Int. J. Nonlinear Anal. Appl. 12 (2021) No. 2, 1157-1161
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.22750.2411

On fixed point in C^*-algebra valued metric spaces using C_*-class function

Deepak Kumar[a], Dhariwal Rishi[b], Choonkil Park[c], Jung Rye Lee[d,*]

[a]*Department of Mathematics, Lovely Professional University, Phagwara, 144411, India*
[b]*Research Scholar, Lovely Professional University, Phagwara, 144411, India*
[c]*Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea*
[d]*Department of Data Science, Daejin University, Kyunggi 11159, Korea*

(Communicated by Madjid Eshaghi Gordji)

Abstract

In the present article, we prove some result using C_*-class function in the notion of C^*-algebra valued metric space which is more general than metric space. The obtained results extend and generalize some of the results in the literature.

Keywords: C^*-algebra valued metric space, C_*-class function, fixed point
2010 MSC: Primary 47H10, 54H25, 46J10, 46J15.

1. Introduction

A very useful and effective tool of fixed point theory is "Banach Contraction Principle [2]". This is widely used in today's world problems. It has many application in many field of pure as well as applied science. Many researcher has generalised the principle in various spaces. From the last few decade, several type of contractive mapping for existence and uniqueness of fixed point has investigated (see [1, 4, 7, 8, 9, 10, 11, 12, 14]). The principle is also gained a remarkable scope in differential equation, partial differential equation and integral equations (see [2, 13]).

Recently, Ma *et al.* [11] introduced the more generalised notion called C^*-algebra valued metric space by replacing real number with positive member of unital C^*-algebra in metric space. See [5] for more information on C^*-algebras. Later, many researcher discussed results in the frame work of C^*-algebra value metric space (see [6, 15, 16, 17]).

In the present manuscript, we prove some result using C_*-class function in the notion of C^*-algebra valued metric space which is more general than metric space.

*Corresponding author
Email addresses: deepakanand@live.in (Deepak Kumar), rishidhariwal51@gmail.com (Dhariwal Rishi), baak@hanyang.ac.kr (Choonkil Park), jrlee@daejin.ac.kr (Jung Rye Lee)

Received: February 2021 *Accepted:* June 2021

2. C^*-Algebra valued metric spaces

The following results and definitions are required in the sequel to prove the results in the main section.

Definition 2.1. *[11] Suppose X is a nonempty set. The mapping $d : X \times X \to \mathbb{A}$ is called a C^*-algebra valued metric on X if it satisfies :*

1. $d(j,k) \succeq \theta_{\mathbb{A}}$;
2. $d(j,k) = \theta_{\mathbb{A}}$ *if and only if* $j = k$;
3. $d(j,k) = d(k,j)$;
4. $d(j,l) \preceq d(j,k) + d(k,l)$ *for all* $j,k,l \in X$.

Then d is called a C^*-algebra metric on X and the triplet $(X, \mathbb{A}, d)$ is called a C^*-algebra valued metric space.

Remark 2.2. *In case, we take $\mathbb{A} = \mathbb{R}$ the notion of C^*-algebra valued metric space becomes equivalent to the definition of the real metric space.*

Definition 2.3. *[11] A sequence $\{t_n\}$ in $(X, \mathbb{A}, d)$ is said to be*

1. *convergent with respect to $\mathbb{A}$, if for given $\epsilon > 0$, there exists a positive integer m such that $\|d(t_n, t)\| < \epsilon$, for all $n > m$;*
2. *Cauchy sequence with respect to $\mathbb{A}$ if for any $\epsilon > 0$, there exists $m \in \mathbb{N}$ such that $\|d(t_n, t_k)\| < \epsilon$, for all $n, k > m$.*

The triplet $(X, \mathbb{A}, d)$ is called a complete C^*-algebra valued metric space if every Cauchy sequence with respect to $\mathbb{A}$ is convergent.

Definition 2.4. *(C_*- class function) [3] Let $\mathbb{A}$ be a unital C^*-algebra. Then a continuous function $F : \mathbb{A}_+ \times \mathbb{A}_+ \to \mathbb{A}_+$ is called a C_*-class function if for any $P, Q \in \mathbb{A}_+$, the following conditions hold:*

1. $F(P,Q) \preceq P$;
2. $F(P,Q) = P$ *implies that either* $P = \theta$ *or* $Q = \theta$.

Here C_* will denote the class of all C_*-class function. The extra condition should be imposed on F if required, i.e., $F(\theta,\theta) = \theta$.

Let Ψ be the set of all continuous functions $\psi : \mathbb{A}_+ \to \mathbb{A}_+$ satisfying the following conditions:

1. *ψ is continuous and nondecreasing;*
2. $\psi(T) = \theta$ *if and only if* $T = \theta$.

3. Main results

In this section, we prove some results using C_*-class function in the framework of C^*-algebra valued metric spaces.

Theorem 3.1. *Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra valued metric space and $T : X \to X$ be a self mapping satisfying the following*

$$\varphi\left(d(Tx,Ty)\right) \preceq F_*\Big(\varphi(d(x,y)), \phi(d(x,y))\Big), \quad \forall x, y \in X,$$

where $\phi, \varphi \in \Psi$ and $F_* \in C_*$. Then T has a unique fixed point.

Proof . Fix $x_0 \in X$. Define $x_n \in T^n(x_0)$ for every $n = 1, 2, 3....$ Then we shall prove that

$$d(x_n, x_{n+1}) \to \theta \quad \text{as} \quad n \to \infty.$$

We have

$$\begin{aligned} \varphi\left(d(x_n, x_{n+1})\right) &= \varphi\left(d(Tx_{n-1}, Tx_n)\right) \\ &\preceq F_*\Big(\varphi(d(x_{n-1}, x_n)), \phi(d(x_{n-1}, x_n))\Big) \\ &\preceq \varphi\left(d(x_{n-1}, x_n)\right). \end{aligned} \tag{3.1}$$

Therefore, we get

$$\varphi(d(x_n, x_{n+1})) \preceq \varphi(d(x_{n-1}, x_n)).$$

Hence φ is non decreasing and so the sequence $(d(x_{n-1}, x_n))$ is monotonically decreasing in $\mathbb{A}_+$. So there exists $\theta \preceq t \in \mathbb{A}_+$ such that

$$d(x_n, x_{n+1}) \to t \quad \text{as} \quad n \to \infty.$$

Letting $n \to \infty$ in (3.1) and by definition of F_* and continuity of φ, ϕ, we have

$$\varphi(t) \preceq F_*(\varphi(t), \phi(t)) \preceq \varphi(t).$$

Thus $F_*\big(\varphi(t), \phi(t)\big) = \varphi(t)$ and so $\varphi(t) = \theta$ or $\phi(t) = \theta$. Hence $t = \theta$. Therefore,

$$d(x_n, x_{n+1}) \to \theta \quad as \quad n \to \infty.$$

Now, we shall show that $\{x_n\}$ is a Cauchy sequence in $(X, \mathbb{A}, d)$. To prove it, we shall prove that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \theta. \tag{3.2}$$

Assume that $\{x_n\}$ is not a Cauchy in $(X, \mathbb{A}, d)$. Then there exist $\epsilon > 0$ and subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ with $n_k > m_k > k$ such that

$$\|d(x_{m_k}, x_{n_k})\| \geq \epsilon.$$

Now, corresponding to m_k, we can choose n_k such that it is the smallest integer with $n_k > m_k$ and satisfying above inequality. Hence

$$\|d(x_{m_k}, x_{n_k-1})\| < \epsilon.$$

So we have

$$\begin{aligned} \epsilon &\leq \|d(x_{m_k}, x_{n_k})\| \leq \|d(x_{m_k}, x_{n_k-1})\| + \|d(x_{n_k-1}, x_{n_k})\| \\ &\leq \epsilon + \|d(x_{n_k-1}, x_{n_k})\|. \end{aligned}$$

Using (3.2, we have

$$\epsilon \leq \lim_{k\to\infty} \|d(x_{m_k}, x_{n_K})\| < \epsilon + \theta.$$

This implies

$$\lim_{k\to\infty} \|d(x_{m_k}, x_{n_k})\| = \epsilon. \tag{3.3}$$

Again,

$$\begin{aligned} \|d(x_{n_k}, x_{m_k})\| &\leq \|d(x_{n_k}, x_{n_k-1})\| + \|d(x_{n_k-1}, x_{m_k})\| \\ &\leq \|d(x_{n_k}, x_{n_k-1})\| + \|d(x_{n_k-1}, x_{m_k-1})\| + \|d(x_{m_k-1}, x_{m_k})\|. \end{aligned} \tag{3.4}$$

Also,

$$\begin{aligned} \|d(x_{n_k-1}, x_{m_k-1})\| &\leq \|d(x_{n_k-1}, x_{n_k})\| + \|d(x_{n_k}, x_{m_k-1})\| \\ &\leq \|d(x_{n_k-1}, x_{n_k})\| + \|d(x_{n_k}, x_{m_k})\| + \|d(x_{m_k}, x_{m_k-1})\|. \end{aligned} \tag{3.5}$$

Letting $k \to \infty$ in (3.4) and (3.5) and using (3.2) and (3.3), we have

$$\lim_{k\to\infty} \|d(x_{n_k-1}, x_{m_k-1})\| = \epsilon.$$

Since $d(x_{n_k-1}, x_{m_k-1}), d(x_{n_k}, x_{m_k}) \in \mathbb{A}_+$ and

$$\lim_{k\to\infty} \|d(x_{n_k-1}, x_{m_k-1})\| = \lim_{k\to\infty} \|d(x_{n_k}, x_{m_k})\| = \epsilon,$$

there exists $s \in \mathbb{A}_+$ with $\|s\| = \epsilon$ such that

$$\lim_{k\to\infty} \|d(x_{n_k-1}, x_{m_k-1})\| = \lim_{k\to\infty} \|d(x_{n_k}, x_{m_k})\| = s. \tag{3.6}$$

Now by (3.6), we have

$$\varphi(s) = \lim_{k\to\infty} \varphi\left(d(x_{n_k}, x_{m_k})\right) \preceq \lim_{k\to\infty} F_*\Bigg(\varphi(d(x_{n_k-1}, x_{m_k-1})), \phi(d(x_{n_k-1}, x_{m_k-1}))\Bigg).$$

Therefore

$$\varphi(s) \preceq F_*\big(\varphi(s), \phi(s)\big) \preceq \varphi(s).$$

Thus $\varphi(s) = \theta$ or $\phi(s) = \theta$ and so $s = \theta$, which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence in $(X, \mathbb{A}, d)$. Hence there exist $z \in X$ such that

$$\lim_{n\to\infty} d(x_n, z) = \theta.$$

Now, we shall show that z is fixed point of T. Using (3.6), we get

$$\varphi(d(x_n, Tz)) \preceq F_*\Bigg(\varphi(d(x_{n-1}, z), \phi(d(x_{n-1}, z)))\Bigg).$$

Letting $n \to \infty$ and using the concept of continuity of the functions φ, ϕ and F_*, we have $d(z, Tz) = \theta$. Hence by Definition 2.1, we have $Tz = z$.

For the uniqueness, let $\alpha, \beta \in X$ be two fixed points of T. Using (3.6), we get

$$\varphi(d(\alpha, \beta)) = \varphi\left(d(T\alpha, T\beta)\right) \preceq F_*\Bigg(\varphi(d(\alpha, \beta)), \phi(d(\alpha, \beta))\Bigg) \preceq \varphi\left(d(\alpha, \beta)\right).$$

Hence $\varphi(d(\alpha, \beta)) = \theta$ or $\phi(d(\alpha, \beta)) = \theta$. Thus we get $d(\alpha, \beta) = \theta$. Hence by Definition 2.1, we have $\alpha = \beta$. This implies uniqueness. □

For $F_* = A - B$, we have the following result.

Corollary 3.2. *Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra valued metric space and $T : X \to X$ be a self mapping satisfying the following*

$$\varphi\left(d(Tx, Ty)\right) \preceq \varphi(d(x, y)) - \phi(d(x, y)) \quad \textit{forall} \quad x, y \in X;$$

where $\phi, \varphi \in \Psi$ and $F_* \in C_*$. Then T has a unique fixed point.

4. Conclusion

In this paper , we have proved some result using C_*-class function in the notion of C^*-algebra valued metric space which is more general than metric space. The obtained results extend and generalize some of the results in the literature.

Acknowledgements

Dedicated to the 70th Birthday of Professor Themistocles M. Rassias

References

[1] M. Asim and M. Imdad, *C^*-Algebra valued symmetric spaces and fixed point results with an application*, Korean J. Math. 28 (2010) 17–30.

[2] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math. 3 (1922, 133–181.

[3] S. Chandok, D. Kumar and C. Park, *C^*-Algebra-valued partial metric space and fixed point theorems*, Proc. Indian Acad. Sci. Math. Sci. 129 (2019) Paper No. 37.

[4] Y. Cho and N. Petrot, *Existence theorems for fixed fuzzy points with closed α-cut sets in complete metric spaces*, Comm. Korean Math. Soc. **26** (2011), 115–124.

[5] K. R. Davidson, C^*-Algebras by Example, Fields Institute Monographs **6**, Am. Math. Soc., Providence, 1996.

[6] X. Hao, L. Liu and Y. Wu, *Positive solutions for second order differential systems with non local conditions*, Fixed Point Theory **13** (2012), 507–516.

[7] S. Khalehoghli, H. Rahimi and M. Eshaghi Gordji, *Fixed point theorems in R-metric spaces with applications*, AIMS Math. **5** (2020), 3125–3137.

[8] M. Kumar, M. Imdad and M. Asim, *Some fixed point theorems under E.A. and (CLR) properties on C^*-algebra valued metric spaces*, Inf. Sci. Lett. **9** (2020), 75–82.

[9] X. Li and Z. Zhao, *On a fixed point theorem of mixed monotone operators and applications*, Electron. J. Qual. Theory Differ. Equ. **2011** (2011), Paper No.94.

[10] L. Liu, X. Zhang, J. Jiang and Y. Wu, *The unique solution of a class of sum mixed monotone operator equations and its application to fractional boundary value problems*, J. Nonlinear Sci. Appl. 9 (2016) 2943–2958.

[11] Z. Ma, L. Jiang and H. Sun, *C^*-Algebra valued metric spaces and related fixed point theorems*, Fixed Point Theory Appl. 2014 (2014) Paper No. 206.

[12] J. Mao, Z. Zhao and N. Xu, *The existence and uniqueness of positive solutions for integral boundary value problems*, Elect. J. Qual. Theory Differ. Equ. 2010 (2010) Paper No.16.

[13] G. J. Murphy, *C^*-Algebras and Operator Theory*, Academic Press, Boston, 1990.

[14] W. Sintunavarat and P. Kumam, *Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces*, J. Appl. Math. 2011 (2011) Article ID 637958.

[15] G. Wang, H.T. Che and H.B. Chen, *Feasibility-solvability theorems for generalized vector equilibrium problem in reflexive Banach spaces*, Fixed Point Theory Appl. 2012 Paper No. 38.

[16] T. Wang and R. Xu, *Bounds for some new integral inequalities with delay on time scales*, J. Math. Inequal. 6 (2012) 355–366.

[17] Q. Xin, L. Jiang and Z. Ma, *Common fixed point theorems in C^*-algebra valued metric spaces*, J. Nonlinear Sci. Appl. 9 (2016) 4617–4627.