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Coefficient bounds for a new class of univalent functions involving Sălăgean operator and the modified sigmoid function

Olubunmi A. Fadipe-Joseph[a,*], W.T. Ademosu[b], G. Murugusundaramoorthy[c]

[a]*Department of Mathematics, University of Ilorin, P.M.B 1515, Ilorin, Nigeria*
[b]*Department of Mathematics, Statistics and Computer Sci., Federal University of Agriculture, P.M.B 2373, Makurdi, Nigeria*
[c]*Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Deemed to be University, Vellore–632 014, India*

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Abstract

We define a new subclass of univalent function based on Sălăgean differential operator and obtain the initial Taylor coefficients using the techniques of Briot–Bouquet differential subordination in association with the modified Sigmoid function. Further, we obtain the classical Fekete–Szegö inequality results.

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1. Introduction

The theory of special functions was overshadowed by many other field like real and functional analysis, topology, algebra and differential equations. The generalized hypergeometric functions plays a major role in geometric function theory after the proof of Bieberbach conjecture by de Branges. Special functions can be categorized into three, namely, Ramp function, threshold function, and

*Corresponding author
Email addresses: famelov@unilorin.edu.ng (Olubunmi A. Fadipe-Joseph), tinuadewuraola114@gmail.com (W.T. Ademosu), gmsmoorthy@yahoo.com (G. Murugusundaramoorthy)

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sigmoid function. The popular type among all is the sigmoid function because of its gradient descendent learning algorithm. It can be evaluated in different ways, most especially by truncated series expansion. The sigmoid function of the form

$$G(s) = \frac{1}{1+e^{-s}} \quad (s \in \mathbb{R}) \tag{1.1}$$

is useful because it is differentiable. Sigmoid function can be evaluated in different ways, it can be done by truncated series expansion [4]. The sigmoid function has very important properties, including the following

- It outputs real numbers between 0 and 1.
- It maps a very large input domain to a small range of outputs.
- It never loses information because it is a one–to–one function.
- It increases monotonically.

With all the properties mentioned in [4] sigmoid function is perfectly useful in geometric function theory.

Let $\mathcal{A}$ denote the class of function of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.2}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$. We say that $f(z)$ is subordinate to $g(z)$ denoted by $f(z) \prec g(z)$ if there exists a Schwarz function $\psi(z)$ where $|\psi(z)| < 1$ and $\psi(0) = 0$ such that

$$f(z) = g(\psi(z)) \quad (z \in \mathbb{U}).$$

Also, let $g(z)$ be univalent in $\mathbb{U}$. Then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$. Let $\mathcal{H} = \mathcal{H}(\mathbb{U})$ denote the class of functions analytic in $\mathbb{U}$. For n a positive integer and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}, f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\},$$

with $\mathcal{H}_\circ \equiv \mathcal{H}[0, 1]$. Let h be a univalent function in $\mathbb{U}$, with $h(0) = a$ and let $p \in \mathcal{H}[a, n]$ satisfying

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z),$$

where β and γ are complex numbers with $\beta \neq 0$. Then the first order differential subordination is called the "Briot–Bouquet differential subordination" and a differential equation of Briot–Bouquet type is given by

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z),$$

where $\beta > 0, \operatorname{Re} \gamma \geq 0$ and if q is the analytic solution of

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = \frac{1+z}{1-z},$$

then

$$\min_{|z|=r} \operatorname{Re} p(z) \geq \min_{|z|=r} \operatorname{Re} q(z),$$

see Hille [6, p. 403]. This particular differential subordination has many important applications in the theory of univalent functions and results concerning dominant and best dominant of the Briot–Bouquet differential subordination was studied by several authors [10, 13].

Theorem 1.1. (Miller and Mocanu, [9]) Let h be convex in $\mathbb{U}$ with $\operatorname{Re}[\beta p(z)+\gamma] > 0$. If p is analytic in $\mathbb{U}$ with $p(0) = h(0)$ each p satisfies then

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \Rightarrow p(z) \prec h(z).$$

A general integral operator of the form

$$\mathcal{I}[f(z)] = \left[\frac{\beta+\gamma}{z^{\gamma}\phi(z)} \int_0^z f^{\alpha}(t)\ t^{\delta-1}\varphi(t)dt\right]^{\frac{1}{\beta}},$$

was introduced by Miller et al [10] (also see [9], page 89) and special case of the integral existence theorem was also considered. Letting $\phi(z) \equiv \varphi \equiv 1$, $\alpha = \beta(\beta > 0)$ and $\delta = \gamma(\gamma \geq 0)$, we have

$$\mathcal{I}[f(z)] = \left[\frac{\beta+\gamma}{z^{\gamma}} \int_0^z f^{\beta}(t)\ t^{\gamma-1}dt\right]^{\frac{1}{\beta}} = z + A_{n+1}z^{n+1} + \cdots .$$

Suitably specializing the parameters, we state various integral operators as illustrated below:

For $\beta = 1$, we have the Bernardi operator [3]

$$\mathcal{B}[f(z)] = \frac{1+\gamma}{z^{\gamma}} \int_0^z f(t)t^{\gamma-1}dt$$

and for $\beta = 1, \gamma = 1$ we have the Libera operator [7] (also see [8])

$$\mathcal{L}[f(z)] = \frac{2}{z} \int_0^z f(t)dt.$$

Further by taking $\beta = 1$ and $\gamma = 0$ we have the Alexander Integral operator [2]

$$\mathcal{T}[f(z)] = \int_0^z \frac{f(t)}{t}dt.$$

There is an important connection between Briot–Bouquet differential equation and the integral operator $\mathcal{F} = \mathcal{I}_{\beta,\gamma}[f]$ which was studied by Miller et al., [10] and defined by

$$\mathcal{F}(z) = \mathcal{I}_{\beta,\gamma}[f(z)] = \left[\frac{\beta+\gamma}{z^{\gamma}} \int_0^z f^{\beta}(t)t^{\gamma-1}dt\right]^{n}.$$

Setting

$$p(z) = \frac{z\mathcal{F}'(z)}{\mathcal{F}(z)},$$

then, the Briot–Bouquet differential equation takes the form

$$p(z) + \frac{nzp'(z)}{\beta p(z) + \gamma} = \frac{zf'(z)}{f(z)}.$$

The relationship allows us to obtain subordination results about integral operators.

We recall the Sălăgean differential operator $\mathcal{D}^n f(z)$ given by Sălăgean [14] as below:

$$\begin{aligned}
&\mathcal{D}^0 f(z) = f(z), \\
&\mathcal{D}^1 f(z) = \mathcal{D} f(z) = zf'(z), \\
&\mathcal{D}^n f(z) = \mathcal{D}(\mathcal{D}^{n-1} f(z)) = z(\mathcal{D}^{n-1} f(z))' \quad n \in \mathbb{N} = \{1, 2, 3, \ldots\}, \\
&\mathcal{D}^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.
\end{aligned} \tag{1.3}$$

For $f(z) \in \mathcal{A}$ of the form (1.2), let $\mathcal{S}_n^*$, be the class of $n-$ starlike function [1], if

$$\mathrm{Re}\left(\frac{\mathcal{D}^{n+1} f(z)}{\mathcal{D}^n f(z)}\right) > 0,$$

where $\mathcal{D}^n f(z)$ be given by (1.3). It is of interest to note that $\mathcal{S}_0^* \equiv \mathcal{S}^*$ and $\mathcal{S}_1^* \equiv \mathcal{K}$ the class of starlike and convex functions respectively.

In this paper, we define a new subclass of univalent function based on Sălăgean differential operator and we obtain the initial Taylor coefficients using the techniques of subordination in association with the modified Sigmoid function. Further, we obtain the classical Fekete-Szegö inequality results.

2. Bounds for the class $\mathcal{S}^*$ based on the modified sigmoid function

Lemma 2.1. (Pommerenke, [12]) If $p \in \mathcal{P}$, then $|p_m| \leq 2$ for each m, where $\mathcal{P}$ is the family of all functions p analytic in $\mathbb{U}$ for which $\mathrm{Re}\,(p(z)) > 0$, where $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ for $z \in \mathbb{U}$.

The function class $\mathcal{P}$ is popularly known as Caratheodory function class.

Let $G(z)$ be a sigmoid function given by (1.1) and $\phi(z)$ be the modified sigmoid function as follows:

$$\phi(z) = \frac{2}{1 + e^{-z}} = 1 + \frac{1}{2}z - \frac{1}{24}z^3 + \frac{1}{240}z^5 - \frac{17}{40320}z^7 + \cdots \tag{2.1}$$

so that $\phi(0) = 1$ and $\mathrm{Re}\ \phi(z) > 0$. Furthermore, $\phi(z)$ is a modified Sigmoid function and belongs to class $\mathcal{P}$. The modified sigmoid function also have series representation of the form

$$\phi(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n\right)^m =: Q_{n,m}(z).$$

For details, see [4]. Let

$$p(z) = 1 + \sum_{n=1}^{\infty} c_m z^m$$

and

$$Q_{1,m}(z) := g(z) = 1 + \sum_{m=1}^{\infty} \frac{1}{2^m} z^m,$$

be analytic and univalent in the unit disk. Then, the convolution of $p(z)$ and $g(z)$ is denoted by

$$p(z) * g(z) = 1 + \sum_{m=1}^{\infty} \frac{c_m}{2^m} z^m =: h(z). \tag{2.2}$$

Theorem 2.2. *Let* $h(z) = 1 + \sum_{m=1}^{\infty} \frac{c_m}{2^m} z^m$*. Then* $h(z) \in \mathcal{P}$.

Proof . Suppose $h(z) = 1 + \sum_{m=1}^{\infty} \frac{c_m}{2^m} z^m$. It is clear that $h(0) = 1$. We now show that Re $(h(z)) > 0$. We have

$$\begin{aligned} \text{Re } (h(z)) &= \frac{1}{2}(h(z) + h(\bar{z})) \\ &= \frac{1}{2}\left(2 + \sum_{m=1}^{\infty} \frac{c_m}{2^m} 2 \text{ Re } z^m\right). \end{aligned}$$

Because $h(z)$ is analytic in the unit disk $\mathbb{U}$ thus, if we choose $z = \frac{1}{2}$, then we have

$$\text{Re } (h(z)) = 1 + \frac{c_1}{4} + \frac{c_2}{16} + \cdots .$$

Consequently

$$\text{Re } (h(z)) > 0,$$

hence, $h(z) \in \mathcal{P}$. $\square$

Since, $h(z) \in \mathcal{P}$, we state the following lemma without proof.

Lemma 2.3. Let $h(z)$ be defined by (2.2). Then $|c_m| \leq 2^{m+1}$.

Theorem 2.4. *If* $f \in \mathcal{S}_h^* = \{f \in \mathcal{A} : zf'/f(z) \prec h(z), z \in \mathbb{U}\}$ *and* $h(z)$ *be given by* (2.2), *then*

$$|a_2| \leq 2, \quad |a_3| \leq 3, \quad |a_4| \leq 4, \quad |a_5| \leq 5,$$

Proof . Let $f \in \mathcal{S}_h^*$. Then there exists a function $h(z) \in \mathcal{P}$ such that

$$zf'(z) = f(z)h(z).$$

By a simple computation, we have

$$\begin{aligned} & z + 2a_2z^2 + 3a_3z^3 + 4a_4z^4 + 5a_5z^5 + \cdots \\ &= z + \left(a_2 + \frac{c_1}{2}\right) z^2 + \left(a_3 + \frac{c_2}{4} + \frac{c_1a_2}{2}\right) z^3 \\ &\quad + \left(a_4 + \frac{c_3}{8} + \frac{c_2}{4}a_2 + \frac{c_1}{2}a_3\right) z^4 \\ &\quad + \left(a_5 + \frac{c_4}{16} + \frac{c_1a_4}{2} + \frac{c_2a_3}{4} + \frac{c_3a_2}{8}\right) z^5 + \cdots . \end{aligned}$$

Comparing coefficients of z^2, z^3, z^4 and z^5, we get

$$2a_2 = a_2 + \frac{c_1}{2} \tag{2.3}$$

$$3a_3 = a_3 + \frac{c_2}{4} + \frac{c_1}{2}a_2 \tag{2.4}$$

$$4a_4 = a_4 + \frac{c_3}{8} + \frac{c_2}{4}a_2 + \frac{c_1}{2}a_3 \tag{2.5}$$

$$5a_5 = a_5 + \frac{c_4}{16} + \frac{c_1a_4}{2} + \frac{c_2a_3}{4} + \frac{c_3a_2}{8}. \tag{2.6}$$

Therefore from (2.3)–(2.6), by using Lemma 2.3, we get the desired result. $\square$

Remark 2.5. Our results coincides with the results obtained in [5].

Theorem 2.6. *If* $f \in \mathcal{S}_h^* = \{f \in \mathcal{A} : zf'/f(z) \prec h(z), z \in \mathbb{U}\}$ *and* $h(z)$ *is given by* (2.2)*, then for any real number* $\lambda \in \mathbb{C}$

$$|a_3 - \lambda a_2^2| \leq \frac{1}{2}\left|\frac{c_2}{4} - \frac{c_1^2}{4}(2\lambda - 1)\right|$$

and

$$|a_2 a_4 - a_3^2| \leq \left|\frac{c_1 c_3}{48} - \frac{c_2^2}{64} - \frac{c_1^4}{192}\right|.$$

Proof . From the Theorem 2.4, we have the desired result. □

3. Bounds for the class $\mathcal{S}_n^*(\beta, \gamma, \phi)$

Now we recall the class of analytic functions of fractional power as follows:

From (1.2) we have $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{A}$ hence, for $\gamma > 0$ by Binomial expansion,

$$[f(z)]^\gamma = z^\gamma \left[1 + \gamma \sum_{k=2}^{\infty} a_k z^{k-1} + \frac{\gamma(\gamma-1)}{2!}\left(\sum_{k=2}^{\infty} a_k z^{k-1}\right)^2 + \frac{\gamma(\gamma-1)(\gamma-2)}{3!}\left(\sum_{k=2}^{\infty} a_k z^{k-1}\right)^3 + \cdots\right]. \quad (3.1)$$

By using (1.3), simple computation yields

$$\mathcal{D}^{n+1}[f(z)]^\gamma = z^\gamma + \gamma 2^{n+1} a_2 z^{\gamma+1} + \left(\gamma 3^{n+1} a_3 + \frac{\gamma(\gamma-1)}{2!} 2^{n+1} a_2^2\right) z^{\gamma+2} + \left(\gamma 4^{n+1} a_4 + \frac{\gamma(\gamma-1)(\gamma-2)}{3!} 2^{n+1} a_2^3 + \frac{\gamma(\gamma-1)}{2} 2 a_2 a_3 2^{n+1} 3^{n+1}\right) z^{\gamma+3} + \cdots. \quad (3.2)$$

Definition 3.1. For $\beta \in [0,1)$, $\gamma \geq 1$, we let $\mathcal{S}_n^*(\beta, \gamma)$, be the subclass of $\mathcal{A}$ if

$$\mathrm{Re}\left(\frac{\mathcal{D}^{n+1}[f(z)]^\gamma}{\mathcal{D}^n[f(z)]^\gamma}\right) > \beta \quad (z \in \mathbb{U}, n \in \mathbb{N}).$$

Remark 3.2. For $\beta = 0, \gamma = 1, f$ is a n–starlike functions class studied in [1].

Definition 3.3. Let $\gamma \geq 1$ and $\phi \in \mathcal{H}[1, n]$ be the sigmoid function of the form (2.1). We say that a function f belongs to the class $\mathcal{S}_n^*(\beta, \gamma, \phi)$, if f satisfies the following equation

$$\frac{\mathcal{D}^{n+1}[f(z)]^\gamma}{\mathcal{D}^n[f(z)]^\gamma} = \beta + (1-\beta)\phi(z) + \frac{(1-\beta)z\phi'(z)}{\beta + (1-\beta)\phi(z) + 1} \quad (0 \leq \beta < 1),$$

where $z \in \mathbb{U}$ and $n \in \mathbb{N}$.

The bound for the coefficients of the functions belonging to the class $\mathcal{S}_n^*(\beta, \gamma, \phi)$ are given in the following theorem.

Theorem 3.4. *Let $f \in \mathcal{S}_n^{\star}(\beta,\gamma,\phi)$ and $\phi(z) \in \mathcal{P}$. Then*

$$|a_2| \le (1-\beta)\left|\frac{1}{2\gamma A}\right|,$$

$$|a_3| \le (1-\beta)^2\left|\frac{\gamma^2 A' - 4\gamma^2 A - \gamma(\gamma-1)}{8\gamma^3 AB}\right|,$$

$$\begin{aligned}|a_4| \le \frac{(1-\beta)^3}{48\gamma^5 A^2 BC}\Big\{ &\left|3\gamma^3(\gamma-1)A'B - 3\gamma^2(\gamma-1)(\gamma A' - 4\gamma A - \gamma + 1)(A'B' - C')\right| \\ &+\left|\gamma^2(\gamma-1)(\gamma-2)B + 12\gamma^3 A^2 B + 12\gamma^3 AB\right| \\ &+\left|(\gamma^4 A' - 4\gamma^4 A - \gamma^3 + \gamma^2)AB'\right| \Big\} + \frac{1-\beta}{24\gamma C},\end{aligned}$$

where

$$A = 2^n, B = 3^n(2), C = 4^n(3) \text{ and } A' = 2^{n+1}, B' = 3^{n+1}, C' = 2^n 3^n,$$

unless otherwise stated.

Proof . Suppose $f \in \mathcal{S}_n^{\star}(\beta,\gamma,\phi)$ *and* $\gamma \ge 1$. By Definition 3.3, and since $\phi(z) \in \mathcal{P}$ we have

$$\frac{\mathcal{D}^{n+1}[f(z)]^\gamma}{\mathcal{D}^n[f(z)]^\gamma} = \beta + (1-\beta)\phi(z) + \frac{(1-\beta)z\phi'(z)}{\beta + (1-\beta)\phi(z) + 1}. \tag{3.3}$$

Also, from (2.1) we have

$$\beta + (1-\beta)\phi(z) = 1 + \frac{1-\beta}{2}z - \frac{1-\beta}{24}z^3 + \frac{1-\beta}{240}z^5 - \frac{17(1-\beta)}{40320}z^7 + \cdots .$$

Now by (1.2) we have $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ hence,

$$f(z)^\gamma = z^\gamma\left[1 + \gamma\sum_{k=2}^{\infty} a_k z^{k-1} + \frac{\gamma(\gamma-1)}{2!}\left(\sum_{k=2}^{\infty} a_k z^{k-1}\right)^2 + \frac{\gamma(\gamma-1)(\gamma-2)}{3!}\left(\sum_{k=2}^{\infty} a_k z^{k-1}\right)^3 + \cdots\right].$$

From (3.3), we have

$$\mathcal{D}^{n+1}[f(z)]^\gamma = \mathcal{D}^n[f(z)]^\gamma\left(\beta + (1-\beta)\phi(z) + \frac{(1-\beta)z\phi'(z)}{1+\beta+(1-\beta)\phi(z)}\right).$$

From (3.2), we have

$$\begin{aligned}\mathcal{D}^{n+1}[f(z)]^\gamma = z^\gamma + \gamma 2^{n+1}a_2 z^{\gamma+1} + \left(\gamma 3^{n+1}a_3 + \frac{\gamma(\gamma-1)}{2!}2^{n+1}a_2^2\right)z^{\gamma+2} \\ + \left(\gamma 4^{n+1}a_4 + \frac{\gamma(\gamma-1)(\gamma-2)}{3!}2^{n+1}a_2^3 + \frac{\gamma(\gamma-1)}{2}2a_2a_3 2^{n+1}3^{n+1}\right)z^{\gamma+3} + \cdots\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
&\mathcal{D}^n[f(z)]^\gamma\left(\beta+(1-\beta)\phi(z)+\frac{(1-\beta)z\phi'(z)}{1+\beta+(1-\beta)\phi(z)}\right)\\
&\quad= z^\gamma+\left(\gamma 2^n a_2+\frac{1-\beta}{2}\right)z^{\gamma+1}+\left(\gamma 3^n a_3+\frac{\gamma(\gamma-1)}{2!}2^n a_2^2+\frac{1-\beta}{2}\gamma 2^n a_2-\frac{(1-\beta)^2}{2}\right)z^{\gamma+2}\\
&\qquad+\left(\gamma 4^n a_4+\frac{\gamma(\gamma-1)(\gamma-2)}{3!}2^n a_2^3+\frac{1-\beta}{2}\gamma 3^n a_3+\frac{(1-\beta)}{2}\frac{\gamma(\gamma-1)}{2!}2^n a_2^2\right.\\
&\qquad\left.+2a_2a_3\frac{\gamma(\gamma-1)}{2!}2^n3^n-\frac{(1-\beta)^2}{2}\gamma 2^n a_2-\frac{(1-\beta)}{24}-\frac{(1-\beta)^3}{4}\right)z^{\gamma+3}. \qquad (3.5)
\end{aligned}$$

Equating the coefficients of $z^{\gamma+1}$, $z^{\gamma+2}$ and $z^{\gamma+3}$ in (3.4) and (3.5), we have

$$\gamma 2^{n+1}a_2=\gamma 2^n a_2+\frac{1-\beta}{2},$$

$$\gamma 3^{n+1}a_3+\frac{\gamma(\gamma-1)}{2!}2^{n+1}a_2^2=\gamma 3^n a_3+\frac{\gamma(\gamma-1)}{2!}2^n a_2^2+\frac{1-\beta}{2}\gamma 2^n a_2-\frac{(1-\beta)^2}{2},$$

$$\begin{aligned}
&\gamma 4^{n+1}a_4+\frac{\gamma(\gamma-1)(\gamma-2)}{3!}2^{n+1}a_2^3+\frac{\gamma(\gamma-1)}{2}2a_2a_3 2^{n+1}3^{n+1}=\\
&\qquad\gamma 4^n a_4+\frac{\gamma(\gamma-1)(\gamma-2)}{3!}2^n a_2^3+\frac{(1-\beta)}{2}\gamma 3^n a_3+\frac{\gamma(\gamma-1)}{2}2a_2a_3 2^n 3^n\\
&\qquad\quad+\frac{(1-\beta)}{2}\frac{\gamma(\gamma-1)}{2!}2^n a_2^2-\frac{(1-\beta)^2}{2}\gamma 2^n a_2-\frac{(1-\beta)}{24}-\frac{(1-\beta)^3}{4}.
\end{aligned}$$

Hence, by a simple computation, we get

$$\begin{aligned}
a_2&=\frac{1-\beta}{2\gamma(2^{n+1}-2^n)},\\
a_3&=(1-\beta)^2\left(\frac{-\gamma(\gamma-1)+\gamma^2 2^{n+1}-4\gamma^2(2^{n+}-2^n}{8\gamma^3(2^{n+1}-2^n)(3^{n+1}-3^n)}\right),
\end{aligned}$$

$$\begin{aligned}
a_4&=\frac{(1-\beta)^3}{\gamma(4^{n+1}-4^n)}\left(\frac{3\gamma^3(\gamma-1)2^{n+1}(3^{n+1}-3^n)}{48\gamma^3(2^{n+1}-2^n)^2(3^{n+1}-3^n)}\right)\\
&\quad+\frac{(1-\beta)^3}{\gamma(4^{n+1}-4^n)}\left(\frac{-3\gamma(\gamma-1)[(\gamma 2^{n+1}-14\gamma(2^{n+1}-2^n)-\gamma+1)(2^{n+1}3^{n+1}-2^n3^n)]}{48\gamma^3(2^{n+1}-2^n)^2(3^{n+1}-3^n)}\right)\\
&\quad+\frac{(1-\beta)^3}{\gamma(4^{n+1}-4^n)}\left(\frac{-12\gamma^3(2^{n+1}-2^n)^2(3^{n+1}-3^n)-12\gamma^3(2^{n+1}-2^n)(3^{n+1}-3^n)}{48\gamma^3(2^{n+1}-2^n)^2(3^{n+1}-3^n)}\right)\\
&\quad+\frac{(1-\beta)^3}{\gamma(4^{n+1}-4^n)}\left(\frac{[(\gamma^3 2^{n+1}-4\gamma^3(2^{n+1}-2^n)-\gamma^3+\gamma^2)](2^{n+1}-2^n)(3^{n+1})}{48\gamma^3(2^{n+1}-2^n)^2(3^{n+1}-3^n)}\right)\\
&\quad+\frac{(1-\beta)^3}{\gamma(4^{n+1}-4^n)}\left(\frac{-\gamma(\gamma-1)(\gamma-2)(3^{n+1}-3^n)}{48\gamma^3(2^{n+1}-2^n)^2(3^{n+1}-3^n)}\right)-\frac{(1-\beta)}{24\gamma(4^{n+1}-4^n)},
\end{aligned}$$

or equivalently

$$a_4 = \frac{(1-\beta)^3}{\gamma(4^{n+1}-4^n)[48\gamma^3(2^{n+1}-2^n)^2\ (3^{n+1}-3^n)]} \Big\{ \big(3\gamma^3(\gamma-1)2^{n+1}(3^{n+1}-3^n)\big)$$
$$+\big(-3\gamma(\gamma-1)[(\gamma 2^{n+1}-14\gamma(2^{n+1}-2^n)-\gamma+1)(2^{n+1}3^{n+1}-2^n3^n)]\big)$$
$$+\big(-12\gamma^3(2^{n+1}-2^n)^2(3^{n+1}-3^n)-12\gamma^3(2^{n+1}-2^n)(3^{n+1}-3^n)\big)$$
$$+\big([\gamma^3 2^{n+1}-4\gamma^3(2^{n+1}-2^n)-\gamma^3+\gamma^2](2^{n+1}-2^n)(3^{n+1})\big)$$
$$+\big(-\gamma(\gamma-1)(\gamma-2)(3^{n+1}-3^n)\big)\Big\} - \frac{(1-\beta)}{24\gamma(4^{n+1}-4^n)}.$$

Now, the assertions follow from the above equations. This is the end of proof. □

Setting $n = 0$ in Theorem 3.4 we state the following bounds for starlikeness:

Corollary 3.5. Let $f \in \mathcal{S}_0^*(\beta, \gamma, \phi)$. Then

$$|a_2| \le \frac{1-\beta}{2\gamma},$$
$$|a_3| \le (1-\beta)^2 \left|\frac{1-3\gamma}{16\gamma^2}\right|,$$
$$|a_4| \le (1-\beta)^3 \left|\frac{(2\gamma^3+63\gamma^2-11\gamma)}{288\gamma^4}\right| + \frac{1-\beta}{72\gamma}.$$

Also, the bounds for a function f which is n–starlike is as stated below:

Corollary 3.6. *Let $f \in \mathcal{S}_0^*(\beta, 1, \phi)$. Then*

$$|a_2| \le \frac{1-\beta}{2},$$
$$|a_3| \le \frac{(1-\beta)^2}{8},$$
$$|a_4| \le \frac{9(1-\beta)^3}{48} + \frac{(1-\beta)}{72}.$$

Proof . Setting $\gamma = 1$ in Corollary 3.5, we get the desired result. □

Remark 3.7. $|a_2|$ and $|a_3|$ agree with the result of Murugusundaramoorthy and Janani [11], but there is a shift on $|a_4|$ as a result of the turning of $f(z)$.

Similarly, we sate the bounds for convexity and n–convexity.

Corollary 3.8. *If $f \in \mathcal{S}_1^*(\beta, \gamma, \phi)$, then*

$$|a_2| \le \frac{1-\beta}{4\gamma},$$
$$|a_3| \le (1-\beta)^2 \left|\frac{1-5\gamma}{96\gamma^2}\right|,$$
$$|a_4| \le (1-\beta)^3 \left|\frac{6\gamma^3+576\gamma^2-78\gamma}{27684\gamma^4}\right| + \frac{1-\beta}{288\gamma}.$$

Proof . Setting $n = 1$ in Theorem 3.4, we get the desired result. □

Corollary 3.9. *Let $f \in \mathcal{S}_1^*(\beta, 1, \phi)$. Then*

$$|a_2| \leq \frac{1-\beta}{4},$$
$$|a_3| \leq \frac{(1-\beta)^2}{24},$$
$$|a_4| \leq \frac{7(1-\beta)^3}{384} + \frac{1-\beta}{288}.$$

Proof . Setting $\gamma = 1$ in Corollary 3.8, the result follows. □

In the following theorem, we obtain the sharp upper bounds of the Fekete-Szegö functional $|a_3 - \sigma a_2^2|$ and $|a_2a_4 - a_3^2|$ for the function class $\mathcal{S}_n^*(\beta, \gamma, \phi)$.

Theorem 3.10. *Let $f \in \mathcal{S}_n^*(\beta, \gamma, \phi)$ and $\sigma \in \mathbb{R}$. Then*

$$\left|a_3 - \sigma a_2^2\right| \leq \frac{(1-\beta)^2}{4\gamma^2 A^2}\left|\frac{(\gamma AA' - 4\gamma A^2 - \gamma A + A)}{2B} - \sigma\right|.$$

Proof . From the Theorem 3.4, we have

$$a_3 - \sigma a_2^2 = \frac{(1-\beta)^2\gamma A' - 4\gamma A - \gamma + 1}{8\gamma^2 AB} - \sigma\left(\frac{(1-\beta)}{(2\gamma A)}\right)^2.$$

Thus

$$|a_3 - \sigma a_2^2| \leq \frac{(1-\beta)^2}{4\gamma^2 A^2}\left|\frac{(\gamma A' - 4\gamma A - \gamma + 1)A}{2B} - \sigma\right|$$

and concluding the proof. □

Remark 3.11. If $f \in \mathcal{S}_n^*(\beta, \gamma, \phi)$, then for $\sigma = 1$, we have

$$\begin{aligned}|a_3 - a_2^2| &\leq \frac{(1-\beta)^2}{4\gamma^2 A^2}\left|\frac{(\gamma A' - 4\gamma A - \gamma + 1)A}{2B} - 1\right| \\ &\leq \frac{(1-\beta)^2}{4\gamma A^2}\left|\frac{(\gamma A' - 4\gamma A - \gamma + 1)A - 2B}{2B}\right|.\end{aligned}$$

Remark 3.12. If $f \in \mathcal{S}_0^*(\beta, \gamma, \phi)$, then for $\sigma = 1$, we have

$$|a_3 - a_2^2| \leq \frac{(1-\beta)^2}{4\gamma^2}\left|\frac{1-7\gamma}{4}\right|.$$

Setting $\gamma = 1$ in the above remark we have the result stated in [11].

Theorem 3.13. *If $f \in \mathcal{S}_n^*(\beta, \gamma, \phi)$, then*

$$|a_2a_4 - a_3^2| \leq \frac{(1-\beta)^2}{48\gamma^2 AC}\left|1 - (1-\beta)^2\frac{6\gamma^2(\gamma-1)A'B + 2\gamma(\gamma-1)(\gamma-2)B}{4\gamma^3 A^2 B}\right|$$

$$+\,((1-\beta)^2\left\{\left|\frac{6\gamma(\gamma-1)(A'B'-C')(\gamma A'-4\gamma A-\gamma+1)}{4\gamma^4A^2B}\right|\right.$$
$$+\left|\frac{24\gamma^3A^2B+24\gamma^3AB-2AB'(\gamma^3A'-4\gamma^3A-\gamma^3+\gamma^2)}{4\gamma^3A^2B}\right|$$
$$+\left|\frac{3\gamma^2[A'(A'-8A-2)+(8A+16A^2+1)]C}{3\gamma^2AB^2}\right|$$
$$\left.+\left|\frac{(6\gamma A'-24\gamma A-6\gamma+1)C}{3\gamma^2AB^2}\right|.\right\}$$

Proof . The proof follows from Theorem 3.4. □

4. Conclusion

The Bieberbach Conjecture and the Fekete–Szegö functional for a certain class of Caratheodory function were presented. Also, by appropriate selection of the values of n, γ, and β, bounds and functionals for a Briot–Bouquet type of differential equation in the space of modified sigmoid function were established.

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