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ON ABSOLUTE GENERALIZED NÖRLUND SUMMABILITY OF DOUBLE ORTHOGONAL SERIES

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Abstract. In the paper [Y. Okuyama, *On the absolute generalized Nörlund summability of orthogonal series*, Tamkang J. Math. Vol. 33, No. 2, (2002), 161-165] the author has found some sufficient conditions under which an orthogonal series is summable $|N, p, q|$ almost everywhere. These conditions are expressed in terms of coefficients of the series. It is the purpose of this paper to extend this result to double absolute summability $|N^{(2)}, \mathfrak{p}, \mathfrak{q}|_k$, $(1 \le k \le 2)$.

1. Introduction and Preliminaries

Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with sequence of partial sums $\{s_n\}$. Then, let p denotes the sequence $\{p_n\}$. For two given sequences p and q, the convolution $(p * q)_n$ is defined by

$$(p * q)_n = \sum_{m=0}^{n} p_m q_{n-m} = \sum_{m=0}^{n} p_{n-m} q_m.$$

When $(p * q)_n \neq 0$ for all n, the generalized Nörlund transform of the sequence $\{s_n\}$ is the sequence $\{t_n^{p,q}\}$ obtained by putting

$$t_n^{p,q} = \frac{1}{(p * q)_n} \sum_{m=0}^{n} p_{n-m} q_m s_m.$$

The infinite series $\sum_{n=0}^{\infty} a_n$ is absolutely summable (N, p, q), if the series

$$\sum_{n=0}^{\infty} |t_n^{p,q} - t_{n-1}^{p,q}|$$

converges $(t_{-1} = 0)$, and we write in brief

$$\sum_{n=0}^{\infty} a_n \in |N, p, q|.$$

The $|N, p, q|$ summability was introduced by Tanaka [2].

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Let $\{\varphi_n(x)\}$ be an orthonormal system defined in the interval (a,b). We assume that $f(x)$ belongs to $L^2(a,b)$ and

$$f(x) \sim \sum_{n=0}^{\infty} a_n \varphi_n(x), \tag{1.1}$$

where $a_n = \int_a^b f(x)\varphi_n(x)dx$, $(n = 0, 1, 2, \dots)$.

Let us write

$$R_n := (p * q)_n, \ R_n^j := \sum_{m=j}^{n} p_{n-m} q_m, \quad \text{and} \quad R_n^{n+1} = 0, \ R_n^0 = R_n.$$

Also we put

$$P_n := (p * 1)_n = \sum_{m=0}^{n} p_m \quad \text{and} \quad Q_n := (1 * q)_n = \sum_{m=0}^{n} q_m.$$

Y. Okuyama [3] among others proved the following two theorems:

Theorem 1.1. *If the series*

$$\sum_{n=0}^{\infty} \left\{ \sum_{j=1}^{n} \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2 |a_j|^2 \right\}^{\frac{1}{2}}$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty} a_n \varphi_n(x)$$

is summable $|N, p, q|$ almost everywhere.

Theorem 1.2. *Let $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$ is a non-increasing sequence and the series $\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)}$ converges. Let $\{p_n\}$ and $\{q_n\}$ be non-negative. If the series $\sum_{n=1}^{\infty} |a_n|^2 \Omega(n) w(n)$ converges, then the orthogonal series $\sum_{n=0}^{\infty} a_n \varphi_n(x) \in |N, p, q|$ almost everywhere, where $w(n)$ is defined by $w(j) := \frac{1}{j} \sum_{n=j}^{\infty} n^2 \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2$.*

Let $\{\phi_{mn}(x) : m, n = 0, 1, \dots\}$ be an orthogonal system on (a,b). The orthogonal development of any real function $f(x)$ of class L^2 with respect to the system $\{\phi_{mn}(x)\}$ is given by

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} \phi_{mn}(x), \tag{1.2}$$

where

$$a_{mn} = \int_a^b f(x)\phi_{mn}(x)dx, \ (m, n = 0, 1, \dots).$$

The series (1.2) shall be referred to as the double Fourier series of $f(x)$.

Our main purpose in this paper is to study the absolute summability with index k, $1 \le k \le 2$, of the series (1.2), and to deduce some corollaries from the main results.

Before doing this we introduce some notions and notations.

Let $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}$ be a given double infinite series with its partial sums $\{s_{mn}\}$. Then, let $\mathfrak{p}$ denotes the sequence $\{p_{mn}\}$. For two given sequences $\mathfrak{p}$ and $\mathfrak{q}$, the convolution $(\mathfrak{p}*\mathfrak{q})_{mn}$ we define by

$$R_{mn} := (\mathfrak{p}*\mathfrak{q})_{mn} = \sum_{i=0}^{m}\sum_{k=0}^{n} p_{ik}q_{m-i,n-k} = \sum_{i=0}^{m}\sum_{k=0}^{n} p_{m-i,n-k}q_{ik}.$$

Likewise we need the following notations:

$$\begin{aligned} R_{mn}^{\nu\mu} &:= \sum_{i=\nu}^{m}\sum_{k=\mu}^{n} p_{m-i,n-k}q_{ik};\\ R_{mn}^{00} &= R_{mn};\\ R_{m,n-1}^{\nu,n} &= R_{m-1,n-1}^{\nu,n} = 0, \quad 0 \le \nu \le m;\\ R_{m,n-1}^{m,\mu} &= R_{m-1,n-1}^{m,\mu} = 0, \quad 0 \le \mu \le n;\\ \bar{\triangle}_{11}\left(\frac{R_{mn}^{\nu\mu}}{R_{mn}}\right) &:= \frac{R_{mn}^{\nu\mu}}{R_{mn}} - \frac{R_{m,n-1}^{\nu\mu}}{R_{m,n-1}} - \frac{R_{m-1,n}^{\nu\mu}}{R_{m-1,n}} + \frac{R_{m-1,n-1}^{\nu\mu}}{R_{m-1,n-1}}. \end{aligned}$$

When $(\mathfrak{p}*\mathfrak{q})_{mn} \neq 0$, the generalized Nörlund transform of the sequence $\{s_{mn}\}$ is the sequence $\{t_{mn}^{\mathfrak{p},\mathfrak{q}}\}$ defined by

$$t_{mn}^{p,q} = \frac{1}{(\mathfrak{p}*\mathfrak{q})_{mn}}\sum_{i=0}^{m}\sum_{k=0}^{n} p_{m-i,n-k}q_{ik}s_{ik}.$$

The double infinite series $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}$ is absolutely summable $(N^{(2)},\mathfrak{p},\mathfrak{q})_k$, $k \geq 1$, if the series

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(mn)^{k-1}|t_{mn}^{\mathfrak{p},\mathfrak{q}} - t_{m,n-1}^{\mathfrak{p},\mathfrak{q}} - t_{m-1,n}^{\mathfrak{p},\mathfrak{q}} + t_{m-1,n-1}^{\mathfrak{p},\mathfrak{q}}|^k$$

converges with the agreement that

$$t_{m,-1}^{\mathfrak{p},\mathfrak{q}} = t_{-1,n}^{\mathfrak{p},\mathfrak{q}} = t_{-1,-1}^{\mathfrak{p},\mathfrak{q}} = 0, \quad m,n = 0,1,\ldots,$$

and we write in brief

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn} \in |N^{(2)},\mathfrak{p},\mathfrak{q}|_k.$$

Throughout this paper K denotes a positive constant that depends on k, and it may be different in different relations.

2. Main Results

The main result is the following.

Theorem 2.1. *If the series*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\left\{\sum_{\nu=1}^{m}\left[\bar{\triangle}_{11}\left(\frac{R_{mn}^{\nu 0}}{R_{mn}}\right)\right]^2|a_{\nu 0}|^2\right\}^{\frac{k}{2}},$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\left\{\sum_{\mu=1}^{n}\left[\bar{\triangle}_{11}\left(\frac{R_{mn}^{0\mu}}{R_{mn}}\right)\right]^2|a_{0\mu}|^2\right\}^{\frac{k}{2}},$$

and

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\left\{\sum_{\nu=1}^{m}\sum_{\mu=1}^{n}\left[\bar{\triangle}_{11}\left(\frac{R_{mn}^{\nu\mu}}{R_{mn}}\right)\right]^2|a_{\nu\mu}|^2\right\}^{\frac{k}{2}}$$

converge for $1 \le k \le 2$*, then the orthogonal series*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{mn}\phi_{mn}(x)$$

is summable $|N^{(2)}, \mathfrak{p}, \mathfrak{q}|_k$ *almost everywhere.*

Proof. Let $1 < k < 2$. For the generalized Nörlund transform $t_{mn}^{\mathfrak{p},\mathfrak{q}}(x)$ of the partial sums of the orthogonal series $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{mn}\phi_{mn}(x)$ we have that

$$\begin{aligned} t_{mn}^{\mathfrak{p},\mathfrak{q}}(x) &= \frac{1}{R_{mn}}\sum_{i=0}^{m}\sum_{k=0}^{n}p_{m-i,n-k}q_{ik}s_{ik}(x) \\ &= \frac{1}{R_{mn}}\sum_{i=0}^{m}\sum_{k=0}^{n}p_{m-i,n-k}q_{ik}\sum_{\nu=0}^{i}\sum_{\mu=0}^{k}a_{\nu\mu}\phi_{\nu\mu}(x) \\ &= \frac{1}{R_{mn}}\sum_{\nu=0}^{m}\sum_{\mu=0}^{n}a_{\nu\mu}\phi_{\nu\mu}(x)\sum_{i=\nu}^{m}\sum_{k=\mu}^{n}p_{m-i,n-k}q_{ik} \\ &= \frac{1}{R_{mn}}\sum_{\nu=0}^{m}\sum_{\mu=0}^{n}R_{mn}^{\nu\mu}a_{\nu\mu}\phi_{\nu\mu}(x), \end{aligned}$$

where $s_{ik}(x)$ are partial sums of order (i,k) of the series (1.2).

Thus, since

$$R_{m,n-1}^{\nu,n} = R_{m-1,n-1}^{\nu,n} = 0, \quad 0 \le \nu \le m,$$

and

$$R_{m,n-1}^{m,\mu} = R_{m-1,n-1}^{m,\mu} = 0, \quad 0 \le \mu \le n,$$

we obtain that

$$
\begin{aligned}
\bar{\triangle}_{11}t^{\mathfrak{p},\mathfrak{q}}_{mn}(x) &= t^{\mathfrak{p},\mathfrak{q}}_{mn}(x)-t^{\mathfrak{p},\mathfrak{q}}_{m,n-1}(x)-t^{\mathfrak{p},\mathfrak{q}}_{m-1,n}(x)+t^{\mathfrak{p},\mathfrak{q}}_{m-1,n-1}(x)\\
&= \frac{1}{R_{mn}}\sum_{\nu=0}^{m}\sum_{\mu=0}^{n}R^{\nu\mu}_{mn}a_{\nu\mu}\phi_{\nu\mu}(x)\\
&\quad -\frac{1}{R_{m,n-1}}\sum_{\nu=0}^{m}\sum_{\mu=0}^{n}R^{\nu\mu}_{m,n-1}a_{\nu\mu}\phi_{\nu\mu}(x)\\
&\quad -\frac{1}{R_{m-1,n}}\sum_{\nu=0}^{m}\sum_{\mu=0}^{n}R^{\nu\mu}_{m-1,n}a_{\nu\mu}\phi_{\nu\mu}(x)\\
&\quad +\frac{1}{R_{m-1,n-1}}\sum_{\nu=0}^{m}\sum_{\mu=0}^{n}R^{\nu\mu}_{m-1,n-1}a_{\nu\mu}\phi_{\nu\mu}(x)\\
&= \sum_{\nu=1}^{m}\left(\frac{R^{\nu0}_{mn}}{R_{mn}}-\frac{R^{\nu0}_{m,n-1}}{R_{m,n-1}}-\frac{R^{\nu0}_{m-1,n}}{R_{m-1,n}}+\frac{R^{\nu0}_{m-1,n-1}}{R_{m-1,n-1}}\right)a_{\nu0}\phi_{\nu0}(x)\\
&\quad +\sum_{\mu=1}^{n}\left(\frac{R^{0\mu}_{mn}}{R_{mn}}-\frac{R^{0\mu}_{m,n-1}}{R_{m,n-1}}-\frac{R^{0\mu}_{m-1,n}}{R_{m-1,n}}+\frac{R^{0\mu}_{m-1,n-1}}{R_{m-1,n-1}}\right)a_{0\mu}\phi_{0\mu}(x)\\
&\quad +\sum_{\nu=1}^{m}\sum_{\mu=1}^{n}\left(\frac{R^{\mu\nu}_{mn}}{R_{mn}}-\frac{R^{\mu\nu}_{m,n-1}}{R_{m,n-1}}-\frac{R^{\mu\nu}_{m-1,n}}{R_{m-1,n}}+\frac{R^{\mu\nu}_{m-1,n-1}}{R_{m-1,n-1}}\right)a_{\nu\mu}\phi_{\nu\mu}(x)\\
&= \sum_{\nu=1}^{m}\bar{\triangle}_{11}\left(\frac{R^{\nu0}_{mn}}{R_{mn}}\right)a_{\nu0}\phi_{\nu0}(x)\\
&\quad +\sum_{\mu=1}^{n}\bar{\triangle}_{11}\left(\frac{R^{0\mu}_{mn}}{R_{mn}}\right)a_{0\mu}\phi_{0\mu}(x)+\sum_{\nu=1}^{m}\sum_{\mu=1}^{n}\bar{\triangle}_{11}\left(\frac{R^{\nu\mu}_{mn}}{R_{mn}}\right)a_{\nu\mu}\phi_{\nu\mu}(x).
\end{aligned}
$$

Applying twice the inequality

$$|\alpha+\beta|^r\le 2^r\left(|\alpha|^r+|\beta|^r\right)\quad \text{for}\quad r\ge 1,$$

then the well-known Hölder's inequality with $p=\frac{2}{k}>1$, q such that $p+q=pq$, and orthogonality we have that

$$\begin{aligned}
\int_a^b |\bar{\triangle} t_{mn}^{\mathfrak{p},\mathfrak{q}}(x)|^k dx &\leq 2^k \int_a^b \left| \sum_{\nu=1}^{m} \bar{\triangle}_{11} \left(\frac{R_{mn}^{\nu 0}}{R_{mn}} \right) a_{\nu 0} \phi_{\nu 0}(x) \right|^k dx \\
&\quad + 4^k \int_a^b \left| \sum_{\mu=1}^{n} \bar{\triangle}_{11} \left(\frac{R_{mn}^{0\mu}}{R_{mn}} \right) a_{0\mu} \phi_{0\mu}(x) \right|^k dx \\
&\quad + 4^k \int_a^b \left| \sum_{\nu=1}^{m} \sum_{\mu=1}^{n} \bar{\triangle}_{11} \left(\frac{R_{mn}^{\nu\mu}}{R_{mn}} \right) a_{\nu\mu} \phi_{\nu\mu}(x) \right|^k dx \\
&\leq K \left(\int_a^b \left| \sum_{\nu=1}^{m} \bar{\triangle}_{11} \left(\frac{R_{mn}^{\nu 0}}{R_{mn}} \right) a_{\nu 0} \phi_{\nu 0}(x) \right|^2 dx \right)^{\frac{k}{2}} \\
&\quad + K \left(\int_a^b \left| \sum_{\mu=1}^{n} \bar{\triangle}_{11} \left(\frac{R_{mn}^{0\mu}}{R_{mn}} \right) a_{0\mu} \phi_{0\mu}(x) \right|^2 dx \right)^{\frac{k}{2}} \\
&\quad + K \left(\int_a^b \left| \sum_{\nu=1}^{m} \sum_{\mu=1}^{n} \bar{\triangle}_{11} \left(\frac{R_{mn}^{\nu\mu}}{R_{mn}} \right) a_{\nu\mu} \phi_{\nu\mu}(x) \right|^2 dx \right)^{\frac{k}{2}} \\
&= K \left\{ \sum_{\nu=1}^{m} \left[\bar{\triangle}_{11} \left(\frac{R_{mn}^{\nu 0}}{R_{mn}} \right) \right]^2 |a_{\nu 0}|^2 \right\}^{\frac{k}{2}} \\
&\quad + K \left\{ \sum_{\mu=1}^{n} \left[\bar{\triangle}_{11} \left(\frac{R_{mn}^{0\mu}}{R_{mn}} \right) \right]^2 |a_{0\mu}|^2 \right\}^{\frac{k}{2}} \\
&\quad + K \left\{ \sum_{\nu=1}^{m} \sum_{\mu=1}^{n} \left[\bar{\triangle}_{11} \left(\frac{R_{mn}^{\nu\mu}}{R_{mn}} \right) \right]^2 |a_{\nu\mu}|^2 \right\}^{\frac{k}{2}}.
\end{aligned}$$

Hence, the series

$$\begin{aligned}
&\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (mn)^{k-1} \int_a^b |\bar{\triangle} t_{mn}^{\mathfrak{p},\mathfrak{q}}(x)|^k dx \leq \\
&\leq K \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (mn)^{k-1} \left\{ \sum_{\nu=1}^{m} \left[\bar{\triangle}_{11} \left(\frac{R_{mn}^{\nu 0}}{R_{mn}} \right) \right]^2 |a_{\nu 0}|^2 \right\}^{\frac{k}{2}} \\
&\quad + K \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (mn)^{k-1} \left\{ \sum_{\mu=1}^{n} \left[\bar{\triangle}_{11} \left(\frac{R_{mn}^{0\mu}}{R_{mn}} \right) \right]^2 |a_{0\mu}|^2 \right\}^{\frac{k}{2}} \\
&\quad + K \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (mn)^{k-1} \left\{ \sum_{\nu=1}^{m} \sum_{\mu=1}^{n} \left[\bar{\triangle}_{11} \left(\frac{R_{mn}^{\nu\mu}}{R_{mn}} \right) \right]^2 |a_{\nu\mu}|^2 \right\}^{\frac{k}{2}} \quad (2.1)
\end{aligned}$$

converges since the last do by the assumption. Since the functions $|\bar{\triangle}t^{\mathfrak{p},\mathfrak{q}}_{mn}(x)|^k$ are nonnegative, then B. Levi's theorem (see [4], page 771) implies that the series

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}|\bar{\triangle}t^{\mathfrak{p},\mathfrak{q}}_{mn}(x)|^k$$

converges almost everywhere. For $k=1$, and $k=2$ we do the same reasoning as above using Schwarz's inequality instead of Hölder's inequality. The proof of the theorem 2.1 is completed. □

Now we shall prove a general theorem that is a consequence of the theorem 2.1. Namely, if we put

$$\begin{aligned} w^{(0,1)}(\nu,\mu;k) &:= \frac{1}{\mu^{\frac{2}{k}-1}}\sum_{n=\mu}^{\infty}n^{\frac{2}{k}}\left[\bar{\triangle}_{11}\left(\frac{R^{0\mu}_{\nu n}}{R_{\nu n}}\right)\right]^2;\\ w^{(1,0)}(\nu,\mu;k) &:= \frac{1}{\nu^{\frac{2}{k}-1}}\sum_{m=\nu}^{\infty}m^{\frac{2}{k}}\left[\bar{\triangle}_{11}\left(\frac{R^{\nu 0}_{m\mu}}{R_{m\mu}}\right)\right]^2;\\ w^{(1,1)}(\nu,\mu;k) &:= \frac{1}{(\nu\mu)^{\frac{2}{k}-1}}\sum_{m=\nu}^{\infty}\sum_{n=\mu}^{\infty}(mn)^{\frac{2}{k}}\left[\bar{\triangle}_{11}\left(\frac{R^{\nu\mu}_{mn}}{R_{mn}}\right)\right]^2, \end{aligned}$$

then the following theorem holds true.

Theorem 2.2. *Let $\{\Omega(m,n)\}$ be a positive sequence such that $\{\Omega(m,n)/mn\}$ is a non-increasing sequence with respect to m and n, and the series $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn\Omega(m,n)}$ converges. Let $\{p_{mn}\}$ and $\{q_{mn}\}$ be two non-negative sequences. If the series*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|a_{m0}|^2 n\Omega^{\frac{2}{k}-1}(m,n)w^{(1,0)}(m,n;k),$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|a_{0n}|^2 m\Omega^{\frac{2}{k}-1}(m,n)w^{(0,1)}(m,n;k),$$

and

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|a_{mn}|^2\Omega(m,n)w^{(1,1)}(m,n)(m,n;k)$$

converge, then the double orthogonal series $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{mn}\phi_{mn}(x)\in|N^{(2)},\mathfrak{p},\mathfrak{q}|_k$, $(1\le k\le 2)$, almost everywhere.

Proof. From (2.1) we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\int_a^b|\bar{\triangle}t_{mn}^{\mathfrak{p},\mathfrak{q}}(x)|^k dx \leq$$
$$\leq K\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\left\{\sum_{\nu=1}^{m}\left[\bar{\triangle}_{11}\left(\frac{R_{mn}^{\nu 0}}{R_{mn}}\right)\right]^2|a_{\nu 0}|^2\right\}^{\frac{k}{2}}$$
$$+K\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\left\{\sum_{\mu=1}^{n}\left[\bar{\triangle}_{11}\left(\frac{R_{mn}^{0\mu}}{R_{mn}}\right)\right]^2|a_{0\mu}|^2\right\}^{\frac{k}{2}}$$
$$+K\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\left\{\sum_{\nu=1}^{m}\sum_{\mu=1}^{n}\left[\bar{\triangle}_{11}\left(\frac{R_{mn}^{\nu\mu}}{R_{mn}}\right)\right]^2|a_{\nu\mu}|^2\right\}^{\frac{k}{2}} := I_1+I_2+I_3 \quad (2.2)$$

Applying Hölder's inequality we obtain

$$\begin{aligned}
I_1 &= K\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(mn\Omega(m,n))^{1-\frac{k}{2}}}\times \\
&\quad\times\left\{mn\Omega^{\frac{2}{k}-1}(m,n)\sum_{\nu=1}^{m}\left[\bar{\triangle}_{11}\left(\frac{R_{mn}^{\nu 0}}{R_{mn}}\right)\right]^2|a_{\nu 0}|^2\right\}^{\frac{k}{2}} \\
&\leq K\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn\Omega(m,n)}\right)^{1-\frac{k}{2}}\times \\
&\quad\times\left\{\sum_{n=1}^{\infty}n\sum_{m=1}^{\infty}m\Omega^{\frac{2}{k}-1}(m,n)\sum_{\nu=1}^{m}\left[\bar{\triangle}_{11}\left(\frac{R_{mn}^{\nu 0}}{R_{mn}}\right)\right]^2|a_{\nu 0}|^2\right\}^{\frac{k}{2}} \\
&\leq K\left\{\sum_{n=1}^{\infty}\sum_{\nu=1}^{\infty}|a_{\nu 0}|^2\times\sum_{m=\nu}^{\infty}mn\Omega^{\frac{2}{k}-1}(m,n)\left[\bar{\triangle}_{11}\left(\frac{R_{mn}^{\nu 0}}{R_{mn}}\right)\right]^2\right\}^{\frac{k}{2}} \\
&\leq K\left\{\sum_{\nu=1}^{\infty}\sum_{\mu=1}^{\infty}|a_{\nu 0}|^2\left(\frac{\Omega(\nu,\mu)}{\nu}\right)^{\frac{2}{k}-1}\mu\sum_{m=\nu}^{\infty}m^{\frac{2}{k}}\left[\bar{\triangle}_{11}\left(\frac{R_{m\mu}^{\nu 0}}{R_{m\mu}}\right)\right]^2\right\}^{\frac{k}{2}} \\
&= K\left\{\sum_{\nu=1}^{\infty}\sum_{\mu=1}^{\infty}|a_{\nu 0}|^2\mu\Omega^{\frac{2}{k}-1}(\nu,\mu)w^{(1,0)}(\nu,\mu;k)\right\}^{\frac{k}{2}}, \qquad (2.3)
\end{aligned}$$

$$\begin{aligned}
I_2 &= K\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(mn\Omega(m,n))^{1-\frac{k}{2}}}\times \\
&\quad\times\left\{mn\Omega^{\frac{2}{k}-1}(m,n)\sum_{\mu=1}^{n}\left[\bar{\triangle}_{11}\left(\frac{R_{mn}^{0\mu}}{R_{mn}}\right)\right]^2|a_{0\mu}|^2\right\}^{\frac{k}{2}} \\
&\leq K\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn\Omega(m,n)}\right)^{1-\frac{k}{2}}\times \\
&\quad\times\left\{\sum_{m=1}^{\infty}m\sum_{n=1}^{\infty}n\Omega^{\frac{2}{k}-1}(m,n)\sum_{\mu=1}^{n}\left[\bar{\triangle}_{11}\left(\frac{R_{mn}^{0\mu}}{R_{mn}}\right)\right]^2|a_{0\mu}|^2\right\}^{\frac{k}{2}} \\
&\leq K\left\{\sum_{m=1}^{\infty}\sum_{\mu=1}^{\infty}|a_{0\mu}|^2\times\sum_{n=\mu}^{\infty}mn\Omega^{\frac{2}{k}-1}(m,n)\left[\bar{\triangle}_{11}\left(\frac{R_{mn}^{0\mu}}{R_{mn}}\right)\right]^2\right\}^{\frac{k}{2}} \\
&\leq K\left\{\sum_{\nu=1}^{\infty}\sum_{\mu=1}^{\infty}|a_{0\mu}|^2\left(\frac{\Omega(\nu,\mu)}{\mu}\right)^{\frac{2}{k}-1}\nu\sum_{n=\mu}^{\infty}n^{\frac{2}{k}}\left[\bar{\triangle}_{11}\left(\frac{R_{\nu n}^{0\mu}}{R_{\nu n}}\right)\right]^2\right\}^{\frac{k}{2}} \\
&= K\left\{\sum_{\nu=1}^{\infty}\sum_{\mu=1}^{\infty}|a_{0\mu}|^2\nu\Omega^{\frac{2}{k}-1}(\nu,\mu)w^{(0,1)}(\nu,\mu;k)\right\}^{\frac{k}{2}}, \qquad (2.4)
\end{aligned}$$

and

$$\begin{aligned}
I_3 &= K\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(mn\Omega(m,n))^{1-\frac{k}{2}}}\times \\
&\quad\times\left\{mn\Omega^{\frac{2}{k}-1}(m,n)\sum_{\nu=1}^{m}\sum_{\mu=1}^{n}\left[\bar{\triangle}_{11}\left(\frac{R_{mn}^{\nu\mu}}{R_{mn}}\right)\right]^2|a_{\nu\mu}|^2\right\}^{\frac{k}{2}} \\
&\leq K\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn\Omega(m,n)}\right)^{1-\frac{k}{2}}\times \\
&\quad\times\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}mn\Omega^{\frac{2}{k}-1}(m,n)\sum_{\nu=1}^{m}\sum_{\mu=1}^{n}\left[\bar{\triangle}_{11}\left(\frac{R_{mn}^{\nu\mu}}{R_{mn}}\right)\right]^2|a_{\nu\mu}|^2\right\}^{\frac{k}{2}} \\
&\leq K\left\{\sum_{\nu=1}^{\infty}\sum_{\mu=1}^{\infty}|a_{\nu\mu}|^2\times\sum_{m=\nu}^{\infty}\sum_{n=\mu}^{\infty}mn\Omega^{\frac{2}{k}-1}(m,n)\left[\bar{\triangle}_{11}\left(\frac{R_{mn}^{\nu\mu}}{R_{mn}}\right)\right]^2\right\}^{\frac{k}{2}}
\end{aligned}$$

$$\leq K\left\{\sum_{\nu=1}^{\infty}\sum_{\mu=1}^{\infty}|a_{\nu\mu}|^2\left(\frac{\Omega(\nu,\mu)}{\nu\mu}\right)^{\frac{2}{k}-1}\sum_{m=\nu}^{\infty}\sum_{n=\mu}^{\infty}(mn)^{\frac{2}{k}}\left[\bar{\triangle}_{11}\left(\frac{R_{mn}^{\nu\mu}}{R_{mn}}\right)\right]^2\right\}^{\frac{k}{2}}$$

$$= K\left\{\sum_{\nu=1}^{\infty}\sum_{\mu=1}^{\infty}|a_{\nu\mu}|^2\Omega^{\frac{2}{k}-1}(\nu,\mu)w^{(1,1)}(\nu,\mu;k)\right\}^{\frac{k}{2}}. \tag{2.5}$$

Using (2.2), (2.3), (2.4), and (2.5) we have that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\int_a^b|\bar{\triangle}t_{mn}^{\mathfrak{p},\mathfrak{q}}(x)|^k dx,$$

is finite by the assumptions. This completes the proof based on the same reasoning as in the proof of theorem 2.1. □

Remark 2.3. For $k=1$ in theorems 2.1 and 2.2 we exactly obtain the two dimensional versions of the theorems 1.1 and 1.2.

3. Corollaries

A double infinite sequence $\{u_{mn}\}$ will be called factorable if there exist sequences $\{c_m\}$ and $\{d_n\}$ such that $u_{mn}=c_m d_n$, and we focus on this case below.

Corollary 3.1. *If the series*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{p_m p'_n(mn)^{k-1}}{P_m P_{m-1}P'_n P'_{n-1}}\left\{\sum_{\nu=1}^{m}\sum_{\mu=1}^{n}p^2_{m-\nu}p'^2_{n-\mu}\left[\left(\frac{P_m}{p_m}-\frac{P_{m-\nu}}{p_{m-\nu}}\right)\left(\frac{P'_n}{p'_n}-\frac{P'_{n-\mu}}{p'_{n-\mu}}\right)\right]^2|a_{\nu\mu}|^2\right\}^{\frac{k}{2}}$$

converges for $1\leq k\leq 2$, then the orthogonal series

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{mn}\phi_{mn}(x)$$

is summable $|N^{(2)},p_n p'_m|_k$ almost everywhere.

Proof. The proof of the corollary follows from theorem 2.1 and the fact that

$$\begin{aligned}
\bar{\triangle}_{11}\left(\frac{R_{mn}^{\nu\mu}}{R_{mn}}\right) &= \frac{R_{mn}^{\mu\nu}}{R_{mn}} - \frac{R_{m,n-1}^{\mu\nu}}{R_{m,n-1}} - \frac{R_{m-1,n}^{\mu\nu}}{R_{m-1,n}} + \frac{R_{m-1,n-1}^{\mu\nu}}{R_{m-1,n-1}} \\
&= \frac{P_{m-\nu}P'_{n-\mu}}{P_mP'_n} - \frac{P_{m-\nu}P'_{n-1-\mu}}{P_mP'_{n-1}} - \frac{P_{m-1-\nu}P'_{n-\mu}}{P_{m-1}P'_n} + \frac{P_{m-1-\nu}P'_{n-1-\mu}}{P_{m-1}P'_{n-1}} \\
&= \left(\frac{P_{m-\nu}}{P_m} - \frac{P_{m-1-\nu}}{P_{m-1}}\right)\left(\frac{P'_{n-\mu}}{P'_n} - \frac{P'_{m-1-\mu}}{P'_{n-1}}\right) \\
&= \frac{1}{P_mP_{m-1}P'_nP'_{n-1}}\left(P_{m-1}P_{m-\nu} - P_mP_{m-1-\nu}\right)\left(P'_{n-1}P'_{n-\mu} - P'_nP'_{m-1-\mu}\right) \\
&= \frac{1}{P_mP_{m-1}P'_nP'_{n-1}}\Big[\left(P_m - p_m\right)P_{m-\nu} - P_m\left(P_{m-\nu} - p_{m-\nu}\right)\Big] \times \\
&\quad \times\Big[\left(P'_n - p'_n\right)P'_{n-\mu} - P'_n\left(P'_{n-\mu} - p'_{n-\mu}\right)\Big] \\
&= \frac{p_mp'_n}{P_mP_{m-1}P'_nP'_{n-1}}\left(\frac{P_m}{p_m} - \frac{P_{m-\nu}}{p_{m-\nu}}\right)\left(\frac{P'_n}{p'_n} - \frac{P'_{n-\mu}}{p'_{n-\mu}}\right)p_{m-\nu}p'_{n-\mu},
\end{aligned}$$

and

$$\bar{\triangle}_{11}\left(\frac{R_{mn}^{0\mu}}{R_{mn}}\right) = 0, \qquad \bar{\triangle}_{11}\left(\frac{R_{mn}^{\nu 0}}{R_{mn}}\right) = 0,$$

for all $q_{mn} = 1$. □

Corollary 3.2. *If the series*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{q_mq'_n(mn)^{k-1}}{Q_mQ_{m-1}Q'_nQ'_{n-1}}\left\{\sum_{\nu=1}^{m}\sum_{\mu=1}^{n}\left(Q_{\nu-1}Q'_{\mu-1}\right)^2|a_{\nu\mu}|^2\right\}^{\frac{k}{2}}$$

converges for $1 \leq k \leq 2$*, then the orthogonal series*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{mn}\phi_{mn}(x)$$

is summable $|\overline{N}^{(2)}, q_nq'_m|_k$ *almost everywhere.*

Proof. We have

$$\begin{aligned}
\bar{\triangle}_{11}\left(\frac{R^{\nu\mu}_{mn}}{R_{mn}}\right) &= \frac{(Q_m - Q_{\nu-1})\left(Q'_n - Q'_{\mu-1}\right)}{Q_m Q'_n} - \frac{(Q_m - Q_{\nu-1})\left(Q'_{n-1} - Q'_{\mu-1}\right)}{Q_m Q'_{n-1}} \\
&\quad - \frac{(Q_{m-1} - Q_{\nu-1})\left(Q'_n - Q'_{\mu-1}\right)}{Q_{m-1} Q'_n} + \frac{(Q_{m-1} - Q_{\nu-1})\left(Q'_{n-1} - Q'_{\mu-1}\right)}{Q_{m-1} Q'_{n-1}} \\
&= \frac{Q_m - Q_{\nu-1}}{Q_m}\left(\frac{Q'_n - Q'_{\mu-1}}{Q'_n} - \frac{Q'_{n-1} - Q'_{\mu-1}}{Q'_{n-1}}\right) \\
&\quad - \frac{Q_{m-1} - Q_{\nu-1}}{Q_{m-1}}\left(\frac{Q'_n - Q'_{\mu-1}}{Q'_n} - \frac{Q'_{n-1} - Q'_{\mu-1}}{Q'_{n-1}}\right) \\
&= \left(\frac{Q_m - Q_{\nu-1}}{Q_m} - \frac{Q_{m-1} - Q_{\nu-1}}{Q_{m-1}}\right)\left(\frac{Q'_n - Q'_{\mu-1}}{Q'_n} - \frac{Q'_{n-1} - Q'_{\mu-1}}{Q'_{n-1}}\right) \\
&= Q_{\nu-1}\left(\frac{1}{Q_m} - \frac{1}{Q_{m-1}}\right) Q'_{\mu-1}\left(\frac{1}{Q'_n} - \frac{1}{Q'_{n-1}}\right) \\
&= \frac{q_m q'_n Q_{\nu-1} Q'_{\mu-1}}{Q_m Q_{m-1} Q'_n Q'_{n-1}}
\end{aligned}$$

for all $p_{mn} = 1$. In this case, with the agreement $Q_{-1} = Q'_{-1} = 0$, we also note that

$$\bar{\triangle}_{11}\left(\frac{R^{0\mu}_{mn}}{R_{mn}}\right) = 0, \qquad \bar{\triangle}_{11}\left(\frac{R^{\nu 0}_{mn}}{R_{mn}}\right) = 0.$$

Now the proof is an immediate result of the theorem 2.1. □

Let $q_{ik} = 1$, for all $0 \le k \le m$, $0 \le i \le n$, $\alpha, \beta > -1$, and

$$p_{m-i,n-k} = E^{\alpha-1}_i E^{\beta-1}_k, \quad \text{where} \quad E^r_j = \binom{r+j}{j}.$$

Then the $mn-$th term of the $(C, \alpha, \beta)-$transform of a sequence $\{s_{ik}\}$ is defined by (see [1])

$$\sigma^{\alpha,\beta}_{mn} = \frac{1}{E^\alpha_m E^\beta_n} \sum_{i=0}^{m} \sum_{k=0}^{n} E^{\alpha-1}_{m-i} E^{\beta-1}_{n-k} s_{ik}.$$

In this way, $|N^{(2)}, \mathfrak{p}, \mathfrak{q}|_k-$summability reduces to the $|(C, \alpha, \beta)|_k-$summability, and for $\alpha = \beta = 1$ the following holds true.

Corollary 3.3. *If the series*

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn} \left\{ \sum_{\nu=1}^{m} \sum_{\mu=1}^{n} |a_{\nu\mu}|^2 \right\}^{\frac{k}{2}}$$

converges for $1 \le k \le 2$*, then the orthogonal series*

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} \phi_{mn}(x)$$

is summable $|(C, 1, 1)|_k$ *almost everywhere.*

Proof. By convention $E^r_{-1}=0$ we have

$$\begin{aligned}
\bar{\triangle}_{11}\left(\frac{R^{\nu\mu}_{mn}}{R_{mn}}\right) &= \sum_{i=\nu}^{m}\sum_{k=\mu}^{n}\frac{E^{\alpha-1}_{m-i}E^{\beta-1}_{n-k}}{E^{\alpha}_{m}E^{\beta}_{n}}-\sum_{i=\nu}^{m}\sum_{k=\mu}^{n-1}\frac{E^{\alpha-1}_{m-i}E^{\beta-1}_{n-1-k}}{E^{\alpha}_{m}E^{\beta}_{n-1}}\\
&\quad -\sum_{i=\nu}^{m-1}\sum_{k=\mu}^{n}\frac{E^{\alpha-1}_{m-1-i}E^{\beta-1}_{n-k}}{E^{\alpha}_{m-1}E^{\beta}_{n}}+\sum_{i=\nu}^{m-1}\sum_{k=\mu}^{n-1}\frac{E^{\alpha-1}_{m-1-i}E^{\beta-1}_{n-1-k}}{E^{\alpha}_{m-1}E^{\beta}_{n-1}}\\
&= \sum_{i=\nu}^{m}\sum_{k=\mu}^{n}\left(\frac{E^{\alpha-1}_{m-i}}{E^{\alpha}_{m}}-\frac{E^{\alpha-1}_{m-1-i}}{E^{\alpha}_{m-1}}\right)\left(\frac{E^{\beta-1}_{n-k}}{E^{\beta}_{n}}-\frac{E^{\beta-1}_{n-1-k}}{E^{\beta}_{n-1}}\right)\\
&= \sum_{i=\nu}^{m}\sum_{k=\mu}^{n}\left(\frac{1}{m+1}-\frac{1}{m}\right)\left(\frac{1}{n+1}-\frac{1}{n}\right)\\
&= \frac{m-\nu+1}{m(m+1)}\cdot\frac{n-\mu+1}{n(n+1)}\\
&= \frac{1}{mn}\left(1-\frac{\nu}{m+1}\right)\left(1-\frac{\mu}{n+1}\right)\\
&\leq \frac{1}{mn},
\end{aligned}$$

for all $q_{ik}=1$. With this we have finished the proof. □

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