Int. J. Nonlinear Anal. Appl. 7 (2016) No. 2, 253-267
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2017.980.1185

Random fractional functional differential equations

Ho Vu

Faculty of Mathematical Economics, Banking University of Ho Chi Minh City, Vietnam

(Communicated by M.B. Ghaemi)

Abstract

In this paper, we prove the existence and uniqueness results to the random fractional functional differential equations under assumptions more general than the Lipschitz type condition. Moreover, the distance between the exact solution and appropriate solution, and the existence extremal solution of the problem is also considered.

Keywords: Sample fractional integral; Sample fractional derivative; Fractional differential equations; random differential equations; Caputo fractional derivative.
2010 MSC: Primary 26A33; Secondary 47H40, 60H25.

1. Introduction

Fractional differential equations have been of great interest recently due to the intensive development of the theory of fractional calculus itself and their application in various sciences, such as physics, chemistry, engineering, automatic control, social sciences, etc. For some fundamental results in the theory of fractional calculus and fractional differential equations (see [11, 15, 17, 18, 19]).

Random differential equations, as natural extensions of deterministic ones, arise in many applications and have been investigated by many mathematicians. There are real world phenomena with anomalous dynamics such as signal transmissions through strong magnetic fields, atmospheric diffusion of pollution, network traffic, the effect of speculations on the protability of stocks in the financial markets and so on where the classical models are not sufficiently good to describe these features. We refer the reader to the monographs [3, 24, 25, 26], the papers [4, 5, 6, 7, 13, 14] and the references therein.

In the time recent, the initial value problems for random fractional differential equation have studied by V. Lupulescu and S.K. Ntouyas [20]. The basic tool in the study of the problems for random fractional differential equation is to treat it as a fractional differential equation in some

Email address: hovumath@gmail.com, vuh@buh.edu.vn (Ho Vu)

Received: October 2015 *Revised:* September 2016

appropriate Banach space. In [23], authors proved the existence results for a random fractional equation under a Carathéodory condition. Existence results for the extremal random solution are also proved. Several other research results [21, 22]. Being motivated by the works of authors [20, 23]. In this paper, we consider the random fractional functional differential equations as follows:

$$\begin{cases} \mathcal{D}^{\alpha}x(t,\omega) \overset{[0,a],\ a.e.\omega\in\Omega}{=} f_{\omega}(t,x_t), \\ x(t,\omega) \overset{[-\sigma,0]}{=} \varphi(t,\omega). \end{cases} \tag{1.1}$$

The fractional differential equations with delay (1.1) are not new to the theory random differential equations. When the random parameter ω is absent, the random (1.1) reduce to the fractional functional differential equations,

$$\begin{cases} \mathcal{D}^{\alpha}x(t) = f(t,x_t) & \text{for } t\in[0,a], \\ x(t) = \varphi(t) & \text{for } t\in[-\sigma,0]. \end{cases} \tag{1.2}$$

The classical fractional functional differential equations (1.2) has been studied in the literature by several authors for different aspects of the solutions. See, for example, M. Benchohra, el. al [2], J. Deng, el. al [10] and the references therein.

The paper is organized as follows. In Section 2 we set up the appropriate framework on random processes and on fractional calculus (see [20, 23, 27]). In section 3, we will prove the existence and uniqueness results to the random fractional functional differential equations under assumptions more general than the Lipschitz type condition. Moreover, the distance between exact solution and appropriate solution, and the existence extremal solution of the problem is also considered.

2. Preliminaries

In this section, we present a few definitions and concepts of the sample path fractional integral, the sample path fractional derivative and the metric space of random elements. The reader can see to detail results in the monographs [24, 25, 26], the papers [20] and the references therein. Let $(\Omega,\mathcal{A},\mathbb{P})$ be a complete probability space. Let $(S,\mathcal{B}(S))$ be a measurable space. If S is a metric space, then the σ-algebra $\mathcal{B}(S)$ will be the σ-algebra of all Borel subsets of S. A measurable function $x:\Omega\to S$ is called a random element in S. A random element in S is called a random variable when $S=\mathbb{R}$, a random vector when $S=\mathbb{R}^d$, and a random or stochastic process when S is a function space.

2.1. The sample path fractional integral

Let $([0,a],\mathcal{L},\lambda)$ be a Lebesgue- measure space where $a>0$ and let $x(\cdot,\cdot):[0,a]\times\Omega\to\mathbb{R}^d$ be a product measurable function. We say that $x(\cdot,\cdot)$ is sample path Lebesgue integrable on $[0,a]$ if $x(\cdot,\omega):[0,a]\to\mathbb{R}^d$ is Legesgue integrable on $[0,a]$ for a.e. $\omega\in\Omega$. Let $\alpha>0$. If $x(\cdot,\cdot):[0,a]\times\Omega\to\mathbb{R}^d$ is sample path Lebesgue integrable on $[0,a]$, then we can consider the fractional integral

$$I^{\alpha}x(t,\omega) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{x(s,\omega)}{(t-s)^{1-\alpha}}ds$$

which will be called the sample path fractional integral of x, where Γ is the Euler's Gamma function

By the properties of convolution product (see [1]), it follows that, if $x(\cdot,\omega):[0,a]\to\mathbb{R}^d$ is Lebesgue integrable on $[0,a]$ for a.e. $\omega\in\Omega$, then $t\mapsto I^{\alpha}x(t,\omega)$ is also Lebesgue integrable on $[0,a]$ for a.e. $\omega\in\Omega$. A function $x(\cdot,\cdot):[0,a]\times\Omega\to\mathbb{R}^d$ is said to be a Carathéodory function if $t\mapsto x(t,\omega)$ is continuous for a.e. $\omega\in\Omega$ and $\omega\mapsto x(t,\omega)$ is measurable for each $t\in[0,a]$. We recall that a Carathéodory function is a product measurable function (see [12]) , then function $(t,\omega)\mapsto I^{\alpha}x(t,\omega)$ is also a Carathéodory function (see [20]).

2.2. The sample path fractional derivative

A function $x(\cdot,\cdot) : [0,a]\times\Omega \to \mathbb{R}^d$ is said to have a sample path derivative at $t \in (0,a)$ if the function $t \mapsto x(t,\omega)$ has a derivative at t for a.e. $\omega \in \Omega$. We will denote by $\frac{d}{dt}x(t,\omega)$ or by $x'(t,\omega)$ the sample path derivative of $x(\cdot,\omega)$ at t.We say that $x(\cdot,\cdot) : [0,a]\times\Omega \to \mathbb{R}^d$ is sample path differentiable on $[0,a]$ if $x(\cdot,\cdot)$ has a sample path derivative for each $t \in (0,a)$ and possesses a one-sided sample path derivative at the end points 0 and a.

If $x(\cdot,\cdot) : [0,a]\times\Omega \to \mathbb{R}^d$ is a sample path absolutely continuous on $[0,a]$, that is, $t \mapsto x(t,\omega)$ is absolutely continuous on $[0,a]$ for a.e. $\omega \in \Omega$, then the sample path derivative $x'(t,\omega)$ exists for λ-a.e. $t \in [0,a]$. Let $x(\cdot,\cdot) : [0,a]\times\Omega \to \mathbb{R}^d$ be a sample path absolutely continuous on $[0,a]$ and let $\alpha \in (0,1]$. Then, for λ-a.e. and for a.e. $\omega \in \Omega$, we define the Caputo sample path fractional derivative of x by

$$\mathcal{D}^\alpha x(t,\omega) = I^{1-\alpha}x'(t,\omega) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{x'(s,\omega)}{(t-s)^\alpha}ds, \quad \text{for every } s > 0.$$

Obviously, if $x(\cdot,\cdot) : [0,a]\times\Omega \to \mathbb{R}$ is the sample path differentiable on $[0,a]$ and $t \mapsto x'(t,\omega)$ is continuous on $[0,a]$, then $\mathcal{D}^\alpha x(t,\omega)$ exists for every $t \in [0,a]$ and $t \mapsto \mathcal{D}^\alpha x(t,\omega)$ is continuous on $[0,a]$.

If $x(\cdot,\cdot) : [0,a]\times\Omega \to \mathbb{R}^d$ is a Carathéodory function then

$$\mathcal{D}^\alpha I^\alpha x(t,\omega) = x(t,\omega)$$

for all $t \in [0,a]$ and for a.e. $\omega \in \Omega$.

If $x(\cdot,\cdot) : [0,a]\times\Omega \to \mathbb{R}$ is the sample path absolutely continuous on $[0,a]$ then

$$I^\alpha \mathcal{D}^\alpha x(t,\omega) = x(t,\omega) - x(0,\omega) \tag{2.1}$$

for λ-a.e. and for a.e. $\omega \in \Omega$.

In particular, if $x(\cdot,\cdot) : [0,a]\times\Omega \to \mathbb{R}^d$ is the sample path differentiable on $[0,a]$ and $t \mapsto x'(t,\omega)$ is continuous on $[0,a]$, then (2.1) holds for all $t \in [0,a]$ and for a.e. $\omega \in \Omega$. If $x(\cdot,\cdot) : [0,a]\times\Omega \to \mathbb{R}^d$ is the sample path absolutely continuous on $[0,a]$ then

$$t \mapsto y(t,\omega) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{x(s,\omega)}{(t-s)^\alpha}ds$$

is also a sample path absolutely continuous on $[0,a]$. Moreover, for λ-a.e. and for a.e. $\omega \in \Omega$ and a.e. $\omega \in \Omega$, we have that

$$y(t,\omega) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{x(s,\omega)}{(t-s)^\alpha}ds = \mathcal{D}^\alpha x(t,\omega) + \frac{x(0,\omega)}{t^\alpha\Gamma(1-\alpha)}, \quad \text{for every } s > 0.$$

2.3. The metric space of random elements

Let the space $C = C([0,a],\mathbb{R}^d)$ of all continuous functions from $[0,a]$ from $\mathbb{R}^d$ endowed with the uniform metric

$$d(x,y) = \sup_{t\in[0,a]} ||x(t) - y(t)||,$$

where $||\cdot||$ is the Euclidean norm on $\mathbb{R}^d$.

Let $\mathcal{B}(C)$ be the σ-algebra of all Borel subsets of C. Then, by Lemma 14.1 in [16], $\mathcal{B}(C)$ coincides with the σ-algebra generated in C by all evaluation maps $\pi_t : C \to \mathbb{R}^d$, $t \in [0, a]$ given by $\pi_t(x) = x(t)$. If $x : \Omega \to C$, then $\pi_t \circ x$ maps Ω into $\mathbb{R}^d$. Therefore, x may also be regarded as a function $x(t, \omega) = (\pi_t \circ x)(\omega)$ from $[0, a] \times \Omega$ into $\mathbb{R}^d$. Then, by Lemma 2.1 in [16], $x : \Omega \to C$ is a random element in C if and only if $\pi_t \circ x : \Omega \to \mathbb{R}^d$ is measurable for all $t \in [0, a]$. In fact, it is easy to see that if $x : \Omega \to C$ is a random element, then the function $x(\cdot, \cdot) : [0, a] \times \Omega \to \mathbb{R}^d$ is a Carathéodory function. If $x : \Omega \to C$ is a random element, then the function $x(\cdot, \omega) : [0, a] \to \mathbb{R}^d$ is said to be a realization or a trajectory of the random element x, corresponding to the outcome $\omega \in \Omega$. Let $M(C)$ be the space of all probability measures on $\mathcal{B}(C)$. If $x : \Omega \to C$ is a random element in C, then the probability measure μ_x, defined by

$$\mu_x = \mathbb{P}(x^{-1}(B)) = \mathbb{P}(\{\omega \in \Omega, x(\omega) \in B\}), B \in \mathcal{B}(C),$$

is called the distribution of x. Since C has been a complete and separable metric space, then it is well known that $M(C)$ is a complete and a separable metric space with respect to the Prohorov metric $D : M(C) \to [0, \infty)$ given by

$$D(\mu, \eta) = \inf\{\epsilon > 0, \mu(B) \leq \eta(B^\epsilon) + \epsilon\}, B \in \mathcal{B}(C),$$

where $B^\alpha = \{x \in C; \inf_{y \in B} d(x, y) < \epsilon\}$. If we identify random elements in C that have the same distribution, then $\rho(x, y) = D(\mu, \eta)$ is a metric on the set of random elements in C (see [9]). Denote by $R(\Omega, C)$ the metric space of all random elements in C. A sequence of random variables $\{x_n\} \subset R(\Omega, C)$ is said to converge almost everywhere (a.e.) to $x \in R(\Omega, C)$ if there exists $N \subset R$ such that $\mathbb{P}(N) = 0$, and $\lim_{n \to \infty} d(x_n(\omega), x(\omega)) = 0$ for every $\omega \in \Omega \setminus N$. We use the notation $x_n \to x$ a.e. for almost everywhere convergence. If $\{x_n\} \subset R(\Omega, C)$ is a ρ convergent sequence, then it is known that $\{x_n\}$ is a ρ-Cauchy sequence.

Theorem 2.1. If $\{x_n\} \subset R(\Omega, C)$ is a $\overline{\rho}$-Cauchy sequence, then we can construct a sequence $\{y_n\} \subset R(\Omega, C)$ and $y \in R(\Omega, C)$ such that $\overline{\rho}(x_n, y_n) = 0$ and $y_n \to y$ for a.e $\omega \in \Omega$.

Definition 2.2. A sequence $\{x_n\} \subset R(\Omega, C)$ is $\overline{\rho}$-relatively compact if every subsequence has a subsequence that converges with respect to metric $\overline{\rho}$. Since C is a complete and separable metric space, then $\overline{\rho}$-relatively compactness and tightness are equivalent.

Theorem 2.3. A sequence $\{x_n\} \subset R(\Omega, C)$ is $\overline{\rho}$-relatively compact if and only if $\{x_n\}$ is a tight; that is, for every $\epsilon > 0$ there is a compact set $K_\epsilon \subset C$ such that $\mathbb{P}(\omega \in \Omega; x_n(\omega) \in K_\epsilon) > 1 - \epsilon$, for all $n \geq 1$.

3. Main results

Let a positive number σ, we denote by C_σ the space $C([-\sigma, 0], \mathbb{R})$. Also, we denote by

$$|x - y|_{C_\sigma} = \sup_{t \in [-\sigma, 0]} |x(t) - y(t)|$$

the metric on the space C_σ. Let $x(\cdot) \in C([-\sigma, \infty), \mathbb{R})$. Then, for each $t \in [0, \infty)$we denote by x_t the element of C_σ defined by $x_t(s) = x(t + s)$ for $s \in [-\sigma, 0]$.

In this section, we consider the random fractional functional differential equations as follows :

$$\begin{cases} \mathcal{D}^{\alpha}x(t,\omega) \overset{[0,a],\ a.e.\,\omega\in\Omega}{=} f_{\omega}(t,x_t), \\ x(t,\omega) \overset{[-\sigma,0]}{=} \varphi(t,\omega), \end{cases} \tag{3.1}$$

where $x_0 : \Omega \to \mathbb{R}$ is a random variable, $\mathcal{D}^{\alpha}x$, $0 < \alpha < 1$, is the Caputo fractional derivative of x with respect to the variable t, and $f : \Omega \times [0,a] \times C_{\sigma} \to \mathbb{R}$ is a given function.

Lemma 3.1. Let $x : [-\sigma,a] \times \Omega \to \mathbb{R}$ be a product measurable function. Then $x(t,\omega)$ is a sample solution of (3.1) if and only if $x(\cdot,\omega)$ is a continuous on $[-\sigma,a]$ for $\mathbb{P}-a.e.\ \omega \in \Omega$ and it satisfies the following random integral equation:

$$\begin{aligned} &x(t,\omega) \overset{[-\sigma,0]}{=} \varphi(t,\omega), \\ &x(t,\omega) \overset{[0,a],\ a.e.\omega\in\Omega}{=} \varphi(0,\omega) + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f_{\omega}(s,x_s)}{(t-s)^{1-\alpha}}ds. \end{aligned} \tag{3.2}$$

Lemma 3.2. Let $v, w : [-\sigma,a] \to \mathbb{R}$ be sample locally continuous random processes and assume that the function $f_{\omega}(t,\varphi)$ satisfies the assumption (f1) . If

(i) $\mathcal{D}^{\alpha}v(t,\omega) \le f_{\omega}(t,v_t)$,

(ii) $\mathcal{D}^{\alpha}w(t,\omega) \ge f_{\omega}(t,w_t)$,

for a.e. $t \in [0,a]$ and a.e. $\omega \in \Omega$, and one of the inequalities (i), (ii) being strict. Then

$$v_0(\omega) \le w_0(\omega), \tag{3.3}$$

implies

$$v(t,\omega) \le w(t,\omega) \tag{3.4}$$

for a.e. $t \in [0,a]$ and a.e. $\omega \in \Omega$.

Proof . Let $I \subset [0,a)$ be such that $\lambda(I) = 0$ and (i)-(ii) hold for all $t \in [0,a) \setminus I$ and $\omega \in \Omega$. Suppose that the claim (3.4) is not true and suppose that the inequality (ii) is strict. Then there exists a $t_1 \in [0,a) \setminus I$ such that $v(t_1,\omega) = w(t_1,\omega)$ for a.e. $\omega \in \Omega$, and $v(t,\omega) \le w(t,\omega)$ for a.e $t \in [0,t_1]$ and a.e. $\omega \in \Omega$. Then, for $h > 0$ small enough, we have

$$\frac{v(t_1,\omega) - v(t_1-h,\omega)}{h} \ge \frac{w(t_1,\omega) - w(t_1-h,\omega)}{h} \tag{3.5}$$

for a.e. $\omega \in \Omega$.

It follows that $v'(t_1,\omega) \ge w'(t_1,\omega)$ for a.e. $\omega \in \Omega$ and hence $\mathcal{D}^{\alpha}v(t_1,\omega) \ge \mathcal{D}^{\alpha}w(t_1,\omega)$ for a.e. $\omega \in \Omega$. Using the relations (i) and (ii), we have

$$f_{\omega}(t_1,v_{t_1}) \ge \mathcal{D}^{\alpha}v(t_1,\omega) \ge \mathcal{D}^{\alpha}w(t_1,\omega) > f_{\omega}(t_1,w_{t_1}) \ge f_{\omega}(t_1,v_{t_1})$$

for a.e. $\omega \in \Omega$. This contradiction proves the claim (3.4) and the proof is complete. □

Lemma 3.3. *Let $L : [0,a] \times \Omega \to \mathbb{R}_+$ be a Carathéodory function. Assume that the conditions Lemma 3.2 hold with non-strict inequalities (i) and (ii). Suppose further that*

$$|f_\omega(t,\Phi) - f_\omega(t,\Psi)| \le \frac{\hat{L}(\omega)}{1+t^\alpha} |\Phi - \Psi|_{C_\sigma} \tag{3.6}$$

for a.e. $t \in [0,a]$ and a.e. $\omega \in \Omega$, $\hat{L}(\omega) = \sup_{t\in[0,a]} L(t,\omega)$, wherever $\Phi(t,\omega) \ge \Psi(t,\omega)$ for a.e. $t \in [0,a]$ and a.e. $\omega \in \Omega$ and $\hat{L}(\omega) > 0$ for a.e. $\omega \in \Omega$. Then

$$v_0(\omega) \le w_0(\omega), \tag{3.7}$$

implies

$$v(t,\omega) \le w(t,\omega) \tag{3.8}$$

for a.e. $t \in [0,a]$ and a.e. $\omega \in \Omega$, provide $\hat{L}(\omega)a^\alpha \le \dfrac{1}{\Gamma(1-\alpha)}$.

Proof . Let $\epsilon > 0$. We set $w_\epsilon(t,\omega) = w(t,\omega) + \epsilon(1+t^\alpha)$ for a.e. $\omega \in \Omega$, so that we have

$$w_{\epsilon,t} = w_t + \epsilon[1+(t+s)^\alpha]$$

and

$$w_{\epsilon,t} \le w_t$$

for a.e. $t \in [0,a]$ and a.e. $\omega \in \Omega$.

For all $s \in [-\sigma, 0]$ and $h > 0$, we have

$$\begin{aligned}\frac{w_{\epsilon,t} - w_{\epsilon,t-h}}{h} &= \frac{w_t + \epsilon[1+(t+s)^\alpha] - w_{t-h} + \epsilon[1+(t+s-h)^\alpha]}{h} \\ &= \frac{w(t+s,\omega) - w(t+s-h,\omega)}{h} + \epsilon\frac{(t+s)^\alpha - (t+s-h)^\alpha}{h},\end{aligned}$$

for a.e. $\omega \in \Omega$.

It follows that

$$\begin{aligned}\mathcal{D}^\alpha w_{\epsilon,t} &= \mathcal{D}^\alpha w(t,\omega) + \epsilon\mathcal{D}^\alpha(1+t^\alpha) \\ &\le f_\omega(t,w_t) + \epsilon\Big[\frac{1}{t^\alpha\Gamma(1-\alpha)} + \Gamma(1+\alpha)\Big] \\ &\ge f_\omega(t,w_{\epsilon,t}) - \frac{\hat{L}(\omega)}{1+t^\alpha}|w_{\epsilon,t} - w_t|_{C_\sigma} + \epsilon\Big[\frac{1}{t^\alpha\Gamma(1-\alpha)} + \Gamma(1+\alpha)\Big] \\ &> f_\omega(t,w_{\epsilon,t}) - \epsilon\hat{L}(\omega) + \frac{\epsilon}{t^\alpha\Gamma(1-\alpha)}.\end{aligned}$$

By (3.8), (3.6) and the assumptions $\hat{L}(\omega)a^\alpha \le \dfrac{1}{\Gamma(1-\alpha)}$ we have $\mathcal{D}^\alpha w_{\epsilon,t} > f_\omega(t,w_{\epsilon,t})$ for a.e. $t \in [0,a]$ and a.e. $\omega \in \Omega$. Now, we use Lemma 3.2 to v and w_ϵ to yield $w_\epsilon(t,\omega) \le v(t,\omega)$, $t \in [0,a]$. Since $\epsilon > 0$ is arbitrary, we conclude that (3.8) is true and the proof is complete. $\square$

Now, we shall prove the local existence result for the random fractional functional differential equations (3.1) assuming that the following assumptions are satisfied.

(f1) The mapping $f_\omega(\cdot,\cdot) : [0,a] \times C_\sigma \to \mathbb{R}$ are measurable and continuous, for every $(t_0, \varphi_0) \in [0,a] \times C_\sigma$, each $\omega \in \Omega$.

(f2) The exists $y_0 \in \mathbb{R}$ such that $x_0(\omega) \in \overline{B}(y_0, \rho)$ for a.e. $\omega \in \Omega$, where

$$B_\rho(y_0) = \{y \in \mathbb{R} : |y - y_0|_{C_\sigma} \leq \rho\}.$$

(f3) There exists a non-negative constants M and ρ such that

$$|f_\omega(t, \Phi)| \leq M,$$

for every $t \in [0,a]$, $\Phi \in B_\rho(y_0)$ and for a.e. $\omega \in \Omega$.

Also, the function $g : [0,a] \times [0,2b] \times \Omega \to \mathbb{R}_+$ satisfy the assumptions as follows :

(g1) the mapping $g_\omega(\cdot,\cdot) : [0,a] \times [0,2b] \to \mathbb{R}_+$ are measurable and continuous, for every $(t_0, \varphi_0) \in [0,a] \times [0,2b]$ and each $\omega \in \Omega$;

(g2) there exists a non-negative constant $\hat{M}$ such that

$$|g_\omega(t, \tau)| \leq \hat{M},$$

for every $t \in [0,a]$, $\tau \in [0,2\rho]$ and for a.e. $\omega \in \Omega$;

(g3) $g_\omega(t, 0) = 0$ for all $t \in [0,a]$ and a.e. $\omega \in \Omega$;

(g4) $\tau(t,\omega) \equiv 0$ is the solution to random fractional initial value problem

$$\mathcal{D}^\alpha \tau(t,\omega) = g_\omega(t, \tau(t,\omega)), \quad \tau(0,\omega) = \tau_0(\omega) \tag{3.9}$$

on $t \in [0,a]$;

(g5) the mapping $g_\omega(t,\cdot) : [0,2\rho] \to \mathbb{R}_+$ is non-decreasing for a.e. $\omega \in \Omega$.

Theorem 3.4. *Let $\varphi \in C_\sigma$ and the mappings $f : [0,a] \times C_\sigma \times \Omega \to \mathbb{R}$, $g : [0,a] \times [0,2b] \times \Omega \to \mathbb{R}_+$ satisfy the assumptions (f1)-(f3), (g1)-(g5), respectively. Suppose that for $\phi, \psi \in B_\rho(x_0)$ it holds*

$$\left| \frac{f_\omega(t,\phi)}{(t_1 - s)^{1-\alpha}} - \frac{f_\omega(t,\psi)}{(t_2 - s)^{1-\alpha}} \right| \leq \left| \frac{1}{(t_1 - s)^{1-\alpha}} - \frac{1}{(t_2 - s)^{1-\alpha}} \right| g_\omega(t, |\phi - \psi|_{C_\sigma}) \tag{3.10}$$

for a.e. $t_1, t_2 \in [0,a]$ and a.e. $\omega \in \Omega$. Then, the successive approximations defined by

$$x^0(t,\omega) = \begin{cases} \varphi(t,\omega), & \text{for } t \in [-\sigma, 0], \\ \varphi(0,\omega), & \text{for } t \in [0,a] \end{cases} \tag{3.11}$$

and for $n \in 0, 1, 2, \ldots$

$$x^{n+1}(t,\omega) = \begin{cases} \varphi(t,\omega), & \text{for } t \in [-\sigma, 0], \\ \varphi(0,\omega) + \dfrac{1}{\Gamma(\alpha)} \int_0^t \dfrac{f_\omega(s, x^n_s)}{(t-s)^{1-\alpha}} ds, & \text{for } t \in [0,a] \end{cases} \tag{3.12}$$

on $[0,\beta]$, are continuous and converge uniformly to the unique solution $x(t,\omega)$ of the problem (3.1) *on $[0,\beta]$ for a.e. $\omega \in \Omega$, where $\beta = \min\left\{a, \sqrt[\alpha]{\dfrac{\rho\Gamma(\alpha+1)}{\max\{M, \hat{M}\}}}\right\}$.*

Proof . First, we shall prove the sequence $\{x^n\}_{n=0}^{\infty}$ are continuous with value $B_\rho(x_0)$ for a.e. $\omega \in \Omega$. For all $0 \le t \le \beta$ and a.e. $\omega \in \Omega$, we have

$$\begin{aligned}\left|x^1(t,\omega) - x^0(t,\omega)\right| &\le \frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-s)^{1-\alpha}} \left|f_\omega(s, x_s^0)\right| ds \\ &\le \frac{\max\{M, \hat{M}\}}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-s)^{1-\alpha}} ds \\ &\le \frac{\max\{M, \hat{M}\} t^\alpha}{\Gamma(\alpha+1)} \le \frac{\max\{M, \hat{M}\} \beta^\alpha}{\Gamma(\alpha+1)} < \rho.\end{aligned}$$

Hence $x^1(t,\omega) \in B_\rho(x_0)$.

Assume that $x^1(t,\omega) \in B_\rho(x_0)$. It is easy to check that $x^{n+1}(t,\omega) \in B_\rho(x_0)$. Indeed, for $0 \le t_1 \le t_2 \le \beta$ and a.e. $\omega \in \Omega$, one obtains

$$\begin{aligned}&|x^n(t_1,\omega) - x^n(t_2,\omega)| \\ &= \left|\frac{1}{\Gamma(\alpha)} \int_0^{t_1} \frac{f_\omega(s, x_s^{n-1})}{(t_1-s)^{1-\alpha}} ds - \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \frac{f_\omega(s, x_s^{n-1})}{(t_2-s)^{1-\alpha}} ds\right| \\ &= \left|\frac{1}{\Gamma(\alpha)} \int_0^{t_1} \Big(\frac{1}{(t_1-s)^{1-\alpha}} - \frac{1}{(t_2-s)^{1-\alpha}}\Big) \left|f_\omega(s, x_s^{n-1})\right| ds - \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \frac{f_\omega(s, x_s^{n-1})}{(t_2-s)^{1-\alpha}} ds\right| \\ &\le \frac{\max\{M, \hat{M}\}}{\Gamma(\alpha)} \int_0^{t_1} \left|\frac{1}{(t_1-s)^{1-\alpha}} - \frac{1}{(t_2-s)^{1-\alpha}}\right| ds + \frac{\max\{M, \hat{M}\}}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left|\frac{1}{(t_2-s)^{1-\alpha}}\right| ds \\ &\le \frac{\max\{M, \hat{M}\}}{\Gamma(\alpha+1)} [t_1^\alpha - t_2^\alpha + 2(t_2-t_1)^\alpha] \le \frac{2\max\{M, \hat{M}\}}{\Gamma(\alpha+1)} (t_2-t_1)^\alpha < \epsilon\end{aligned}$$

provided $|t_2 - t_1| \le \delta = \sqrt[\alpha]{\frac{\epsilon\Gamma(\alpha+1)}{2\max\{M, \hat{M}\}}}$. Therefore, we obtain the sequence $x^n(\cdot,\omega)$ are continuous for a.e. $\omega \in \Omega$.

For $h > 0$ small enough, we have

$$\begin{aligned}x^{n+1}(t+h,\omega) - x^{n+1}(t,\omega) &= \frac{1}{\Gamma(\alpha)} \int_0^{t+h} \frac{f_\omega(s, x_s^{n-1})}{(t-s)^{1-\alpha}} ds - \frac{1}{\Gamma(\alpha)} \int_0^{t} \frac{f_\omega(s, x_s^{n-1})}{(t-s)^{1-\alpha}} ds \\ &= \frac{1}{\Gamma(\alpha)} \int_t^{t+h} \frac{f_\omega(s, x_s^{n-1})}{(t-s)^{1-\alpha}} ds\end{aligned}$$

for a.e. $\omega \in \Omega$. This implies that

$$\begin{cases}\mathcal{D}^\alpha x^{n+1}(t,\omega) = f_\omega(t, x_t^n), & \text{for } t \in [0,a], \\ x^{n+1}(0,\omega) = \varphi(t,\omega), & \text{for } t \in [-\sigma, 0]\end{cases}$$

for $n = 0, 1, \ldots$.

Now, let us define the successive approximations $\{\tau_n\}_{n=0}^{\infty}$ for the problem (3.9) and $r = \min\left\{\beta, \sqrt[\alpha]{\frac{\Gamma(1+\alpha)}{\max\{M, \hat{M}\}}}\right\}$ as follows:

$$\begin{aligned}\tau_0(t,\omega) &= \frac{\max\{M, \hat{M}\} t^\alpha}{\Gamma(1+\alpha)}, \\ \tau_{n+1}(t,\omega) &= \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g_\omega(s, \tau_n(s,\omega))}{(t-s)^\alpha} ds\end{aligned} \tag{3.13}$$

for $t \in [0, r]$ and $n = 0, 1, \ldots$.

Then, we get immediately

$$\tau_1(t,\omega) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g_\omega(s, \tau_0(s,\omega))}{(t-s)^\alpha} ds \le \frac{\hat{M} t^\alpha}{\Gamma(1+\alpha)} \le \tau_0(t,\omega) \le \rho,$$

for $t \in [0, r]$. Hence, by the assumption (g2), (g5) and proceeding recursively, we obtain

$$0 \le \tau_{n+1}(t,\omega) \le \tau_n(t,\omega) \le \beta, \quad n = 0, 1, \ldots. \tag{3.14}$$

Moreover,

$$|\mathcal{D}^\alpha \tau_n(t,\omega)| = g_\omega(t, \tau_{n-1}(t,\omega)) \le \hat{M}. \tag{3.15}$$

for a.e $t \in [0, \beta]$ and a.e. $\omega \in \Omega$;

By (3.14) and (3.15), we infer that the sequence $\{\tau_n(\cdot,\omega)\}$ is sample uniformly bounded and sample equicontinuous for a.e. $\omega \in \Omega$. Hence, by the Ascoli- Arzela Theorem, the sequence $\{\tau_n(\cdot,\omega)\}$ is a relatively compact subset of $C_\eta = C([0,\eta], \mathbb{R})$, $\eta < r$ for a.e $\omega \in \Omega$. Now, by Theorem 2.3, the sequence $\{\tau_n(\cdot,\omega)\} \subset R(\Omega, C_\eta)$ is d-relatively compact. Thus , $\{\tau_n(\cdot,\omega)\}$ has a d-Cauchy subsequence. Further, by Theorem 2.1, we can construct a sequence $\{\kappa_n(\cdot,\omega)\} \subset R(\Omega, C_\eta)$ and $\kappa_\omega \in R(\Omega, C_\eta)$ such that $d(\tau_n(\cdot,\omega), \kappa_n(\cdot,\omega)) = 0$ and $\tau_n(\cdot,\omega) \to \kappa_\omega$, where κ_ω is a solution of the fractional initial value problem: $\mathcal{D}^\alpha \kappa(t) = g_\omega(t, \kappa_\omega(t))$, $\kappa_\omega(0) = 0$. By the assumption (g4) and (3.14), we have $\tau_n(t,\omega) \to 0$ as $n \to \infty$ for a.e. $\omega \in \Omega$.

Note for $t \in [0, \beta]$ and a.e $\omega \in \Omega$, we have

$$\left| x^1(t,\omega) - x^0(t,\omega) \right| \le \frac{M t^\alpha}{\Gamma(\alpha+1)} \le \tau_0(t,\omega)$$

and

$$\begin{aligned}
\sup_{\upsilon \in [-\sigma,0]} \left| x^1(t+\upsilon,\omega) - x^0(t+\upsilon,\omega) \right| &\le \frac{1}{\Gamma(\alpha)} \sup_{\upsilon \in [-\sigma,0]} \int_0^{t+\upsilon} \frac{1}{(t-s)^{1-\alpha}} \left| f_\omega(s, x_s^0) \right| ds \\
&\le \frac{M}{\Gamma(\alpha)} \sup_{\upsilon \in [-\sigma,0]} \int_0^{t+\upsilon} \frac{1}{(t-s)^{1-\alpha}} ds \\
&= \frac{M}{\Gamma(\alpha)} \sup_{\theta \in [t-\sigma,t]} \int_0^{\theta} \frac{1}{(t-s)^{1-\alpha}} ds \\
&= \frac{M}{\Gamma(\alpha+1)} \sup_{\theta \in [t-\sigma,t]} [t^\alpha - (t-\theta)^\alpha] \\
&\le \frac{M t^\alpha}{\Gamma(\alpha+1)} \le \tau_0(t,\omega).
\end{aligned}$$

Assume that for $k \ge 1$, we have

$$\left| x^k(t,\omega) - x^{k-1}(t,\omega) \right| \le \tau_{k-1}(t,\omega)$$

and

$$\sup_{\upsilon \in [-\sigma,0]} \left| x^k(t+\upsilon,\omega) - x^{k-1}(t+\upsilon,\omega) \right| \le \tau_{k-1}(t,\omega).$$

Since

$$\begin{aligned}\left|x^{k+1}(t,\omega)-x^{k}(t,\omega)\right| &= \left|\frac{1}{\Gamma(\alpha)}\int_0^t \frac{f_\omega(s,x_s^k)}{(t-s)^{1-\alpha}}ds - \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f_\omega(s,x_s^{k-1})}{(t-s)^{1-\alpha}}ds\right| \\ &\le \frac{1}{\Gamma(\alpha)}\int_0^t \frac{1}{(t-s)^{1-\alpha}}\left|f_\omega(s,x_s^k)-f_\omega(s,x_s^{k-1})\right|ds.\end{aligned}$$

Using the condition (3.10) and assumption (g5), we get

$$\begin{aligned}\left|x^{k+1}(t,\omega)-x^{k}(t,\omega)\right| &\le \frac{1}{\Gamma(\alpha)}\int_0^t \frac{1}{(t-s)^{1-\alpha}} g_\omega\big(s,\sup_{\xi\in[s-\sigma,s]}|x^k(\xi,\omega)-x^{k-1}(\xi,\omega)|\big)ds \\ &= \frac{1}{\Gamma(\alpha)}\int_0^t \frac{1}{(t-s)^{1-\alpha}} g_\omega\big(s,\sup_{\xi\in[0,s]}|x^k(\xi,\omega)-x^{k-1}(\xi,\omega)|\big)ds \\ &\le \tau_k(t,\omega).\end{aligned}$$

Thus, by mathematical induction, we obtain

$$\left|x^{n+1}(t,\omega)-x^{n}(t,\omega)\right| \le \tau_n(t,\omega), \quad n=0,1,2,\dots,$$

for a.e. $t\in[0,\beta]$ and a.e. $\omega\in\Omega$.

Moreover, we have

$$\begin{aligned}\left|\mathcal{D}^\alpha x^{n+1}(t,\omega)-\mathcal{D}^\alpha x^{n}(t,\omega)\right| &\le \left|f_\omega(t,x_t^n)-f_\omega(t,x_t^{n-1})\right| \\ &\le g_\omega(t,|x_t^n-x_t^{n-1}|_{C_\sigma}) \le g_\omega(t,\tau_{n-1}(t,\omega))\end{aligned}$$

for a.e. $\omega\in\Omega$.

Let $m>n$, then one can easily obtain

$$\begin{aligned}\mathcal{D}^\alpha\left|x^{m}(t,\omega)-x^{n}(t,\omega)\right| &\le \mathcal{D}^\alpha\left|x^{m}(t,\omega)-x^{m+1}(t,\omega)\right| + \mathcal{D}^\alpha\left|x^{m+1}(t,\omega)-x^{n+1}(t,\omega)\right| \\ &\quad + \mathcal{D}^\alpha\left|x^{n+1}(t,\omega)-x^{n}(t,\omega)\right| \\ &\le g_\omega(t,\tau_{n-1}(t,\omega))+g_\omega(t,\tau_{m-1}(t,\omega))+g_\omega(t,|x_t^m-x_t^n|_{C_\sigma})\end{aligned}$$

Since, $\tau_{n+1}(t,\omega)\le\tau_n(t,\omega)$ for a.e. $t\in[0,\beta]$ and a.e. $\omega\in\Omega$, we have

$$\mathcal{D}^\alpha\left|x^{m}(t,\omega)-x^{n}(t,\omega)\right| \le 2g_\omega(t,\tau_{n-1}(t,\omega))+g_\omega(t,|x_t^m-x_t^n|_{C_\sigma}), \quad n=0,1,\dots.$$

Therefore, we obtain the Dini derivative $\mathcal{D}^{+\alpha}$ of the function $|x^m(t,\omega)-x^n(t,\omega)|$ as follows:

$$\mathcal{D}^{+\alpha}\left|x^{m}(t,\omega)-x^{n}(t,\omega)\right| \le 2g_\omega(t,\tau_{n-1}(t,\omega))+g_\omega(t,|x_t^m-x_t^n|_{C_\sigma}), \quad n=0,1,\dots.$$

for a.e. $\omega\in\Omega$. As the sequence of the functions $g_\omega(\cdot,\tau_{n-1}(\cdot,\omega))$ converge uniformly to 0, then for a.e. $\omega\in\Omega$, for every $\epsilon>0$ there exists a natural number n_0 such that

$$\mathcal{D}^{+\alpha}\left|x^{m}(t,\omega)-x^{n}(t,\omega)\right| \le 2g_\omega(t,\tau_{n-1}(t,\omega))+\epsilon$$

for $m>n>n_0$.

This fact together with $\mathcal{D}^{+\alpha}|x^m(0,\omega)-x^n(0,\omega)|\le\epsilon$ for a.e. $\omega\in\Omega$ and by Theorem 2.5.1 in [24] we have

$$\mathcal{D}^{+\alpha}\left|x^{m}(t,\omega)-x^{n}(t,\omega)\right| \le v_\epsilon(t,\omega), \quad t\in[0,\beta],\ m>n>n_0, \tag{3.16}$$

where $v_\epsilon(t,\omega)$ denotes the maximal solution of the fractional initial value problem as follows:

$$\mathcal{D}^\alpha v_\epsilon(t,\omega) = g_\omega(t, v_\epsilon(t,\omega)) + \epsilon, \quad v_\epsilon(0,\omega) = 0,$$

for a.e. $t \in [0,\beta]$ and a.e. $\omega \in \Omega$.

In [24] it has been showed that $\{v_\epsilon(\cdot,\omega)\}$ converges uniformly to maximal solution of the problem (3.9) on $[0,\beta]$ as $\epsilon \to 0$ which by assumption is constants equal to 0. Hence, by virtue of (3.16), we infer that $\{x^n\}$ converges uniformly to a sample continuous function x for a.e. $\omega \in \Omega$. Since $x^n(t,\omega) \in B_\rho(x_0)$, we have $x(t,\omega) \in B_\rho(x_0)$.

Note that $x(t,\omega)$ is the desired solution to (3.1) for a.e $t \in [0,\beta]$ a.e $\omega \in \Omega$. Indeed, we have

$$\begin{aligned}
&\left| x(t,\omega) - \varphi(0,\omega) - \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f_\omega(s,x_s)}{(t-s)^{1-\alpha}} ds \right| \\
&\qquad \le |x(t,\omega) - x^n(t,\omega)| + \left| x^n(t,\omega) - \varphi(0,\omega) - \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f_\omega(s,x_s^{n-1})}{(t-s)^{1-\alpha}} ds \right| \\
&\qquad + \left| \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f_\omega(s,x_s)}{(t-s)^{1-\alpha}} ds - \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f_\omega(s,x_s^{n-1})}{(t-s)^{1-\alpha}} ds \right|.
\end{aligned}$$

The first term of the right-hand side of the inequality uniformly converges to zero, whereas the second is equal to zero. Now, we consider the third term. By the Lebesgue dominated convergence theorem and assumption (g1), we are easy to prove that

$$\left| \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f_\omega(s,x_s)}{(t-s)^{1-\alpha}} ds - \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f_\omega(s,x_s^{n-1})}{(t-s)^{1-\alpha}} ds \right| \to 0.$$

for a.e. $t \in [0,\beta]$ and a.e. $\omega \in \Omega$. Therefore,

$$\left| x(t,\omega) - \varphi(0,\omega) - \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f_\omega(s,x_s)}{(t-s)^{1-\alpha}} ds \right| \to 0,$$

for a.e. $t \in [0,\beta]$ and a.e. $\omega \in \Omega$. Hence, by Lemma 3.2 the function $x(t,\omega)$ is a solution to (3.1).

For the uniqueness let us assume that $x, y : [-\sigma,\beta] \times \Omega \to \mathbb{R}$ are the two solutions to (3.1). Denote $m(t,\omega) = |x(t,\omega) - y(t,\omega)|$. Then $m(0,\omega) = 0$ and

$$\mathcal{D}^{+\alpha} m(t,\omega) = |\mathcal{D}^{+\alpha} x(t,\omega) - \mathcal{D}^{+\alpha} y(t,\omega)| \le g_\omega\big(t, |x_t - y_t|_{C_\sigma}\big) \le g_\omega\big(t, |m(t,\omega)|_{C_\sigma}\big).$$

Hence, by Theorem 2.5.1 in [24], we infer that

$$m(t,\omega) \le \tau(t,\omega)$$

for a.e. $t \in [0,\beta]$ and a.e. $\omega \in \Omega$, where $\tau(t,\omega)$ is a maximal solution to the problem (3.9). Thus, by the assumption (g3) then $m(t,\omega) \le 0$ for a.e. $\omega \in \Omega$. Therefore, $x(t,\omega)$ is a unique solution to the problem (3.1). This proof is completed. $\square$

Corollary 3.5. *Let f, g be as in Theorem 3.4 and suppose that for $\phi, \psi \in B_\rho(x_0)$, the function $L : [0,a] \times \Omega \to \mathbb{R}^+$ is a sample continuous processes such that $\sup_{t\in[0,a]} L(t,\omega) = \hat{L}(\omega)$ with a.e. $\omega \in \Omega$, it holds*

$$\left| \frac{f_\omega(t,\phi)}{(t_1-s)^{1-\alpha}} - \frac{f_\omega(t,\psi)}{(t_2-s)^{1-\alpha}} \right| \le \left| \frac{1}{(t_1-s)^{1-\alpha}} - \frac{1}{(t_2-s)^{1-\alpha}} \right| L(t,\omega) |\phi - \psi|_{C_\sigma} \tag{3.17}$$

and for a.e. $t_1, t_2 \in [0,a]$ *and a.e.* $\omega \in \Omega$*. Then, the successive approximations defined by* (3.11) *and* (3.12) *on* $[0,\beta]$*, are continuous and converge uniformly to the unique solution* $x(t,\omega)$ *of the problem* (3.1) *on* $[0,\beta]$ *for a.e.* $\omega \in \Omega$*, where* $\beta = \min\left\{a,\ \sqrt[\alpha]{\frac{\rho\Gamma(\alpha+1)}{M}},\ \sqrt[\alpha]{\frac{\rho\Gamma(\alpha+1)}{\hat{L}(\omega)}}\right\}$.

Proof . The proof is obtained immediately by taking $g_\omega(t,\omega) = L(t,\omega)|\phi-\psi|_{C_\sigma}$ in Theorem 3.4 . □

In the sequel, we will show that the distance between exact solution and approximate solution is bounded by maximal solution to fractional random initial value problem. Let us consider two the problems as follows:

$$\begin{cases} \mathcal{D}^\alpha x(t,\omega) \overset{[0,a],\ a.e.\,\omega\in\Omega}{=} f_\omega(t,x_t), \\ x(t,\omega) \overset{[-\sigma,0]}{=} \varphi_1(t,\omega). \end{cases} \tag{3.18}$$

and

$$\begin{cases} \mathcal{D}^\alpha x(t,\omega) \overset{[0,a],\ a.e.\,\omega\in\Omega}{=} f_\omega(t,x_t), \\ x(t,\omega) \overset{[-\sigma,0]}{=} \varphi_2(t,\omega). \end{cases} \tag{3.19}$$

Theorem 3.6. *Let* f, g *be as in Theorem* 3.4*, and let* $\varphi_1, \varphi_2 \in C_\sigma$ *and*

$$\max_{s\in[-\sigma,0]} |\varphi_1(s,\omega) - \varphi_2(s,\omega)| \le \tau_0(\omega)$$

for a.e. $\omega \in \Omega$*. Assume that* x, y *are a solution to* (3.18) *and* (3.19)*, respectively. Then*

$$|x(t,\omega) - y(t,\omega)| \le \tau(t,\omega),$$

for a.e. $t \in [0,a]$ *and a.e.* $\omega \in \Omega$*, where* $\tau(t,\omega)$ *is the solution to random fractional initial value problem* (3.9)*.*

Proof . Note that for $t \in [0,a]$ and $h > 0$ small enough, we have

$$\begin{aligned} &|x(t+h,\omega) - y(t+h,\omega)| \\ &\qquad \le |x(t+h,\omega) - x(t,\omega) - h^\alpha f_\omega(t,x_t)| + |x(t,\omega) + h^\alpha f_\omega(t,x_t) - y(t+h,\omega)| \\ &\qquad \le |x(t+h,\omega) - x(t,\omega) - h^\alpha f_\omega(t,x_t)| + |y(t,\omega) + h^\alpha f_\omega(t,y_t) - y(t+h,\omega)| \\ &\qquad + |x(t,\omega) + h^\alpha f_\omega(t,x_t) - y(t,\omega) - h^\alpha f_\omega(t,y_t)| \\ &\qquad \le |x(t+h,\omega) - x(t,\omega) - h^\alpha f_\omega(t,x_t)| + |y(t+h,\omega) - y(t,\omega) - h^\alpha f_\omega(t,y_t)| \\ &\qquad + |x(t,\omega) - y(t,\omega)| + h^\alpha\,|f_\omega(t,x_t) - f_\omega(t,y_t)| \end{aligned}$$

for a.e. $\omega \in \Omega$.

Now let us define $m(t,\omega) = \sup\limits_{\theta\in[0,t]} |x(\theta,\omega) - y(\theta,\omega)|$ for a.e. $t \in [0,a]$ and a.e. $\omega \in \Omega$. Then we have

$$\begin{aligned}
\mathcal{D}^{+\alpha} m(t,\omega) &\le \limsup_{h\to 0^+} \left| \frac{x(t+h,\omega) - x(t,\omega)}{h^\alpha} - f_\omega(t,x_t) \right| + |f_\omega(t,x_t) - f_\omega(t,y_t)| \\
&+ \limsup_{h\to 0^+} \left| \frac{y(t+h,\omega) - y(t,\omega)}{h^\alpha} - f_\omega(t,y_t) \right| \\
&\le (t-s)^{\alpha-1} g_\omega(t, |x_t - y_t|_{C_\sigma}) \le g_\omega(t, \sup_{\theta\in[t-\sigma,t]} |x(\theta,\omega) - y(\theta,\omega)|) \\
&\le g_\omega(t, \sup_{\theta\in[0,t]} |x(\theta,\omega) - y(\theta,\omega)|)
\end{aligned}$$

where $\mathcal{D}^{+\alpha}$ is the corresponding Caputo fractional Dini derivative to $\mathcal{D}^+$. By Theorem 2.5.1 in [24], then we obtain

$$|x(t,\omega) - y(t,\omega)| \le \tau(t,\omega),$$

for a.e. $t \in [0,a]$ and a.e. $\omega \in \Omega$. This proof is completed. □

Next, we shall consider the existence of extremal solutions to the problem (3.1).

Let $l(\cdot,\cdot) : [0,\beta] \times \Omega \to \mathbb{R}$ be a sample solution process on $[0,\beta]$ of the problem as follows:

$$\mathcal{D}^\alpha x(t,\omega) \overset{[0,\beta]\, a.e.\omega\in\Omega}{=} f_\omega(t,x_t), \quad x(t,\omega) \overset{[-\sigma,0]}{=} \varphi(t,\omega). \tag{3.20}$$

Then $l(\cdot,\cdot)$ is said to be sample maximal solution process of the problem (3.20) on $[0,\beta]$, if for every sample solution of the problem (3.20), the inequality $x(t,\omega) \le l(t,\omega)$ for a.e $\omega \in \Omega$.

Theorem 3.7. *Let f be as in Theorem 3.4 and suppose that $f_\omega(t,\Phi)$ is sample nondecreasing in Φ for fixed t and a.e. $\omega \in \Omega$. Then given $\Phi(0,\omega) \in C_\sigma$ at $t = 0$, there exists an $\beta_1 > 0$ such that the problem (3.1) admits extremal solution on $[0,\beta_1]$, where $\beta_1 = \min\left\{\beta, \sqrt[\alpha]{\dfrac{\rho\Gamma(\alpha+1)}{2M+\rho}}\right\}$.*

Proof . Let $0 < \epsilon < \rho/2$ and consider the random fractional initial value problem as follows:

$$\mathcal{D}^\alpha x(t,\omega) \overset{[0,\beta]\, a.e.\omega\in\Omega}{=} f_\omega(t,x_t) + \epsilon, \quad x(t,\omega) \overset{[-\sigma,0]}{=} \varphi(t,\omega) + \epsilon. \tag{3.21}$$

Note that $f^\epsilon_\omega(t,x_t) = f_\omega(t,x_t) + \epsilon$ is define and sample continuous on

$$R_\epsilon = \Big\{(t,\psi) : 0 \le t \le \beta, \text{ and } \psi \in B_\rho(x_0), |\psi - (\varphi(0,\omega) + \epsilon)| \le \rho\Big\}.$$

It is easy to see that $R_\epsilon \subset R_0$ and $|f^\epsilon_\omega(t,\psi) + \epsilon| \le M + \rho/2$ on R_ϵ. Then by Theorem 3.4 the problem (3.21) has a sample solution $x^\epsilon(\cdot,\omega)$ on $[0,\beta_1]$.

Let $\{\epsilon_n\}$ be a strict decreasing sequence such that $0 < \epsilon_n < \epsilon$ for all $n \ge 1$ and $\lim\limits_{n\to\infty} \epsilon_n = 0$. Also, for each $n \ge 1$, let $x^{\epsilon_n}(\cdot,\cdot)$ be solution of

$$\mathcal{D}^\alpha x^{\epsilon_n}(t,\omega) \overset{[0,\beta]\, a.e.\omega\in\Omega}{=} f_\omega(t,x^{\epsilon_n}_t) + \epsilon_n, \quad x^{\epsilon_n}(t,\omega) \overset{[-\sigma,0]}{=} \varphi(t,\omega) + \epsilon_n. \tag{3.22}$$

This implies that $x^{\epsilon_{n+1}}(t,\omega) \le x^{\epsilon_n}(t,\omega)$ on $t \in [-\sigma,0]$ and

$$\mathcal{D}^\alpha x^{\epsilon_{n+1}}(t,\omega) = f^{\epsilon_{n+1}}_\omega(t,x^{\epsilon_{n+1}}_t) = f_\omega(t,x^{\epsilon_{n+1}}_t) + \epsilon_{n+1} < f_\omega(t,x^{\epsilon_{n+1}}_t) + \epsilon_n = f_\omega(t,x^{\epsilon_{n+1}}_t)$$

for $t \in [0, \beta_1]$. Then, by Lemma 3.2, it follows that $x^{\epsilon_{n+1}}(t,\omega) \le x^{\epsilon_n}(t,\omega)$ for a.e $t \in [0,\beta_1]$ and a.e $\omega \in \Omega$.

Now, we consider the family of sample continuous functions $\{x^{\epsilon_n}(\cdot,\cdot)\}$ on $[0,\beta_1]$. Note that for a.e. $t \in [0,\beta_1]$ and a.e. $\omega \in \Omega$, we have

$$\begin{aligned} |x^{\epsilon_n}(t,\omega) - \varphi(0,\omega) - \epsilon_n| &\le \frac{1}{\Gamma(\alpha)} \left| \int_0^t \frac{f_\omega(t, x_t^{\epsilon_n})}{(t-s)^{1-\alpha}} ds \right| = \frac{1}{\Gamma(\alpha)} \left| \int_0^t \frac{f_\omega(t, x_t) + \epsilon_n}{(t-s)^{1-\alpha}} ds \right| \\ &\le \frac{2M+\rho}{2\Gamma(\alpha)} \int_0^t \frac{1}{(t-s)^{1-\alpha}} ds \le \frac{(2M+\rho)\beta_1^\alpha}{2\Gamma(\alpha+1)} < \rho. \end{aligned}$$

Hence, the family of functions $\{x^{\epsilon_n}(\cdot,\cdot)\}$ is sample uniformly bounded for a.e. $t \in [0,\beta_1]$ and a.e. $\omega \in \Omega$.

Moreover, by Lemma 2.2 in [19] and $|\mathcal{D}^\alpha x^{\epsilon_n}(t,\omega)| = |f_\omega(t, x_t^{\epsilon_n}) + \epsilon_n| \le M + \rho/2$, we infer that the family of functions also equicontinuous for a.e. $t \in [0,\beta_1]$ and a.e. $\omega \in \Omega$. Therefore, there exists a decreasing sequence $\{\epsilon_n\}$ such that $\epsilon_n \to 0$ as $n \to \infty$ and the uniform limit $l(t,\omega) = \lim_{n\to\infty} x^{\epsilon_n}(t,\omega)$ exists on $t \in [0,\beta_1]$ and a.e. $\omega \in \Omega$. It is easy to check that $l(0,\omega) = \varphi(0,\omega)$ for a.e. $\omega \in \Omega$.

Since $x(t,\omega)$ is a sample continuous functions for fixed $t \in [0,\beta_1]$, then $f_\omega(t, x_t^{\epsilon_n}) \to f_\omega(t, l_t)$ as $n \to \infty$ for a.e $t \in [0,\beta_1]$. By Theorem 3.4, we infer that $l(t,\omega)$ is a solution of the problem (3.1) on $[0,\beta_1]$.

In the sequel, we will show that $l(t,\omega)$ is the maximal solution of the problem (3.1) on $[-\sigma,\beta_1]$. Let us $x(t,\omega)$ be any solution of the problem (3.1) on $[-\sigma,\beta_1]$. We have

$$\begin{aligned} &\varphi(t,\omega) < \varphi(t,\omega) + \epsilon, \\ &\mathcal{D}^\alpha x(t,\omega) < f_\omega(t, x_t), \\ &\mathcal{D}^\alpha l(t,\omega) \ge f_\omega(t, l_t) \end{aligned}$$

for a.e. $t \in [0,\beta_1]$ and a.e. $\omega \in \Omega$. By Lemma 3.2, we obtain

$$x(t,\omega) \le l(t,\omega)$$

for all $t \in [0,\beta_1]$ and a.e. $\omega \in \Omega$. It follows that $x(t,\omega) \le \lim_{n\to\infty} x^{\epsilon_n}(t,\omega)$ uniformly on $[0,\beta_1]$. This proof is completed. $\square$

Acknowledgements

The authors would like to thank the referees and the respected editor for the important comments that lead to a greatly improved paper.

References

[1] W. Arendt, C.J.K. Batty, M. Hiber and F. Neubrander, *Vector-valued Laplace Transforms and Cauchy Problems*, Monographs in Mathematics, Vol. 96, Birkhäuser, Basel, 2011.

[2] M. Benchohra, J. Henderson, S.K. Ntouyas and A. Ouahab, *Existence results for fractional order functional differential equations with innite delay*, J. Math. Anal. Appl. 338 (2008) 1340–1350.

[3] A.T. Bharucha-Reid, *Random Integral Equations*, Academic Press, New York, 1972.

[4] B.C. Dhage, *Some algebraic and topological random xed point theorem with applications to nonlinear random intefralequations*, Tamkang J. Math. 35 (2004) 321–345.

[5] B.C. Dhage, *A random version of a Schaefer type xed point theorem with applications to functional random integral equations*, Tamkang J. Math. 35 (2004) 197–205.

[6] B.C. Dhage, *Monotone iterative technique for Carathéodory theorem of nonlinear functional random integral equations*, Tamkang J. Math. 35 (2002) 341–351.

[7] B.C. Dhage, S.V. Badgire and S.K. Ntouyas, *Periodic boundary value problems of second order random differential equations*, Electron. J. Qaul. Theory Diff. Equ. 21 (2009) 1–14.

[8] S.S. Dragomir, R.P. Agarwal and N.S. Barnett, *Inequalities for Beta and Gamma functions via some classical and new integral inequalities*, J. Inequal. Appl. 5 (2000) 103–165.

[9] R.M. Dudley, *Real analysis and probability*, Vol. 74 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2002.

[10] J. Deng and H. Qu, *New uniqueness results of solutions for fractional differential equations with innite delay*, Comput. Math. Appl. 60 (2010) 2253–2259.

[11] J. Deng and L. Ma, *Existence and uniqueness of solutions of initial value problems for nonlinear fractional differential equations*, Appl. Math. Lett. 23 (2010) 676–680.

[12] C.J. Himmelberg, *Measurable relations*, Fund. Math. 87 (1975) 53–72.

[13] X. Han, X. Ma and G. Dai, *solutions to fourth-order random differential equations with periodic boundary conditions*, Elect. J. Diff. Eq., Vol. 2012 (2012), No. 235, pp. 1–9.

[14] S. Itoh, *Random xed point theorems with applications to random differential equations in Banach spaces*, J. Math. Anal. Appl. 67 (1979) 261–273.

[15] A.A. Kilbas, H. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies*, Volume 204, 2006.

[16] O. Kallenberg, *Foundations of modern probability*, 2nd ed. Springer-Verlag, New York, 2002.

[17] V. Lakshmikantham and A.S. Vatsala, *Basic theory of fractional differential equations*, Nonlinear Anal. 69 (2008) 2677–2682.

[18] V. Lakshmikantham and A.S. Vatsala, *Theory of fractional differential inequalities and applications*, Commun. Appl. Anal., Vol-II (2007) 395–402.

[19] V. Lakshmikantham, *Theory of fractional functional differential equations*, Nonlinear Anal. 69 (2008) 3337–3343.

[20] V. Lupulescu and S.K. Ntouyas, *Random fractional differential equations*, Int. Electron. J. Pure Appl. Math. 4 (2012) 119–136.

[21] V. Lupulescu and C. Lungan, *Random integral equation on time scales*, Opuscula Math. 33 (2013) 323–335.

[22] V. Lupulescu and C. Lungan, *Random dynamical systems on time scales*, Electron. J. Diff. Equ., Vol. 2012 (2012), No. 86, pp. 1–14.

[23] V. Lupulescu and Donal O'Regan, *Ghaus ur Rahman Existence results for random fractional differential equations*, Opuscula Math. 34 (2014) 813–825.

[24] G.S. Ladde, V. Lakshmikantham, Random Differential Inequalities, Academic Press, New York (1980).

[25] T.T. Soong, *Random Dfferential Equations in Science and Engineering*, Academic Press, New York, 1973.

[26] C.P. Tsokos and W.J. Padgett, *Random Integral Equations with Applications to Life Sciences and Engineering*, Academic Press, New York, 1974.

[27] H.Vu, N.N. Phung and N. Phuong, *On fractional random differential equations with delay*, Opuscula Math. 36 (2016) 541–556.

[28] H. Ye, J. Gao and Y. Ding, *A generalized Gronwall inequality and its application to a fractional differential equation*, J. Math. Anal. Appl. 328 (2007) 1075–1081.