq& \frac{\Lambda_1\theta^*\|u\|}{\Gamma(\alpha)} + \frac{\Lambda_1\theta^*}{\Gamma(\alpha)}\int_0^1 (t-s)^{\alpha-1}\int_0^s \lambda(s,w\Big[|g(w,u(w))-g(w,0)|+|g(w,0)|\Big]dwds\\ &+ \frac{\Lambda_1\theta^*}{\Gamma(\alpha)}\int_0^1 (t-s)^{\alpha-1}\int_0^s \delta(s,w\Big[|g(w,u(w))-g(w,0)|+|g(w,0)|\Big]dwds + \frac{\Lambda_1 f^*}{\Gamma(\alpha)} + \frac{\Lambda_2\theta^*\|vu\|}{\Gamma(\alpha-1)} + \frac{\Lambda_2 f^*}{\Gamma(\alpha-1)}\\ &+ \frac{\Lambda_2\theta^*}{\Gamma(\alpha-1)}\int_0^1 (t-s)^{\alpha-2}\int_0^s (\lambda(s,w)+\delta(s,w))|g(w,u(w))-g(w,0)|dwds\\ &+ \frac{\Lambda_2\theta^*}{\Gamma(\alpha-1)}\int_0^1 (t-s)^{\alpha-2}\int_0^s (\lambda(s,w)+\delta(s,w))|g(w,0)|dwds + \frac{|\lambda_1|\theta^*\|u\|}{|1-\lambda_1|\Gamma(\alpha+\beta)} + \frac{|\lambda_1|f^*}{|1-\lambda_1|\Gamma(\alpha+\beta)}\\ &+ \frac{|\lambda_1|\theta^*}{|1-\lambda_1|\Gamma(\alpha+\beta)}\int_0^1 (t-s)^{\alpha+\beta-1}\int_0^s (\lambda(s,w)+\delta(s,w))|g(w,u(w))-g(w,0)|dwds\\ &+ \frac{|\lambda_1|\theta^*}{|1-\lambda_1|\Gamma(\alpha+\beta)}\int_0^1 (t-s)^{\alpha+\beta-1}\int_0^s (\lambda(s,w)+\delta(s,w))|g(w,0)|dwds + \frac{\Lambda_3\theta^*\|u\|}{\Gamma(\alpha+\beta-1)}\\ &+ \frac{\Lambda_3 f^*}{\Gamma(\alpha+\beta-1)} + \frac{\Lambda_3\theta^*}{\Gamma(\alpha+\beta-1)}\int_0^1 (t-s)^{\alpha+\beta-2}\int_0^s (\lambda(s,w)+\delta(s,w))|g(w,u(w))-g(w,0)|dwds\\ &+ \frac{\Lambda_3\theta^*}{\Gamma(\alpha+\beta-1)}\int_0^1 (t-s)^{\alpha+\beta-2}\int_0^s (\lambda(s,w)+\delta(s,w))|g(w,0)|dwds,\end{aligned}$$

which implies that

$$\begin{aligned}|\mathbf{A}u(t)| \leq\ & \Big[\frac{\Lambda_1}{\Gamma(\alpha)}+\frac{\Lambda_2}{\Gamma(\alpha-1)}+\frac{|\lambda_1|}{|1-\lambda_1|\Gamma(\alpha+\beta)}+\frac{\Lambda_3}{\Gamma(\alpha+\beta-1)}\Big]\theta^*\|u\|+\frac{\lambda_0+\delta_0}{\Gamma(\alpha)}\Lambda_1\theta^*\Lambda^*\|u\|\\ &+\frac{\Lambda_1(\lambda_0+\delta_0)\theta^*g^*+\Lambda_1 f^*}{\Gamma(\alpha)}+\frac{\lambda_0+\delta_0}{\Gamma(\alpha-1)}\Lambda_2\theta^*\Lambda^*\|u\|+\frac{\Lambda_2(\lambda_0+\delta_0)\theta^*g^*+\Lambda_2 f^*}{\Gamma(\alpha-1)}\\ &+\frac{\lambda_0+\delta_0}{\Gamma(\alpha+\beta)}|\lambda_1|\theta^*\Lambda^*\|u\|+\frac{|\lambda_1|(\lambda_0+\delta_0)\theta^*g^*+|\lambda_1|f^*}{\Gamma(\alpha+\beta)}\\ &+\frac{\lambda_0+\delta_0}{\Gamma(\alpha+\beta-1)}\Lambda_3\theta^*\Lambda^*\|u\|+\frac{\Lambda_3(\lambda_0+\delta_0)\theta^*g^*+\Lambda_3 f^*}{\Gamma(\alpha+\beta-1)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\mathbf{A}u\| \leq\ & \Big[\frac{\Lambda_1}{\Gamma(\alpha)}+\frac{\Lambda_2}{\Gamma(\alpha-1)}+\frac{|\lambda_1|}{|1-\lambda_1|\Gamma(\alpha+\beta)}+\frac{\Lambda_3}{\Gamma(\alpha+\beta-1)}\Big](1+\lambda_0\Lambda^*+\delta_0\Lambda^*)\theta^*\|u\|\\ &+\Big[\frac{\Lambda_1}{\Gamma(\alpha)}+\frac{\Lambda_2}{\Gamma(\alpha-1)}+\frac{|\lambda_1|}{|1-\lambda_1|\Gamma(\alpha+\beta)}+\frac{\Lambda_3}{\Gamma(\alpha+\beta-1)}\Big](\theta^*\lambda_0 g^*+\theta^*\delta_0 g^*+f^*).\end{aligned}$$

Similarly, for any $v\in\mathbb{B}_r$ and all $t\in[0,1]$, using the hypothesis (H1), we get

$$\|\mathbf{B}v\|\leq\frac{1+\Lambda^*\lambda_0+\Lambda^*\delta_0}{\Gamma(\alpha+\beta)}\theta^*\|v\|+\frac{\theta^*\Lambda^*g^*(\lambda_0+\delta_0)+f^*}{\Gamma(\alpha+\beta)}.$$

From the above inequalities, for any $u$, $v\in\mathbb{B}_r$ and $t\in[0,1]$, we get

$$\begin{aligned}\|\mathbf{A}u+\mathbf{B}v\| \leq\ & \Big[\frac{\Lambda_1}{\Gamma(\alpha)}+\frac{\Lambda_2}{\Gamma(\alpha-1)}+\frac{|\lambda_1||1-\lambda_1|}{|1-\lambda_1|\Gamma(\alpha+\beta)}+\frac{\Lambda_3}{\Gamma(\alpha+\beta-1)}\Big](1+\lambda_0\Lambda^*+\delta_0\Lambda^*)\theta^* r\\ &+\Big[\frac{\Lambda_1}{\Gamma(\alpha)}+\frac{\Lambda_2}{\Gamma(\alpha-1)}+\frac{|\lambda_1||1-\lambda_1|}{|1-\lambda_1|\Gamma(\alpha+\beta)}+\frac{\Lambda_3}{\Gamma(\alpha+\beta-1)}\Big](\theta^*\lambda_0 g^*+\theta^*\delta_0 g^*+f^*)\end{aligned}$$

Using the inequality (4.3) and hypothesis (H2), we obtain

$$\|\mathbf{A}u+\mathbf{B}v\|\leq r.$$

Therefore, $\mathbf{A}\mathbb{B}_r+\mathbf{B}\mathbb{B}_r\subseteq\mathbb{B}_r$. This achieves the proof. □ □

In the second lemma, we show that The operator $\mathbf{A}$ is compact.

**Lemma 4.3.** The operator $\mathbf{A}$ is compact and continuous on $X$.

**Proof .** Continuity of $f$ and $g$ implies that the operator $\mathbf{A}$ is continuous. Also, $\mathbf{A}$ is uniformly bounded on $\mathbb{B}_r$ as

$$\begin{aligned}\|\mathbf{A}u\| \leq\ & \Big[\frac{\Lambda_1}{\Gamma(\alpha)}+\frac{\Lambda_2}{\Gamma(\alpha-1)}+\frac{|\lambda_1|}{|1-\lambda_1|\Gamma(\alpha+\beta)}+\frac{\Lambda_3}{\Gamma(\alpha+\beta-1)}\Big](1+\lambda_0\Lambda^*+\delta_0\Lambda^*)\theta^* r\\ &+\Big[\frac{\Lambda_1}{\Gamma(\alpha)}+\frac{\Lambda_2}{\Gamma(\alpha-1)}+\frac{|\lambda_1|}{|1-\lambda_1|\Gamma(\alpha+\beta)}+\frac{\Lambda_3}{\Gamma(\alpha+\beta-1)}\Big](\theta^*\lambda_0 g^*+\theta^*\delta_0 g^*+f^*).\end{aligned}$$

Let $u\in\mathbb{B}_r$, $t_1$, $t_2\in[0,1]$, such that $t_2<t_1$, we have

$$\begin{aligned}|\mathbf{A}u(t_1)-\mathbf{A}u(t_2)| \leq\ & \frac{|\Lambda_1(t_1)-\Lambda_1(t_2)|}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}|f(s,u(s),Tu(s),Gu(s))|ds\\ &+\frac{|\Lambda_2(t_1)-\Lambda_2(t_2)|}{\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-1}|f(s,u(s),Tu(s),Gu(s))|ds\\ &+\frac{|\Lambda_3(t_1)-\Lambda_3(t_2)|}{\Gamma(\alpha+\beta-1)}\int_0^1(1-s)^{\alpha+\beta-1}|f(s,u(s),Tu(s),Gu(s))|ds\end{aligned}$$

Hence, if $t_1 \longrightarrow t_2$, then $|\mathbf{A}u(t_1) - \mathbf{A}u(t_2)| \longrightarrow 0$ regardless of $u \in \mathbb{B}_r$. Then $\mathbf{A}$ is equicontinuous and so, by Arzela-Ascoli theorem [8], we deduce that $\mathbf{A}$ is compact on $\mathbb{B}_r$. So the operator $\mathbf{A}$ is completely continuous. This ends the proof. □ □ In the next lemma, we show that $\mathbf{B}$ is a contraction,

**Lemma 4.4.** Assume that the hypotheses (H1) and (H3) are satisfied. Then, the operator $\mathbf{B}$ is a contraction.

**Proof .** For $u, v \in \mathbb{B}_r$, and any $t \in [0,1]$, using (H1), we have

$$\begin{aligned}
|\mathbf{B}u(t) - \mathbf{B}v(t)| &\leq \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} |f(s,u(s),Tu(s),G(us)) - f(s,v(s),Tv(s),G(vs))| ds \\
&\leq \frac{\theta^*}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} \Big( |u(s)-v(s)| + |Tu(s)-Tv(s)| + |Gu(s)-Gv(s)| \Big) ds \\
&\leq \frac{\theta^*}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} |u(s)-v(s)| ds \\
&\quad + \frac{\theta^* \Lambda^*}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} \int_0^s (\lambda(s,w)+\delta(s,w)) |u(w)-v(w)| dw ds \\
&\leq \frac{\theta^*(1+\Lambda^*\lambda_0+\Lambda^*\delta_0)}{\Gamma(\alpha+\beta)} \|u-v\|.
\end{aligned}$$

Therefore,

$$\|\mathbf{B}u - \mathbf{B}v\| \leq C^* \|u-v\|,$$

where

$$C^* = \frac{\theta^*(1+\Lambda^*\lambda_0+\Lambda^*\delta_0)}{\Gamma(\alpha+\beta)}.$$

By the hypothesis (H3), we conclude that $\mathbf{B}$ is a contraction. □

Now, we are in a position to prove the existence result of the problem (3.1)-(3.2).

**Theorem 4.5.** Assume that the hypotheses (H) are satisfied. Then, for all $r > 0$ satisfying

$$r \geq 2\Big[\frac{\Lambda_1}{\Gamma(\alpha)} + \frac{\Lambda_2}{\Gamma(\alpha-1)} + \frac{|\lambda_1||1-\lambda_1|}{|1-\lambda_1|\Gamma(\alpha+\beta)} + \frac{\Lambda_3}{\Gamma(\alpha+\beta-1)}\Big](\theta^*\lambda_0 g^* + \theta^*\delta_0 g^* + f^*), \tag{4.3}$$

the problem (3.1)-(3.2) has at least one solution on $\mathbb{B}_r$.

**Proof .** Accordingly to lemma 4.2, we have $\mathbf{A}\mathbb{B}_r + \mathbf{B}\mathbb{B}_r \subseteq \mathbb{B}_r$, next the operator $\mathbf{A}$ is compact by lemma 4.3. Finally, from lemma 4.4, the operator $\mathbf{B}$ is a contraction. Hence, for each $r$ satisfying the condition (4.3), the operators $\mathbf{A}$ and $\mathbf{B}$ satisfies the conditions of the Krasnosel'skii fixed point theorem (Theorem 2.6) in the convex set $\mathbb{B}_r$. Therefore, problem (3.1)-(3.2) has at least one solution. This completes the proof. □

## 5 Example

We consider the following fractional problem:

$$\begin{cases}
{}^cD^{\frac{11}{7}}({}^cD^{\frac{9}{7}})u(t) = \dfrac{t}{3000}\left(\dfrac{1}{1+|u(t)|} + \dfrac{2}{3000}\displaystyle\int_0^t t^5 s^5 |u(t)| ds\right), \ t \in ]0,1[, \\
u(0) = \dfrac{1}{200}u(1), \quad u'(0) = \dfrac{1}{200}u'(1), \\
{}^cD^{\frac{9}{7}}u(0) = \dfrac{1}{200}{}^cD^{\frac{9}{7}}u(1), \\
{}^cD^{\frac{16}{7}}u(0) = \dfrac{1}{200}{}^cD^{\frac{16}{7}}u(1).
\end{cases} \tag{5.1}$$

In this example, we have

$$f(t,x,y,z) = \frac{t}{3000}\Big(\frac{1}{1+|x|} + y + z\Big), \quad g(t,x) = t|x| \text{ and } \lambda(t,s) = \delta(t,s) = \frac{t^5 s^4}{3000}.$$

We have

$$|f(t,x_1,y_1,z_1)-f(t,x_2,y_2,z_2)| \le \frac{t}{3000}\Big(|x_1-x_2|+|y_1-y_2|+|z_1-z_2|\Big),\ t\in]0,1[,\ x_i,y_i,z_i\in\mathbb{R},$$

and

$$|g(t,x)-g(t,y)| \le t|x-y|,\ t\in]0,1[,\ x,y\in\mathbb{R}.$$

We obtain

$$f^* = \theta^* = \frac{1}{3000},\ g^*=0,\ \Lambda^*=1,\ \text{ and } \lambda_0=\delta_0=\frac{1}{15000}.$$

Hance, $C^* \simeq 2.4 * 10^{-8} < 1$. Ten, the problem (5.1) has at least one solution.

## Conclusion

In this paper, we have successfully established the existence result of a nonlinear sequential fractional integr–differential equations with nonseparated boundary conditions. Our result is obtained by Krasnosel'skii fixed point theorem. Since theoretical results can help to get an in-depth understanding for the fractional order model, motivated by the mentioned equation models. It should be noted that we used Caputo fractional derivative because of its applicability to real world physical problems.

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