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Ostrowski type inequalities via $(\alpha, \beta, \gamma, \delta)$-convex function

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Abstract

In this paper, we are introducing very first time the class of $(\alpha, \beta, \gamma, \delta)-$ convex (concave) function in mixed kind, which is the generalization of many classes of functions given in [2, 3, 4, 15, 16, 17]. We would like to state the well-known Ostrowski inequality via generalized Montgomery identity [14] for $(\alpha, \beta, \gamma, \delta)-$ convex (concave) function in mixed kind. In addition, we establish some Ostrowski-type inequalities for the class of functions whose derivatives in absolute values at certain powers are $(\alpha, \beta, \gamma, \delta)$-convex (concave) functions in mixed kind by using different techniques including Hölder's inequality [27] and power mean inequality [26]. Also, various established results would be captured as special cases. Moreover, some applications in terms of special means would also be given.

Keywords: Ostrowski inequality, Montgomery identity, convex functions, special means
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1 Introduction

From literature, we recall and introduce some definitions for various convex (concave) functions.

Definition 1.1. [3] A function $\eta : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex (concave) function, if

$$\eta(tx + (1-t)y) \leq (\geq) t\eta(x) + (1-t)\eta(y),$$

for all $x, y \in I, t \in [0, 1]$.

We recall here definition of $P-$convex(concave) function from [15]:

Definition 1.2. Let $\eta : I \subset \mathbb{R} \to \mathbb{R}$ is a $P-$convex(concave) function, if η is a non-negative and $\forall x, y \in I$ and $t \in [0, 1]$, we have

$$\eta(tx + (1-t)y) \leq (\geq) \eta(x) + \eta(y).$$

Here we also have definition of quasi-convex (concave) function, for detailed discussion see [17].

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Definition 1.3. A function $\eta : I \subset \mathbb{R} \to \mathbb{R}$ is known as quasi$-$convex(concave), if

$$\eta(tx + (1-t)y) \leq (\geq) \max\{\eta(x), \eta(y)\}$$

for all $x, y \in I$, $t \in [0,1]$.

Now we present definition of $s-$convex functions in the first kind as follows which are extracted from [24]:

Definition 1.4. [4] Let $s \in [0,1]$. A function $\eta : I \subset [0,\infty) \to [0,\infty)$ is said to be $s-$convex (concave) function in the 1^{st} kind, if

$$\eta\left(tx + (1-t)y\right) \leq (\geq) t^s \eta(x) + (1-t^s)\eta(y),$$

for all $x, y \in I, t \in [0,1]$.

Remark 1.5. Note that in this definition we also included $s = 0$. Further, if we put $s = 0$, we get quasi$-$convexity (see Definition 1.3).

For second kind convexity we recall definition from [24].

Definition 1.6. Let $s \in [0,1]$. A function $\eta : I \subset [0,\infty) \to [0,\infty)$ is said to be $s-$convex (concave) function in the 2^{nd} kind, if

$$\eta\left(tx + (1-t)y\right) \leq (\geq) t^s \eta(x) + (1-t)^s \eta(y),$$

for all $x, y \in I, t \in [0,1]$.

Remark 1.7. In the similar manner, we have slightly improved definition of second kind convexity by including $s = 0$. Further if we put $s = 0$, we easily get $P-$convexity (see Definition 1.2).

Now we introduce a new class of function which would be called class of $(s,r)-$ convex (concave) functions in the mixed kind:

Definition 1.8. Let $(s,r) \in [0,1]^2$. A function $\eta : I \subset [0,\infty) \to [0,\infty)$ is said to be $(s,r)-$convex (concave) function in mixed kind, if

$$\eta\left(tx + (1-t)y\right) \leq (\geq) t^{rs} \eta(x) + (1-t^r)^s \eta(y),$$

for all $x, y \in I, t \in [0,1]$.

Definition 1.9. [16] Let $(\alpha,\beta) \in [0,1]^2$. A function $\eta : I \subset [0,\infty) \to [0,\infty)$ is said to be $(\alpha,\beta)-$convex(concave) in the 1^{st} kind, if

$$\eta\left(tx + (1-t)y\right) \leq (\geq) t^{\alpha} \eta(x) + (1-t^{\beta})\eta(y),$$

for all $x, y \in I, t \in [0,1]$.

Definition 1.10. [16] Let $(\alpha,\beta) \in [0,1]^2$. A function $\eta : I \subset [0,\infty) \to [0,\infty)$ is said to be $(\alpha,\beta)-$convex(concave) function in the 2^{nd} kind, if

$$\eta\left(tx + (1-t)y\right) \leq (\geq) t^{\alpha} \eta(x) + (1-t)^{\beta} \eta(y),$$

for all $x, y \in I, t \in [0,1]$.

Next we introduce $(\alpha,\beta,\gamma,\delta)-$convex(concave) in mixed kind.

Definition 1.11. Let $(\alpha, \beta, \gamma, \delta) \in [0,1]^4$. A function $\eta : I \subset [0,\infty) \to [0,\infty)$ is said to be $(\alpha, \beta, \gamma, \delta)-$convex(concave) function in mixed kind, if

$$\eta(tx + (1-t)y) \leq (\geq) t^{\alpha\gamma}\eta(x) + (1 - t^{\beta\gamma})^{\delta}\eta(y), \tag{1.1}$$

for all $x, y \in I, t \in [0,1]$.

Remark 1.12. In Definition 1.11, we have the following cases.

1. If we choose $\gamma = \delta = 1$ in (1.1), we get $(\alpha, \beta)-$convex (concave) in 1^{st} kind function.
2. If we choose $\beta = \gamma = 1$ in (1.1), we get $(\alpha, \beta)-$convex (concave) in 2^{nd} kind function.
3. If we choose $\alpha = \delta = s$, $\beta = 1, \gamma = r$, where $s, r \in [0,1]$ in (1.1), we get $(s,r)-$convex (concave) in mixed kind function.
4. If we choose $\alpha = \beta = s$ and $\gamma = \delta = 1$ where $s \in [0,1]$ in (1.1), we get $s-$convex (concave) in 1^{st} kind function.
5. If we choose $\alpha = \beta = 0$, and $\gamma = \delta = 1$, in (1.1), we get quasi$-$convex (concave) function.
6. If we choose $\alpha = \delta = s$, $\beta = \gamma = 1$ where $s \in [0,1]$ in (1.1), we get $s-$convex (concave) in 2^{nd} kind function.
7. If we choose $\alpha = \delta = 0$, and $\beta = \gamma = 1$, in (1.1), we get $P-$convex (concave) function.
8. If we choose $\alpha = \beta = \gamma = \delta = 1$ in (1.1), gives us ordinary convex (concave) function.

In almost every field of science, inequalities play a significant role. Although it is very vast discipline but our focus is mainly on Ostrowski type inequalities. In 1938, Ostrowski established the following interesting integral inequality for differentiable mappings with bounded derivatives [25]. This inequality is well known in the literature as Ostrowski inequality.

Theorem 1.13. Let $\varphi : [\rho_a, \rho_b] \to \mathbb{R}$ be differentiable function on (ρ_a, ρ_b) with the property that $|\varphi'(t)| \leq M$ for all $t \in (\rho_a, \rho_b)$. Then

$$\left|\varphi(x) - \frac{1}{\rho_b - \rho_a}\int_{\rho_a}^{\rho_b} \varphi(t)dt\right| \leq M(\rho_b - \rho_a)\left[\frac{1}{4} + \left(\frac{x - \frac{\rho_a+\rho_b}{2}}{\rho_b - \rho_a}\right)^2\right],$$

for all $x \in (\rho_a, \rho_b)$. The constant $\frac{1}{4}$ is the best possible in the kind that it cannot be replaced by a smaller quantity.

The generalization of Montgomery identity via parameter is introduced in [14] by Dragomir.

Theorem 1.14. [9] If $\varphi : [\rho_a, \rho_b] \to \mathbb{R}$ is differentiable on $[\rho_a, \rho_b]$ with φ' integrable on $[\rho_a, \rho_b]$, where $\epsilon \in [0,1]$, then the generalized Montgomery identity holds

$$(1-\epsilon)\varphi(x) + \epsilon\frac{\varphi(\rho_a) + \varphi(\rho_b)}{2} = \frac{1}{\rho_b - \rho_a}\int_{\rho_a}^{\rho_b} \varphi(t)dt + \frac{1}{\rho_b - \rho_a}\int_{\rho_a}^{\rho_b} P(x,t)\varphi'(t)dt,$$

where $P(x,t)$ is the Generalized Peano Kernel defined by:

$$P(x,t) = \begin{cases} t - \mu, & \text{if} \quad t \in [\rho_a, x], \\ t - \nu, & \text{if} \quad t \in (x, \rho_b]. \end{cases}$$

for all $x \in [\mu, \nu]$ for $\mu = \rho_a + \epsilon\frac{\rho_b - \rho_a}{2}$ and $\nu = \rho_b - \epsilon\frac{\rho_b - \rho_a}{2}$.

Ostrowski inequality has applications in numerical integration, probability and optimization theory, statistics, information and integral operator theory. Until now, a large number of research papers and books have been written on generalizations of Ostrowski inequalities and their numerous applications in [9]-[14] and [18]-[22]. Let $[\rho_a, \rho_b] \subseteq (0, +\infty)$, we may define special means as follows:

(a) The arithmetic mean

$$A = A(\rho_a, \rho_b) := \frac{\rho_a + \rho_b}{2};$$

(b) The geometric mean

$$G = G(\rho_a, \rho_b) := \sqrt{\rho_a \rho_b};$$

(c) The harmonic mean

$$H = H(\rho_a, \rho_b) := \frac{2}{\frac{1}{\rho_a} + \frac{1}{\rho_b}};$$

(d) The logarithmic mean

$$L = L(\rho_a, \rho_b) := \begin{cases} \rho_a & \text{if} \quad \rho_a = \rho_b \\ \dfrac{\rho_b - \rho_a}{\ln \rho_b - \ln \rho_a}, & \text{if} \quad \rho_a \neq \rho_b \end{cases};$$

(e) The identric mean

$$I = I(\rho_a, \rho_b) := \begin{cases} \rho_a & \text{if} \quad \rho_a = \rho_b \\ \dfrac{1}{e}\left(\dfrac{{\rho_b}^{\rho_b}}{{\rho_a}^{\rho_a}}\right)^{\frac{1}{\rho_b - \rho_a}}, & \text{if} \quad \rho_a \neq \rho_b. \end{cases};$$

(f) The $p-$logarithmic mean

$$L_p = L_p(\rho_a, \rho_b) := \begin{cases} \rho_a & \text{if} \quad \rho_a = \rho_b \\ \left[\dfrac{\rho_b^{p+1} - {\rho_a}^{p+1}}{(p+1)(\rho_a - \rho_b)}\right]^{\frac{1}{p}}, & \text{if} \quad \rho_a \neq \rho_b. \end{cases};$$

where $p \in \mathbb{R} \setminus \{0, -1\}$.

In order to prove our main results, we need the following lemma that has been obtained in [5].

Lemma 1.15. Let $\varphi : [\rho_a, \rho_b] \to \mathbb{R}$ be an absolutely continuous mapping on (ρ_a, ρ_b) with $a < b$. If $\varphi' \in L_1([\rho_a, \rho_b])$, then $x \in (\rho_a, \rho_b)$ the following identity holds:

$$\varphi(x) - \frac{1}{\rho_b - \rho_a}\int_{\rho_a}^{\rho_b} \varphi(t)dt = \frac{(x-\rho_a)^2}{\rho_b - \rho_a}\int_0^1 t\varphi'(tx + (1-t)\rho_a)dt - \frac{(\rho_b - x)^2}{\rho_b - \rho_a}\int_0^1 t\varphi'(tx + (1-t)\rho_b)dt. \quad (1.2)$$

We make use of the beta function of Euler type, which is for $x, y > 0$ defined as

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

where $\Gamma(x) = \int_0^\infty e^{-u}u^{x-1}du$.

The main aim of our study is to generalize the ostrowski inequality (1.2) for $(\alpha, \beta, \gamma, \delta)-$convex(concave) in mixed kind, which is given in Section 2. Moreover we establish some Ostrowski type inequalities for the class of functions whose derivatives in absolute values at certain powers are $(\alpha, \beta, \gamma, \delta)$-convex (concave) functions in mixed kind by using different techniques including Hölder's inequality [27] and power mean inequality [26]. Also we give the special cases of our results. The application of midpoint inequalities in the special means, some particular cases of these inequalities given in Section 3. The last section gives us conclusion with some remarks and future ideas.

2 Generalization of Ostrowski inequalities

Theorem 2.1. Suppose all the assumptions of Lemma 1.15 hold. Additionally, assume that $|\varphi'|$ is $(\alpha, \beta, \gamma, \delta)-$convex function on $[\rho_a, \rho_b]$ and $|\varphi'(x)| \leq M$. Then for each $x \in (\rho_a, \rho_b)$ the following inequality holds:

$$\left|\varphi(x) - \frac{1}{\rho_b - \rho_a}\int_{\rho_a}^{\rho_b} \varphi(t)dt\right| \leq M\left(\frac{1}{\alpha\gamma + 2} + \frac{B\left(\frac{2}{\beta\gamma}, \delta + 1\right)}{\beta\gamma}\right) I(x), \tag{2.1}$$

where $I(x) = \frac{(x-\rho_a)^2 + (\rho_b - x)^2}{\rho_b - \rho_a}$.

Proof . From the Lemma 1.15, we have

$$\left|\varphi(x) - \frac{1}{\rho_b - \rho_a}\int_{\rho_a}^{\rho_b} \varphi(t)dt\right| \leq \frac{(x-\rho_a)^2}{\rho_b - \rho_a}\int_0^1 t\left|\varphi'(tx + (1-t)\rho_a)\right| dt + \frac{(\rho_b - x)^2}{\rho_b - \rho_a}\int_0^1 t\left|\varphi'(tx + (1-t)\rho_b)\right| dt. \tag{2.2}$$

Since $|\varphi'|$ is $(\alpha, \beta, \gamma, \delta)-$convex on $[\rho_a, \rho_b]$ and $|\varphi'(x)| \leq M$, we have

$$\int_0^1 t\left|\varphi'(tx + (1-t)\rho_a)\right| dt \leq M\int_0^1 t\left(t^{\alpha\gamma} + (1 - t^{\beta\gamma})^{\delta}\right) dt \tag{2.3}$$

and similarly

$$\int_0^1 t\left|\varphi'(tx + (1-t)\rho_b)\right| dt \leq M\int_0^1 t\left(t^{\alpha\gamma} + (1 - t^{\beta\gamma})^{\delta}\right) dt. \tag{2.4}$$

By using inequalities (2.3) and (2.4) in (2.2), we get

$$\left|\varphi(x) - \frac{1}{\rho_b - \rho_a}\int_{\rho_a}^{\rho_b} \varphi(t)dt\right| \leq M\left(\frac{1}{\alpha\gamma + 2} + \frac{B\left(\frac{2}{\beta\gamma}, \delta + 1\right)}{\beta\gamma}\right)\left[\frac{(x-\rho_a)^2 + (\rho_b - x)^2}{\rho_b - \rho_a}\right],$$

which completes the proof. □

Corollary 2.2. In Theorem 2.1, one can see the following.

1. If one takes $\gamma = \delta = 1$, $\alpha \in [0, 1]$ and $\beta \in (0, 1]$, in (2.1), one has the Ostrowski inequality for $(\alpha, \beta)-$convex functions in 1^{st} kind:

$$\left|\varphi(x) - \frac{1}{\rho_b - \rho_a}\int_{\rho_a}^{\rho_b} \varphi(t)dt\right| \leq M\left(\frac{1}{\alpha + 2} + \frac{B\left(\frac{2}{\beta}, 2\right)}{\beta}\right) I(x).$$

2. If one takes $\beta = \gamma = 1$, $\alpha \in [0, 1]$ and $\delta \in [0, 1]$, in (2.1), then one has the Ostrowski inequality for $(\alpha, \delta)-$convex functions in 2^{nd} kind:

$$\left|\varphi(x) - \frac{1}{\rho_b - \rho_a}\int_{\rho_a}^{\rho_b} \varphi(t)dt\right| \leq M\left(\frac{1}{\alpha + 2} + \frac{1}{(\delta + 1)(\delta + 2)}\right) I(x).$$

3. If one takes $\alpha = \delta = s$, $\beta = 1, \gamma = r$, where $s \in [0, 1]$ and $r \in (0, 1]$ in (2.1), then one has the Ostrowski inequality for $(s, r)-$convex functions in mixed kind:

$$\left|\varphi(x) - \frac{1}{\rho_b - \rho_a}\int_{\rho_a}^{\rho_b} \varphi(t)dt\right| \leq M\left(\frac{1}{rs + 2} + \frac{B\left(\frac{2}{r}, s + 1\right)}{r}\right) I(x).$$

4. If one takes $\alpha = \beta = s$ and $\gamma = \delta = 1$, where $s \in (0,1]$ in (2.1), then one has the Ostrowski inequality for $s-$convex functions in 1^{st} kind:

$$\left|\varphi(x) - \frac{1}{\rho_b - \rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right| \leq M\left(\frac{1}{s+2} + \frac{B\left(\frac{2}{s},2\right)}{s}\right)I(x).$$

5. If one takes $\alpha = \delta = 0$ and $\beta = \gamma = 1$ in (2.1), then one has the Ostrowski inequality for $P-$convex functions:

$$\left|\varphi(x) - \frac{1}{\rho_b - \rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right| \leq MI(x).$$

6. If one takes $\beta = \gamma = 1$, $\alpha = \delta = s$ where $s \in [0,1]$, then (2.1) reduces to the inequality (2.1) of Theorem 2 in [1].
7. If one takes $\alpha = \beta = \gamma = \delta = 1$, then (2.1) reduces to the inequality (1.2).

Theorem 2.3. Suppose all the assumptions of Lemma 1.15 hold. Additionally, assume that $|\varphi'|^q$ is $(\alpha,\beta,\gamma,\delta)-$convex function on $[\rho_a,\rho_b], q \geq 1$ and $|\varphi'(x)| \leq M$, then for each $x \in (\rho_a,\rho_b)$ the following inequality holds:

$$\left|\varphi(x) - \frac{1}{\rho_b - \rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right| \leq \frac{M}{(2)^{1-\frac{1}{q}}}\left(\frac{1}{\alpha\gamma+2} + \frac{B\left(\frac{2}{\beta\gamma},\delta+1\right)}{\beta\gamma}\right)^{\frac{1}{q}} I(x).$$

Proof . From the Lemma 1.15 and power mean inequality [26], we have

$$\begin{aligned}\left|\varphi(x) - \frac{1}{\rho_b - \rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right| &\leq \frac{(x-\rho_a)^2}{\rho_b-\rho_a}\int_0^1 t\left|\varphi'(tx+(1-t)\rho_a)\right|dt + \frac{(\rho_b-x)^2}{\rho_b-\rho_a}\int_0^1 t\left|\varphi'(tx+(1-t)\rho_b)\right|dt \\ &\leq \frac{(x-\rho_a)^2}{\rho_b-\rho_a}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|\varphi'\left(tx+(1-t)\rho_a\right)\right|^q dt\right)^{\frac{1}{q}} \\ &\quad + \frac{(\rho_b-x)^2}{\rho_b-\rho_a}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|\varphi'\left(tx+(1-t)\rho_b\right)\right|^q dt\right)^{\frac{1}{q}}. \end{aligned} \tag{2.5}$$

Since $|\varphi'|^q$ is $(\alpha,\beta,\gamma,\delta)-$convex on $[\rho_a,\rho_b]$. and $|\varphi'(x)| \leq M$, we get

$$\int_0^1 t\left|\varphi'\left(tx+(1-t)\rho_a\right)\right|^q dt \leq M^q\int_0^1 t\left(t^{\alpha\gamma} + (1-t^{\beta\gamma})^{\delta}\right)dt \tag{2.6}$$

and

$$\int_0^1 t\left|\varphi'\left(tx+(1-t)\rho_b\right)\right|^q dt \leq M^q\int_0^1 t\left(t^{\alpha\gamma} + (1-t^{\beta\gamma})^{\delta}\right)dt. \tag{2.7}$$

Using the inequalities $(2.5)-(2.7)$, we get

$$\left|\varphi(x) - \frac{1}{\rho_b - \rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right| \leq \frac{M}{(2)^{1-\frac{1}{q}}}\left(\frac{1}{\alpha\gamma+2} + \frac{B\left(\frac{2}{\beta\gamma},\delta+1\right)}{\beta\gamma}\right)^{\frac{1}{q}} I(x),$$

which completes the proof. □

Corollary 2.4. In Theorem 2.3, one can see the following.

1. If one takes $q = 1$, one has the Theorem 2.1.

2. If one takes $\gamma=\delta=1$, $\alpha\in[0,1]$ and $\beta\in(0,1]$, in (2.5), one has the Ostrowski inequality for $(\alpha,\beta)-$convex functions in 1^{st} kind:

$$\left|\varphi(x)-\frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right|\leq\frac{M}{(2)^{1-\frac{1}{q}}}\left(\frac{1}{\alpha+2}+\frac{B\left(\frac{2}{\beta},2\right)}{\beta}\right)^{\frac{1}{q}}I(x).$$

3. If one takes $\beta=\gamma=1$, $\alpha\in[0,1]$ and $\delta\in[0,1]$, in (2.5), then one has the Ostrowski inequality for $(\alpha,\delta)-$convex functions in 2^{nd} kind:

$$\left|\varphi(x)-\frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right|\leq\frac{M}{(2)^{1-\frac{1}{q}}}\left(\frac{1}{(\alpha+2)}+\frac{1}{(\delta+1)(\delta+2)}\right)^{\frac{1}{q}}I(x).$$

4. If one takes $\alpha=\delta=s,\beta=1,\gamma=r$, where $s\in[0,1]$ and $r\in(0,1]$ in (2.5), then one has the Ostrowski inequality for $(s,r)-$convex functions in mixed kind:

$$\left|\varphi(x)-\frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right|\leq\frac{M}{(2)^{1-\frac{1}{q}}}\left(\frac{1}{rs+2}+\frac{B\left(\frac{2}{r},s+1\right)}{r}\right)^{\frac{1}{q}}I(x).$$

5. If one takes $\alpha=\beta=s$ and $\gamma=\delta=1$, where $s\in(0,1]$ in (2.5), then one has the Ostrowski inequality for $s-$convex functions in 1^{st} kind:

$$\left|\varphi(x)-\frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right|\leq\frac{M}{(2)^{1-\frac{1}{q}}}\left(\frac{1}{s+2}+\frac{B\left(\frac{2}{s},2\right)}{s}\right)^{\frac{1}{q}}I(x).$$

6. If one takes $\alpha=\delta=0$ and $\beta=\gamma=1$ in (2.5), then one has the Ostrowski inequality for $P-$convex functions:

$$\left|\varphi(x)-\frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right|\leq\frac{M}{(2)^{1-\frac{1}{q}}}I(x).$$

7. If one takes $\beta=\gamma=1$, $\alpha=\delta=s$ where $s\in[0,1]$, then (2.5) reduces to the inequality (2.3) of Theorem 4 in [1].

8. If one takes $\alpha=\beta=\gamma=\delta=1$, then (2.5) reduces to the inequality (1.2).

Theorem 2.5. Suppose all the assumptions of Lemma 1.15 hold. Additionally, assume that $|\varphi'|^q$ is $(\alpha,\beta,\gamma,\delta)-$convex function on $[\rho_a,\rho_b], q>1$ and $|\varphi'(x)|\leq M$, then for each $x\in(\rho_a,\rho_b)$, the following inequality holds:

$$\left|\varphi(x)-\frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right|\leq\frac{M}{(p+1)^{\frac{1}{p}}}\left(\frac{1}{\alpha\gamma+1}+\frac{B\left(\frac{1}{\beta\gamma},\delta+1\right)}{\beta\gamma}\right)^{\frac{1}{q}}I(x),\tag{2.8}$$

where $p^{-1}+q^{-1}=1$.

Proof . From the Lemma 1.15 and Hölder's inequality [27], we have

$$\begin{aligned}\left|\varphi(x)-\frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right|\leq&\frac{(x-\rho_a)^2}{\rho_b-\rho_a}\left(\int_0^1 t^{\zeta p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|\varphi'\left(tx+(1-t)\rho_a\right)\right|^q dt\right)^{\frac{1}{q}}\\&+\frac{(\rho_b-x)^2}{\rho_b-\rho_a}\left(\int_0^1 t^{\zeta p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|\varphi'\left(tx+(1-t)\rho_b\right)\right|^q dt\right)^{\frac{1}{q}}.\end{aligned}\tag{2.9}$$

Since $|\varphi'|^q$ is $(\alpha,\beta,\gamma,\delta)-$convex, we have

$$\int_0^1\left|\varphi'\left(tx+(1-t)\rho_a\right)\right|^q dt\leq M^q\int_0^1 t^{\alpha\gamma}+(1-t^{\beta\gamma})^{\delta}dt\tag{2.10}$$

and

$$\int_0^1 |\varphi'(tx+(1-t)\rho_b)|^q\,dt \le M^q \int_0^1 t^{\alpha\gamma} + (1-t^{\beta\gamma})^{\delta}dt. \tag{2.11}$$

Using inequalities (2.9) – (2.11), we get

$$\left|\varphi(x) - \frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right| \le \frac{M}{(p+1)^{\frac{1}{p}}}\left(\frac{1}{\alpha\gamma+1} + \frac{B\left(\frac{1}{\beta\gamma},\delta+1\right)}{\beta\gamma}\right)^{\frac{1}{q}} I(x),$$

which completes the proof. □

Corollary 2.6. In Theorem 2.5, one can see the following.

1. If one takes $\gamma=\delta=1$, $\alpha\in[0,1]$ and $\beta\in(0,1]$, in (2.8), one has the Ostrowski inequality for $(\alpha,\beta)-$convex functions in 1^{st} kind:

$$\left|\varphi(x) - \frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right| \le \frac{M}{(p+1)^{\frac{1}{p}}}\left(\frac{1}{\alpha+1} + \frac{B\left(\frac{1}{\beta},2\right)}{\beta}\right)^{\frac{1}{q}} I(x).$$

2. If one takes $\beta=\gamma=1$, $\alpha\in[0,1]$ and $\delta\in[0,1]$, in (2.8), then one has the Ostrowski inequality for $(\alpha,\delta)-$convex functions in 2^{nd} kind:

$$\left|\varphi(x) - \frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right| \le \frac{M}{(p+1)^{\frac{1}{p}}}\left(\frac{1}{\alpha+1} + \frac{1}{\delta+1}\right)^{\frac{1}{q}} I(x).$$

3. If one takes $\alpha=\delta=s$, $\beta=1,\gamma=r$, where $s\in[0,1]$ and $r\in(0,1]$ in (2.8), then one has the Ostrowski inequality for $(s,r)-$convex functions in mixed kind:

$$\left|\varphi(x) - \frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right| \le \frac{M}{(p+1)^{\frac{1}{p}}}\left(\frac{1}{rs+1} + \frac{B\left(\frac{1}{r},s+1\right)}{r}\right)^{\frac{1}{q}} I(x).$$

4. If one takes $\alpha=\beta=s$ and $\gamma=\delta=1$, where $s\in(0,1]$ in (2.8), then one has the Ostrowski inequality for $s-$convex functions in 1^{st} kind:

$$\left|\varphi(x) - \frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right| \le \frac{M}{(p+1)^{\frac{1}{p}}}\left(\frac{1}{s+1} + \frac{B\left(\frac{1}{s},2\right)}{s}\right)^{\frac{1}{q}} I(x).$$

5. if one takes $\beta=\gamma=1$, $\alpha=\delta=s$ where $s\in[0,1]$, then (2.8) reduces to the inequality (2.2) of Theorem 3 in [1].

6. If one takes $\alpha=\delta=0$ and $\beta=\gamma=1$ in (2.8), then one has the Ostrowski inequality for $P-$convex functions:

$$\left|\varphi(x) - \frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right| \le \frac{(2)^{\frac{1}{q}}M}{(p+1)^{\frac{1}{p}}} I(x).$$

7. If one takes $\alpha=\beta=\gamma=\delta=1$ in (2.8), then one has the Ostrowski inequality for convex functions:

$$\left|\varphi(x) - \frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right| \le \frac{M}{(p+1)^{\frac{1}{p}}} I(x).$$

Theorem 2.7. Let $\varphi:[\rho_a,\rho_b]\to\mathbb{R}$ be an absolutely continuous. If $\eta: I\subset[0,\infty)\to\mathbb{R}$ is $(\alpha,\beta,\gamma,\delta)-$convex(concave) in mixed kind, then we have the inequalities:

$$\eta\left((1-\epsilon)\varphi(x)+\epsilon\frac{\varphi(\rho_a)+\varphi(\rho_b)}{2}-\frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)\le(\ge)\left(\frac{x-\rho_a}{\rho_b-\rho_a}\right)^{\alpha\gamma}\left[\frac{1}{x-\rho_a}\int_{\rho_a}^{x}\eta\left[(t-\mu)\varphi'(t)\right]dt\right]$$
$$+\left(1-\left(\frac{x-\rho_a}{\rho_b-\rho_a}\right)^{\beta\gamma}\right)^{\delta}\left[\frac{1}{\rho_b-x}\int_{x}^{\rho_b}\eta\left[(t-\nu)\varphi'(t)\right]dt\right], \tag{2.12}$$

for all $x\in[\mu,\nu]$ for $\epsilon\in[0,1]$.

Proof . Utilizing the Generalized Montgomery identity

$$(1-\epsilon)\varphi(x)+\epsilon\frac{\varphi(\rho_a)+\varphi(\rho_b)}{2}-\frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt=\left(\frac{x-\rho_a}{\rho_b-\rho_a}\right)\left[\frac{1}{x-\rho_a}\int_{\rho_a}^{x}(t-\mu)\varphi'(t)dt\right]$$
$$+\left(1-\left(\frac{x-\rho_a}{\rho_b-\rho_a}\right)\right)\left[\frac{1}{\rho_b-x}\int_{x}^{\rho_b}(t-\nu)\varphi'(t)dt\right], \tag{2.13}$$

using the $(\alpha,\beta,\gamma,\delta)-$convexity(concavity) in mixed kind of $\eta: I\subset[0,\infty)\to\mathbb{R}$, we have

$$\eta\left((1-\epsilon)\varphi(x)+\epsilon\frac{\varphi(\rho_a)+\varphi(\rho_b)}{2}-\frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)\le(\ge)\left(\frac{x-\rho_a}{\rho_b-\rho_a}\right)^{\alpha\gamma}\eta\left[\frac{1}{x-\rho_a}\int_{\rho_a}^{x}(t-\mu)\varphi'(t)dt\right]$$
$$+\left(1-\left(\frac{x-\rho_a}{\rho_b-\rho_a}\right)^{\beta\gamma}\right)^{\delta}\eta\left[\frac{1}{\rho_b-x}\int_{x}^{\rho_b}(t-\nu)\varphi'(t)dt\right], \tag{2.14}$$

for all $x\in[\mu,\nu]$ for $\epsilon\in[0,1]$, which is an inequality of interest in itself as well. If we use Jensen's Integral identity

$$\eta\left(\frac{1}{\rho_d-\rho_c}\int_{\rho_c}^{\rho_d}g(t)dt\right)\le(\ge)\frac{1}{\rho_d-\rho_c}\int_{\rho_c}^{\rho_d}\eta[g(t)]dt,$$

we have

$$\eta\left(\frac{1}{x-\rho_a}\int_{\rho_a}^{x}(t-\mu)\varphi'(t)dt\right)\le(\ge)\frac{1}{x-\rho_a}\int_{\rho_a}^{x}\eta\left[(t-\nu)\varphi'(t)\right]dt, \tag{2.15}$$

and

$$\eta\left(\frac{1}{\rho_b-x}\int_{x}^{\rho_b}(t-\mu)\varphi'(t)dt\right)\le(\ge)\frac{1}{\rho_b-x}\int_{x}^{\rho_b}\eta\left[(t-\nu)\varphi'(t)\right]dt, \tag{2.16}$$

for all $x\in[\mu,\nu]$ for $\epsilon\in[0,1]$. Making use of (2.14) – (2.16), we get

$$\eta\left((1-\epsilon)\varphi(x)+\epsilon\frac{\varphi(\rho_a)+\varphi(\rho_b)}{2}-\frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)\le(\ge)\left(\frac{x-\rho_a}{\rho_b-\rho_a}\right)^{\alpha\gamma}\left[\frac{1}{x-\rho_a}\int_{\rho_a}^{x}\eta\left[(t-\mu)\varphi'(t)\right]dt\right]$$
$$+\left(1-\left(\frac{x-\rho_a}{\rho_b-\rho_a}\right)^{\beta\gamma}\right)^{\delta}\left[\frac{1}{\rho_b-x}\int_{x}^{\rho_b}\eta\left[(t-\nu)\varphi'(t)\right]dt\right], \tag{2.17}$$

for all $x\in[\mu,\nu]$ for $\epsilon\in[0,1]$. This completes the proof. □

Remark 2.8. In Theorem 2.7, if we choose $\epsilon=0$, then (2.12) reduces to the inequality:

$$\eta\left(\varphi(x)-\frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)\le(\ge)\left(\frac{x-\rho_a}{\rho_b-\rho_a}\right)^{\alpha\gamma}\left[\frac{1}{x-\rho_a}\int_{\rho_a}^{x}\eta\left[(t-\rho_a)\varphi'(t)\right]dt\right]$$
$$+\left(1-\left(\frac{x-\rho_a}{\rho_b-\rho_a}\right)^{\beta\gamma}\right)^{\delta}\left[\frac{1}{\rho_b-x}\int_{x}^{\rho_b}\eta\left[(t-\rho_b)\,\varphi'(t)\right]dt\right].$$

Corollary 2.9. In Theorem 2.1, one can see the following.

1. If one takes $\gamma = \delta = 1$, $\alpha \in [0,1]$ and $\beta \in [0,1]$ in (2.12), one has the functional generalization of Ostrowski inequality for $(\alpha,\beta)-$convex(concave) functions in 1^{st} kind:

$$\begin{aligned}&\eta\left((1-\epsilon)\varphi(x)+\epsilon\frac{\varphi(\rho_a)+\varphi(\rho_b)}{2}-\frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)\\&\leq(\geq)\left(\frac{x-\rho_a}{\rho_b-\rho_a}\right)^{\alpha}\left[\frac{1}{x-\rho_a}\int_{\rho_a}^{x}\eta\left[(t-\mu)\varphi'(t)\right]dt\right]+\left(1-\left(\frac{x-\rho_a}{\rho_b-\rho_a}\right)^{\beta}\right)\left[\frac{1}{\rho_b-x}\int_{x}^{\rho_b}\eta\left[(t-\nu)\varphi'(t)\right]dt\right].\end{aligned} \tag{2.18}$$

Remark 2.10. If one choose $\epsilon = 0$, then (2.18) reduces to the inequality:

$$\begin{aligned}\eta\left(\varphi(x)-\frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)&\leq(\geq)\left(\frac{x-\rho_a}{\rho_b-\rho_a}\right)^{\alpha}\left[\frac{1}{x-\rho_a}\int_{\rho_a}^{x}\eta[(t-\rho_a)\varphi'(t)]dt\right]\\&+\left(1-\left(\frac{x-\rho_a}{\rho_b-\rho_a}\right)^{\beta}\right)\left[\frac{1}{\rho_b-x}\int_{x}^{\rho_b}\eta[(t-\rho_b)\varphi'(t)]dt\right].\end{aligned}$$

2. If one takes $\beta = \gamma = 1$, $\alpha \in [0,1]$ and $\delta \in [0,1]$ in (2.12), then one has the functional generalization of Ostrowski inequality for $(\alpha,\delta)-$convex(concave) functions in 2^{nd} kind:

$$\begin{aligned}\eta\left((1-\epsilon)\varphi(x)+\epsilon\frac{\varphi(\rho_a)+\varphi(\rho_b)}{2}-\frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)&\leq(\geq)\left(\frac{x-\rho_a}{\rho_b-\rho_a}\right)^{\alpha}\left[\frac{1}{x-\rho_a}\int_{\rho_a}^{x}\eta\left[(t-\mu)\varphi'(t)\right]dt\right]\\&+\left(1-\left(\frac{x-\rho_a}{\rho_b-\rho_a}\right)\right)^{\delta}\left[\frac{1}{\rho_b-x}\int_{x}^{\rho_b}\eta\left[(t-\nu)\varphi'(t)\right]dt\right].\end{aligned} \tag{2.19}$$

Remark 2.11. If one choose $\epsilon = 0$, then (2.19) reduces to the inequality:

$$\begin{aligned}\eta\left(\varphi(x)-\frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)&\leq(\geq)\left(\frac{x-\rho_a}{\rho_b-\rho_a}\right)^{\alpha}\left[\frac{1}{x-\rho_a}\int_{\rho_a}^{x}\eta[(t-\rho_a)\varphi'(t)]dt\right]\\&+\left(1-\left(\frac{x-\rho_a}{\rho_b-\rho_a}\right)\right)^{\delta}\left[\frac{1}{\rho_b-x}\int_{x}^{\rho_b}\eta[(t-\rho_b)\varphi'(t)]dt\right].\end{aligned}$$

3. If one takes $\alpha = \delta = s$, $\beta = 1, \gamma = r$, where $s \in [0,1]$ and $r \in [0,1]$ in (2.12), then one has the functional generalization of Ostrowski inequality for $(s,r)-$convex(concave) functions in mixed kind:

$$\begin{aligned}\eta\left((1-\epsilon)\varphi(x)+\epsilon\frac{\varphi(\rho_a)+\varphi(\rho_b)}{2}-\frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)&\leq(\geq)\left(\frac{x-\rho_a}{\rho_b-\rho_a}\right)^{rs}\left[\frac{1}{x-\rho_a}\int_{\rho_a}^{x}\eta\left[(t-\mu)\varphi'(t)\right]dt\right]\\&+\left(1-\left(\frac{x-\rho_a}{\rho_b-\rho_a}\right)^{r}\right)^{s}\left[\frac{1}{\rho_b-x}\int_{x}^{\rho_b}\eta\left[(t-\nu)\varphi'(t)\right]dt\right].\end{aligned} \tag{2.20}$$

Remark 2.12. If one choose $\epsilon = 0$, then (2.20) reduces to the inequality:

$$\begin{aligned}&\eta\left(\varphi(x)-\frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)\\&\leq(\geq)\left(\frac{x-\rho_a}{\rho_b-\rho_a}\right)^{rs}\left[\frac{1}{x-\rho_a}\int_{\rho_a}^{x}\eta[(t-\rho_a)\varphi'(t)]dt\right]+\left(1-\left(\frac{x-\rho_a}{\rho_b-\rho_a}\right)^{r}\right)^{s}\left[\frac{1}{\rho_b-x}\int_{x}^{\rho_b}\eta[(t-\rho_b)\varphi'(t)]dt\right].\end{aligned}$$

4. If one takes $\alpha = \beta = s$ and $\gamma = \delta = 1$, where $s \in [0,1]$ in (2.12), then one has the functional generalization of Ostrowski inequality for $s-$convex (concave) functions in 1^{st} kind:

$$\begin{aligned}&\eta\left((1-\epsilon)\varphi(x)+\epsilon\frac{\varphi(\rho_a)+\varphi(\rho_b)}{2}-\frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)\\&\leq(\geq)\left(\frac{x-\rho_a}{\rho_b-\rho_a}\right)^{s}\left[\frac{1}{x-\rho_a}\int_{\rho_a}^{x}\eta\left[(t-\mu)\varphi'(t)\right]dt\right]+\left(1-\left(\frac{x-\rho_a}{\rho_b-\rho_a}\right)^{s}\right)\left[\frac{1}{\rho_b-x}\int_{x}^{\rho_b}\eta\left[(t-\nu)\varphi'(t)\right]dt\right].\end{aligned} \tag{2.21}$$

Remark 2.13. If one choose $\epsilon = 0$, then (2.21) reduces to the inequality:

$$\eta\left(\varphi(x) - \frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)$$
$$\leq (\geq)\left(\frac{x-\rho_a}{\rho_b-\rho_a}\right)^s\left[\frac{1}{x-\rho_a}\int_{\rho_a}^{x}\eta[(t-\rho_a)\varphi'(t)]dt\right] + \left(1-\left(\frac{x-\rho_a}{\rho_b-\rho_a}\right)^s\right)\left[\frac{1}{\rho_b-x}\int_{x}^{\rho_b}\eta[(t-\rho_b)\varphi'(t)]dt\right].$$

5. If one takes $\alpha = \beta = 0$ and $\gamma = \delta = 1$ in (2.12), then one has the functional generalization of Ostrowski inequality for quasi$-$convex(concave) functions:

$$\eta\left((1-\epsilon)\varphi(x) + \epsilon\frac{\varphi(\rho_a)+\varphi(\rho_b)}{2} - \frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right) \leq (\geq)\frac{1}{x-\rho_a}\int_{\rho_a}^{x}\eta\left[(t-\mu)\varphi'(t)\right]dt. \tag{2.22}$$

Remark 2.14. If one choose $\epsilon = 0$, then (2.22) reduces to the inequality:

$$\eta\left(\varphi(x) - \frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right) \leq (\geq)\frac{1}{x-\rho_a}\int_{\rho_a}^{x}\eta[(t-\rho_a)\varphi'(t)]dt.$$

6. If one takes $\alpha = \delta = s$, $\beta = \gamma = 1$, where $s \in [0,1]$ in (2.12), then one has the functional generalization of Ostrowski inequality for $s-$convex(concave) functions in 2^{nd} kind:

$$\eta\left((1-\epsilon)\varphi(x) + \epsilon\frac{\varphi(\rho_a)+\varphi(\rho_b)}{2} - \frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)$$
$$\leq (\geq)\left(\frac{x-\rho_a}{\rho_b-\rho_a}\right)^s\left[\frac{1}{x-\rho_a}\int_{\rho_a}^{x}\eta\left[(t-\mu)\varphi'(t)\right]dt\right] + \left(\frac{\rho_b-x}{\rho_b-\rho_a}\right)^s\left[\frac{1}{\rho_b-x}\int_{x}^{\rho_b}\eta\left[(t-\nu)\varphi'(t)\right]dt\right]. \tag{2.23}$$

Remark 2.15. If one choose $\epsilon = 0$, then (2.23) reduces to the inequality:

$$\eta\left(\varphi(x) - \frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)$$
$$\leq (\geq)\left(\frac{x-\rho_a}{\rho_b-\rho_a}\right)^s\left[\frac{1}{x-\rho_a}\int_{\rho_a}^{x}\eta[(t-\rho_a)\varphi'(t)]dt\right] + \left(\frac{\rho_b-x}{\rho_b-\rho_a}\right)^s\left[\frac{1}{\rho_b-x}\int_{x}^{\rho_b}\eta[(t-\rho_b)\varphi'(t)]dt\right].$$

7. If one takes $\alpha = \delta = 0$ and $\beta = \gamma = 1$ in (2.12), then one has the functional generalization of Ostrowski inequality for $P-$convex(concave) functions:

$$\eta\left((1-\epsilon)\varphi(x) + \epsilon\frac{\varphi(\rho_a)+\varphi(\rho_b)}{2} - \frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)$$
$$\leq (\geq)\frac{1}{x-\rho_a}\int_{\rho_a}^{x}\eta\left[(t-\mu)\varphi'(t)\right]dt + \frac{1}{\rho_b-x}\int_{x}^{\rho_b}\eta\left[(t-\nu)\varphi'(t)\right]dt. \tag{2.24}$$

Remark 2.16. If one choose $\epsilon = 0$, then (2.24) reduces to the inequality:

$$\eta\left(\varphi(x) - \frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right) \leq (\geq)\frac{1}{x-\rho_a}\int_{\rho_a}^{x}\eta[(t-\rho_a)\varphi'(t)]dt + \frac{1}{\rho_b-x}\int_{x}^{\rho_b}\eta[(t-\rho_b)\varphi'(t)]dt.$$

8. If one takes $\alpha = \beta = \gamma = \delta = 1$ in (2.12), then one has the functional generalization of Ostrowski inequality for convex(concave) functions:

$$\eta\left((1-\epsilon)\varphi(x) + \epsilon\frac{\varphi(\rho_a)+\varphi(\rho_b)}{2} - \frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)$$
$$\leq (\geq)\frac{1}{\rho_b-\rho_a}\left[\int_{\rho_a}^{x}\eta\left[(t-\mu)\varphi'(t)\right]dt + \int_{x}^{\rho_b}\eta\left[(t-\nu)\varphi'(t)\right]dt\right]. \tag{2.25}$$

Remark 2.17. If one choose $\epsilon = 0$, then (2.25) reduces to the inequality (2.1) of Theorem 7 in [6].

If we replace φ by $-\varphi$ and $x = \frac{\rho_a+\rho_b}{2}$ in Theorem 2.7, we get

Theorem 2.18. Let $\varphi : [\rho_a, \rho_b] \to \mathbb{R}$ be an absolutely continuous. If $\eta : I \subset [0,\infty) \to \mathbb{R}$ is $(\alpha,\beta,\gamma,\delta)-$convex(concave) in mixed kind, then we have the inequalities

$$\eta\left((\epsilon-1)\varphi\left(\frac{\rho_a+\rho_b}{2}\right)-\epsilon\frac{\varphi(\rho_a)+\varphi(\rho_b)}{2}+\frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)$$
$$\leq(\geq)\frac{1}{\rho_b-\rho_a}\left[\frac{1}{2^{\alpha\gamma-1}}\int_{\rho_a}^{\frac{\rho_a+\rho_b}{2}}\eta\left[(\mu-t)\varphi'(t)\right]dt+\frac{\left(2^{\beta\gamma}-1\right)^{\delta}}{2^{\beta\gamma\delta-1}}\int_{\frac{\rho_a+\rho_b}{2}}^{\rho_b}\eta\left[(\nu-t)\varphi'(t)\right]dt\right], \tag{2.26}$$

for all $\epsilon \in [0,1]$.

Remark 2.19. In Theorem 2.18, if we choose $\epsilon = 0$, then (2.26) reduces to the inequality:

$$\eta\left(\frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt-\varphi\left(\frac{\rho_a+\rho_b}{2}\right)\right)$$
$$\leq(\geq)\frac{1}{\rho_b-\rho_a}\left[\frac{1}{2^{\alpha\gamma-1}}\int_{\rho_a}^{\frac{\rho_a+\rho_b}{2}}\eta[(\rho_a-t)\varphi'(t)]dt+\frac{\left(2^{\beta\gamma}-1\right)^{\delta}}{2^{\beta\gamma\delta-1}}\int_{\frac{\rho_a+\rho_b}{2}}^{\rho_b}\eta[(\rho_b-t)\varphi'(t)]dt\right].$$

Corollary 2.20. In Theorem 2.1, one can see the following.

1. If one takes $\gamma = \delta = 1$, $\alpha \in [0,1]$ and $\beta \in [0,1]$ in (2.26), one has the midpoint inequality for $(\alpha,\beta)-$convex(concave) functions in 1^{st} kind:

$$\eta\left((\epsilon-1)\varphi\left(\frac{\rho_a+\rho_b}{2}\right)-\epsilon\frac{\varphi(\rho_a)+\varphi(\rho_b)}{2}+\frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)$$
$$\leq(\geq)\frac{1}{\rho_b-\rho_a}\left[\frac{1}{2^{\alpha-1}}\int_{\rho_a}^{\frac{\rho_a+\rho_b}{2}}\eta\left[(\mu-t)\varphi'(t)\right]dt+\frac{2^{\beta}-1}{2^{\beta-1}}\int_{\frac{\rho_a+\rho_b}{2}}^{\rho_b}\eta\left[(\nu-t)\varphi'(t)\right]dt\right]. \tag{2.27}$$

Remark 2.21. If one choose $\epsilon = 0$, then (2.27) reduces to the inequality:

$$\eta\left(\frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt-\varphi\left(\frac{\rho_a+\rho_b}{2}\right)\right)$$
$$\leq(\geq)\frac{1}{\rho_b-\rho_a}\left[\frac{1}{2^{\alpha-1}}\int_{\rho_a}^{\frac{\rho_a+\rho_b}{2}}\eta[(\rho_a-t)\varphi'(t)]dt+\frac{2^{\beta}-1}{2^{\beta-1}}\int_{\frac{\rho_a+\rho_b}{2}}^{\rho_b}\eta[(\rho_b-t)\varphi'(t)]dt\right].$$

2. If one takes $\beta = \gamma = 1$, $\alpha \in [0,1]$ and $\delta \in [0,1]$ in (2.26), then one has the midpoint inequality for $(\alpha,\delta)-$convex(concave) functions in 2^{nd} kind:

$$\eta\left((\epsilon-1)\varphi\left(\frac{\rho_a+\rho_b}{2}\right)-\epsilon\frac{\varphi(\rho_a)+\varphi(\rho_b)}{2}+\frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)$$
$$\leq(\geq)\frac{1}{\rho_b-\rho_a}\left[\frac{1}{2^{\alpha-1}}\int_{\rho_a}^{\frac{\rho_a+\rho_b}{2}}\eta\left[(\mu-t)\varphi'(t)\right]dt+\frac{1}{2^{\delta-1}}\int_{\frac{\rho_a+\rho_b}{2}}^{\rho_b}\eta\left[(\nu-t)\varphi'(t)\right]dt\right]. \tag{2.28}$$

Remark 2.22. If one choose $\epsilon = 0$, then (2.28) reduces to the inequality:

$$\eta\left(\frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt-\varphi\left(\frac{\rho_a+\rho_b}{2}\right)\right)$$
$$\leq(\geq)\frac{1}{\rho_b-\rho_a}\left[\frac{1}{2^{\alpha-1}}\int_{\rho_a}^{\frac{\rho_a+\rho_b}{2}}\eta[(\rho_a-t)\varphi'(t)]dt+\frac{1}{2^{\delta-1}}\int_{\frac{\rho_a+\rho_b}{2}}^{\rho_b}\eta[(\rho_b-t)\varphi'(t)]dt\right].$$

3. If one takes $\alpha=\delta=s$, $\beta=1,\gamma=r$, where $s\in[0,1]$ and $r\in[0,1]$ in (2.26), then one has the midpoint inequality for $(s,r)-$convex(concave) functions in mixed kind:

$$\begin{aligned}&\eta\left((\epsilon-1)\varphi\left(\frac{\rho_a+\rho_b}{2}\right)-\epsilon\frac{\varphi(\rho_a)+\varphi(\rho_b)}{2}+\frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)\\&\leq(\geq)\frac{2^{1-rs}}{\rho_b-\rho_a}\left[\int_{\rho_a}^{\frac{\rho_a+\rho_b}{2}}\eta\left[(\mu-t)\varphi'(t)\right]dt+(2^r-1)^s\int_{\frac{\rho_a+\rho_b}{2}}^{\rho_b}\eta\left[(\nu-t)\varphi'(t)\right]dt\right].\end{aligned}\tag{2.29}$$

Remark 2.23. If one choose $\epsilon=0$, then (2.29) reduces to the inequality:

$$\begin{aligned}&\eta\left(\frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt-\left(\frac{\rho_a+\rho_b}{2}\right)\right)\\&\leq(\geq)\frac{2^{1-rs}}{\rho_b-\rho_a}\left[\int_{\rho_a}^{\frac{\rho_a+\rho_b}{2}}\eta[(\rho_a-t)\varphi'(t)]dt+(2^r-1)^s\int_{\frac{\rho_a+\rho_b}{2}}^{\rho_b}\eta[(\rho_b-t)\varphi'(t)]dt\right].\end{aligned}$$

4. If one takes $\alpha=\beta=s$ and $\gamma=\delta=1$, where $s\in[0,1]$ in (2.26), then one has the midpoint inequality for $s-$convex(concave) functions in 1^{st} kind:

$$\begin{aligned}&\eta\left((\epsilon-1)\varphi\left(\frac{\rho_a+\rho_b}{2}\right)-\epsilon\frac{\varphi(\rho_a)+\varphi(\rho_b)}{2}+\frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)\\&\leq(\geq)\frac{2^{1-s}}{\rho_b-\rho_a}\left[\int_{\rho_a}^{\frac{\rho_a+\rho_b}{2}}\eta\left[(\mu-t)\varphi'(t)\right]dt+(2^s-1)\int_{\frac{\rho_a+\rho_b}{2}}^{\rho_b}\eta\left[(\nu-t)\varphi'(t)\right]dt\right].\end{aligned}\tag{2.30}$$

Remark 2.24. If one choose $\epsilon=0$, then (2.30) reduces to the inequality:

$$\begin{aligned}&\eta\left(\frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt-\varphi\left(\frac{\rho_a+\rho_b}{2}\right)\right)\\&\leq(\geq)\frac{2^{1-s}}{\rho_b-\rho_a}\left[\int_{\rho_a}^{\frac{\rho_a+\rho_b}{2}}\eta[(\rho_a-t)\varphi'(t)]dt+(2^s-1)\int_{\frac{\rho_a+\rho_b}{2}}^{\rho_b}\eta[(\rho_b-t)\varphi'(t)]dt\right].\end{aligned}$$

5. If one takes $\alpha=\beta=0$ and $\gamma=\delta=1$ in (2.26), then one has the midpoint inequality for quasi$-$convex(concave) functions:

$$\eta\left((\epsilon-1)\varphi\left(\frac{\rho_a+\rho_b}{2}\right)-\epsilon\frac{\varphi(\rho_a)+\varphi(\rho_b)}{2}+\frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)\leq(\geq)\frac{2}{\rho_b-\rho_a}\left[\int_{\rho_a}^{\frac{\rho_a+\rho_b}{2}}\eta\left[(\mu-t)\varphi'(t)\right]dt\right].\tag{2.31}$$

Remark 2.25. If one choose $\epsilon=0$, then (2.31) reduces to the inequality:

$$\eta\left(\frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt-\varphi\left(\frac{\rho_a+\rho_b}{2}\right)\right)\leq(\geq)\frac{2}{\rho_b-\rho_a}\int_{\rho_a}^{\frac{\rho_a+\rho_b}{2}}\eta\left[(\rho_a-t)\varphi'(t)\right]dt.$$

6. If one takes $\alpha=\delta=s$, $\beta=\gamma=1$, where $s\in[0,1]$ in (2.26), then one has the midpoint inequality for $s-$convex(concave) functions in 2^{nd} kind:

$$\begin{aligned}&\eta\left((\epsilon-1)\varphi\left(\frac{\rho_a+\rho_b}{2}\right)-\epsilon\frac{\varphi(\rho_a)+\varphi(\rho_b)}{2}+\frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)\\&\leq(\geq)\frac{2^{1-s}}{\rho_b-\rho_a}\left[\int_{\rho_a}^{\frac{\rho_a+\rho_b}{2}}\eta\left[(\mu-t)\varphi'(t)\right]dt+\int_{\frac{\rho_a+\rho_b}{2}}^{\rho_b}\eta\left[(\nu-t)\varphi'(t)\right]dt\right].\end{aligned}\tag{2.32}$$

Remark 2.26. If one choose $\epsilon = 0$, then (2.32) reduces to the inequality:

$$\eta\left(\frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt-\varphi\left(\frac{\rho_a+\rho_b}{2}\right)\right)\leq(\geq)\frac{2^{1-s}}{\rho_b-\rho_a}\left[\int_{\rho_a}^{\frac{\rho_a+\rho_b}{2}}\eta[(\rho_a-t)\varphi'(t)]dt+\int_{\frac{\rho_a+\rho_b}{2}}^{\rho_b}\eta[(\rho_b-t)\varphi'(t)]dt\right].$$

7. If one takes $\alpha=\delta=0$ and $\beta=\gamma=1$ in (2.26), then one has the midpoint inequality for $P-$convex(concave) functions:

$$\eta\left((\epsilon-1)\varphi\left(\frac{\rho_a+\rho_b}{2}\right)-\epsilon\frac{\varphi(\rho_a)+\varphi(\rho_b)}{2}+\frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)$$
$$\leq(\geq)\frac{2}{\rho_b-\rho_a}\left[\int_{\rho_a}^{\frac{\rho_a+\rho_b}{2}}\eta\left[(\mu-t)\varphi'(t)\right]dt+\int_{\frac{\rho_a+\rho_b}{2}}^{\rho_b}\eta\left[(\nu-t)\varphi'(t)\right]dt\right]. \tag{2.33}$$

Remark 2.27. If one choose $\epsilon = 0$, then (2.33) reduces to the inequality:

$$\eta\left(\frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt-\varphi\left(\frac{\rho_a+\rho_b}{2}\right)\right)\leq(\geq)\frac{2}{\rho_b-\rho_a}\left[\int_{\rho_a}^{\frac{\rho_a+\rho_b}{2}}\eta[(\rho_a-t)\varphi'(t)]dt+\int_{\frac{\rho_a+\rho_b}{2}}^{\rho_b}\eta[(\rho_b-t)\varphi'(t)]dt\right].$$

8. If one takes $\alpha=\beta=\gamma=\delta=1$ in (2.26), then one has the midpoint inequality for convex(concave) functions:

$$\eta\left((\epsilon-1)\varphi\left(\frac{\rho_a+\rho_b}{2}\right)-\epsilon\frac{\varphi(\rho_a)+\varphi(\rho_b)}{2}+\frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)$$
$$\leq(\geq)\frac{1}{\rho_b-\rho_a}\left[\int_{\rho_a}^{\frac{\rho_a+\rho_b}{2}}\eta\left[(\mu-t)\varphi'(t)\right]dt+\int_{\frac{\rho_a+\rho_b}{2}}^{\rho_b}\eta\left[(\nu-t)\varphi'(t)\right]dt\right]. \tag{2.34}$$

Remark 2.28. If one choose $\epsilon = 0$, then (2.25) reduces to the inequality (5.1) of Proposition 1 in [6].

Remark 2.29. Assume that $\eta : I \subset [0,\infty) \to \mathbb{R}$ be an $(\alpha,\beta,\gamma,\delta)-$convex(concave) function in mixed kind:

1. If we take $\varphi(t) = \frac{1}{t}$ in inequality (2.26) where $t \in [\rho_a, \rho_b] \subset (0,\infty)$, then we have

$$\eta\left[\frac{A(\rho_a,\rho_b)+(\epsilon-1)L(\rho_a,\rho_b)}{A(\rho_a,\rho_b)L(\rho_a,\rho_b)}-\epsilon\frac{A(\rho_a,\rho_b)}{G^2(\rho_a,\rho_b)}\right]$$
$$\leq(\geq)\frac{1}{\rho_b-\rho_a}\left[\frac{1}{2^{\alpha\gamma-1}}\int_{\rho_a}^{\frac{\rho_a+\rho_b}{2}}\eta\left[\frac{t-\mu}{t^2}\right]dt+\frac{\left(2^{\beta\gamma}-1\right)^{\delta}}{2^{\beta\gamma\delta-1}}\int_{\frac{\rho_a+\rho_b}{2}}^{\rho_b}\eta\left[\frac{t-\nu}{t^2}\right]dt\right].$$

2. If we take $\varphi(t) = -\ln t$ in inequality (2.26), where $t \in [\rho_a, \rho_b] \subset (0,\infty)$, then we have

$$\eta\left[\ln\left(\frac{\exp\left[\epsilon A(\ln\rho_a,\ln\rho_b)\right]A^{(1-\epsilon)}(\rho_a,\rho_b)}{I(\rho_a,\rho_b)}\right)\right]$$
$$\leq(\geq)\frac{1}{\rho_b-\rho_a}\left[\frac{1}{2^{\alpha\gamma-1}}\int_{\rho_a}^{\frac{\rho_a+\rho_b}{2}}\eta\left[\frac{t-\mu}{t}\right]dt+\frac{\left(2^{\beta\gamma}-1\right)^{\delta}}{2^{\beta\gamma\delta-1}}\int_{\frac{\rho_a+\rho_b}{2}}^{\rho_b}\eta\left[\frac{t-\nu}{t}\right]dt\right].$$

3. If we take $\varphi(t) = t^p, p \in \mathbb{R}-\{0,-1\}$ in inequality (2.26), where $t \in [\rho_a, \rho_b] \subset (0,\infty)$, then we have

$$\eta\left[L_p^p(\rho_a,\rho_b)-(\epsilon-1)A^p(\rho_a,\rho_b)-\epsilon A(a^p,b^p)\right]$$
$$\leq(\geq)\frac{1}{\rho_b-\rho_a}\left[\frac{1}{2^{\alpha\gamma-1}}\int_{\rho_a}^{\frac{\rho_a+\rho_b}{2}}\eta\left[\frac{p\,(\mu-t)}{t^{1-p}}\right]dt+\frac{\left(2^{\beta\gamma}-1\right)^{\delta}}{2^{\beta\gamma\delta-1}}\int_{\frac{\rho_a+\rho_b}{2}}^{\rho_b}\eta\left[\frac{p\,(\nu-t)}{t^{1-p}}\right]dt\right].$$

Remark 2.30. In Theorem 2.3, one can see the following.

1. Let $x=\frac{\rho_a+\rho_b}{2}$, $0<\rho_a<\rho_b$, $q\geq 1$ and $\varphi:\mathbb{R}\to\mathbb{R}^+$, $\varphi(x)=x^n$ in (2.5). Then

$$\left|A\left(\rho_a,\rho_b\right)-L_n^n\left(\rho_a,\rho_b\right)\right|\leq \frac{M\left(\rho_b-\rho_a\right)}{(2)^{2-\frac{1}{q}}}\left(\frac{1}{\alpha\gamma+2}+\frac{B\left(\frac{2}{\beta\gamma},\delta+1\right)}{\beta\gamma}\right)^{\frac{1}{q}}.$$

2. Let $x=\frac{\rho_a+\rho_b}{2}$, $0<\rho_a<\rho_b$, $q\geq 1$ and $\varphi:(0,1]\to\mathbb{R}$, $\varphi(x)=-\ln x$ in (2.5). Then

$$\left|\ln I\left(\rho_a,\rho_b\right)-\ln A\left(\rho_a,\rho_b\right)\right|\leq \frac{M\left(\rho_b-\rho_a\right)}{(2)^{2-\frac{1}{q}}}\left(\frac{1}{\alpha\gamma+2}+\frac{B\left(\frac{2}{\beta\gamma},\delta+1\right)}{\beta\gamma}\right)^{\frac{1}{q}}.$$

Remark 2.31. In Theorem 2.5, one can see the following.

1. Let $x=\frac{\rho_a+\rho_b}{2}$, $0<\rho_a<\rho_b$, $q\geq 1$ and $\varphi:\mathbb{R}\to\mathbb{R}^+$, $\varphi(x)=x^n$ in (2.8). Then

$$\left|A\left(\rho_a,\rho_b\right)-L_n^n\left(\rho_a,\rho_b\right)\right|\leq \frac{M\left(\rho_b-\rho_a\right)}{2\left(p+1\right)^{\frac{1}{p}}}\left(\frac{1}{\alpha\gamma+1}+\frac{B\left(\frac{1}{\beta\gamma},\delta+1\right)}{\beta\gamma}\right)^{\frac{1}{q}}.$$

2. Let $x=\frac{\rho_a+\rho_b}{2}$, $0<\rho_a<\rho_b$, $q\geq 1$ and $\varphi:(0,1]\to\mathbb{R}$, $\varphi(x)=-\ln x$ in (2.8). Then

$$\left|\ln I\left(\rho_a,\rho_b\right)-\ln A\left(\rho_a,\rho_b\right)\right|\leq \frac{M\left(\rho_b-\rho_a\right)}{2\left(p+1\right)^{\frac{1}{p}}}\left(\frac{1}{\alpha\gamma+1}+\frac{B\left(\frac{1}{\beta\gamma},\delta+1\right)}{\beta\gamma}\right)^{\frac{1}{q}}.$$

3 Conclusion and Remarks

3.1 Conclusion

Ostrowski inequality is one of the most celebrated inequalities, we can find its various generalizations and variants in literature. In this paper, we presented the generalized notion of $(\alpha,\beta,\gamma,\delta)-$convex (concave) functions in mixed kind. This class of functions contains many important classes including class of $(\alpha,\beta)-$convex (concave) functions in 1^{st} and 2^{nd} kind [16], $(s,r)-$convex (concave) functions in mixed kind [2], $s-$convex (concave) functions in 1^{st} and 2^{nd} kind [4], $P-$convex (concave) functions [15], quasi convex(concave) functions [17] and the class of convex (concave) functions [3]). We have stated our first main result in section 2, the generalization of Ostrowski inequality [25] via generalized Montgomery identity [14] with $(\alpha,\beta,\gamma,\delta)-$convex (concave) functions in mixed kind. Further, we used different techniques including Hölder's inequality [27] and power mean inequality [26] for generalization of Ostrowski inequality [25]. Finally we have given some applications in terms of special means including arithmetic, geometric, harmonic, logarithmic, identric and $p-$logarithmic means by using the midpoint inequalities.

3.2 Remarks and Future Ideas

1. One may do similar work to generalize all results stated in this article by applying weights.
2. One may also do similar work by using various different classes of functions.
3. One may also generalize this work in fractional integral form.
4. One may try to state all results stated in this article for fractional integral with respect to another function.
5. One may also state all results stated in this article for higher dimensions.

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