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Solvability, continuous dependence and asymptotic expansion of solutions in a small parameter of Dirichlet problem for a nonlinear Kirchhoff wave equation

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Abstract

We study the existence, uniqueness, continuous dependence, and asymptotic expansion of solutions of the Dirichlet problem for a nonlinear Kirchhoff wave equation. At first, we state and prove a theorem involving the local existence and uniqueness of a weak solution. Next, we establish a sufficient condition to get an estimate of the continuous dependence of the solution with respect to the nonlinear terms. Finally, an asymptotic expansion of high order in a small parameter of a weak solution is also discussed.

Keywords: Faedo-Galerkin method, Linear recurrent sequence, Continuous dependence, Asymptotic expansion
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1 Introduction

In this paper, we study the following Dirichlet problem for a nonlinear Kirchhoff wave equation with strong damping and nonlinear memory

$$\begin{cases} u_{tt} - \lambda u_{txx} - \dfrac{\partial^2}{\partial x^2}\left[\mu\left(x, t, u(x,t), \|u(t)\|^2, \|u_x(t)\|^2\right)\right] = f(x,t),\ 0 < x < 1,\ 0 < t < T, \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \end{cases} \tag{1.1}$$

where $\lambda > 0$ is given constant; μ, f, $\tilde{u}_0$, $\tilde{u}_1$ are given functions; $\|u(t)\|^2 = \int_0^1 u^2(x,t)dx$, $\|u_x(t)\|^2 = \int_0^1 u_x^2(x,t)dx$.

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When $\mu := \mu_1\left(\|u_x(t)\|^2\right)u$, Prob. (1.1) is related to the Kirchhoff equation

$$\rho h u_{tt} = \left(P_0 + \frac{Eh}{2L}\int_0^L u_x^2(y,t)dy\right)u_{xx}, \tag{1.2}$$

presented by Kirchhoff in 1876 (see [14]). This equation is an extension of the classical D'Alembert wave equation which considers the effects of the changes in the length of the string during the vibrations. The parameters in (1.2) have the following meanings: u is the lateral deflection, L is the length of the string, h is the area of the cross - section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension.

The Kirchhoff equations of the form Eq. $(1.1)_1$ has been studied by many authors, for example, we refer to [1], [9], [10], [12]-[15], [17] - [21], [23]. By using different methods together with various techniques in functional analysis, several results concerning the existence/global existence and the properties of solutions of viscoelastic problems such as blow-up, decay, stability have been established.

For more details, there have been a lot of investigations dedicated to the following viscoelastic Kirchhoff equation

$$u_{tt} - M\left(\|\nabla u\|^2\right)\Delta u + \int_0^t g(t-s)\Delta u(x,s)ds - \lambda\Delta u_t + \gamma h(u_t) = \mathcal{F}(x,t,u), \tag{1.3}$$

where positive function M, the kernel g and the source $\mathcal{F}$ are C^1 functions satisfying some appropriate hypotheses, and h is a linear or nonlinear function of u_t.

When $g \equiv 0$ and M is not a constant function, the equation (1.3) without damping and the source terms is often called the Kirchhoff type equation; it was first introduced by Kirchhoff [14] in order to describe the nonlinear vibrations of an elastic string. In this regard, the existence and nonexistence of solutions have been discussed by many authors and references cited therein ([22], [24]-[26]).

On the contrary, when $g \neq 0$ and $M \equiv 1$, (1.3) becomes a semilinear viscoelastic wave equation. In [1], Cavalcanti et al. proved that, as $\lambda = 0,\ \gamma = 0,\ \mathcal{F} = 0$ and together with nonlinear boundary damping, the energy of solutions of the corresponding problem went uniformly to zero at infinity. In [20], Messaoudi considered Eq. (1.3) with $\lambda = 0$, $\gamma = 0$, $\mathcal{F} = |u|^{p-2}u$, and showed that, for certain class of relaxation functions and certain initial data, the solution energy decayed at a similar rate of decay of the relaxation function, which was not necessarily decaying in a polynomial or exponential fashion. In [19], Messaoudi studied Eq. (1.3) in case of $\lambda = 0,\ h = a|u_t|^{m-2}u_t,\ \mathcal{F} = b|u|^{p-2}u$, and proved a blow-up result for solutions with negative initial energy if $p > m$ and a global existence result for $p \leq m$. In the presence of the strong damping $-\Delta u_t$ and the linear damping u_t ($m = 2$), Li and He [15] proved the global existence of solutions and established a general decay rate estimate for the corresponding problem given by

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds - \Delta u_t + u_t = u|u|^{p-2}, \tag{1.4}$$

where the relaxation g is a C^1 function satisfying some suitable hypotheses.

There are few works devoted to the study of wave equations with nonlinear memory, for example, we can see [2], [3], [11], [23]. In [23], Ngoc et al. proved the local existence of the wave equation with strong damping and nonlinear viscoelastic term as follows

$$\begin{aligned} & u_{tt} - \lambda u_{xxt} - \frac{\partial}{\partial x}\left[\mu_1(x,t,u(x,t),\|u(t)\|^2,\|u_x(t)\|^2)u_x\right] + \int_0^t g(t-s)\frac{\partial}{\partial x}\left[\mu_2(x,s,u(x,s),\|u(s)\|^2,\|u_x(s)\|^2)u_x(x,s)\right]ds \\ & = F(x,t,u,u_x,u_t,\|u(t)\|^2,\|u_x(t)\|^2),\ 0 < x < 1,\ 0 < t < T, \end{aligned} \tag{1.5}$$

associated with Robin-Dirichlet boundary conditions and initial conditions, where $\lambda > 0$ is a constant, $\mu_1,\ \mu_2,\ g,\ f$ are given functions which satisfy some certain conditions. Moreover, the authors established an asymptotic expansion of solutions, i.e., the solutions of (1.5) can be approximated by a N-order polynomial in small parameter.

The topic of continuous dependence on datum has received important attention since 1960, with the earlier works of Douglis [4] and Fritz [8]. Recently, Quynh et al. [27] discussed the continuous dependence of solutions for a wave equation with a nonlinear memory term

$$\begin{cases} u_{tt} - \lambda u_{txx} - \dfrac{\partial^2}{\partial x^2}\left(\mu(x,t,u(x,t))\right) + \displaystyle\int_0^t g(t-s)\frac{\partial^2}{\partial x^2}\left(\bar{\mu}(x,s,u(x,s))\right)ds = f(x,t,u,u_t,u_x,u_{tx}),\ 0 < x < 1,\ 0 < t < T \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x). \end{cases} \tag{1.6}$$

The authors defined the continuous dependence of solutions in sense, if $u = u(\mu, \bar{\mu}, f, g)$ and $u_j = u(\mu_j, \bar{\mu}_j, f_j, g_j)$ are the solutions of Prob. (1.6) respectively depending on the datum $(\mu, \bar{\mu}, f, g)$ and $(\mu_j, \bar{\mu}_j, f_j, g_j)$, such that

$$\begin{cases} \sup\limits_{M>0} \max\limits_{|\beta|\leq 3} \left\|D^\beta \mu_j - D^\beta \mu\right\|_{C^0(A_M)} \to 0, \text{ as } j \to \infty, \\ \sup\limits_{M>0} \max\limits_{|\beta|\leq 3} \left\|D^\beta \bar{\mu}_j - D^\beta \bar{\mu}\right\|_{C^0(A_M)} \to 0, \text{ as } j \to \infty, \\ \sup\limits_{M>0} \max\limits_{|\alpha|\leq 1} \left\|D^\alpha f_j - D^\alpha f\right\|_{C^0(\tilde{A}_M)} \to 0, \text{ as } j \to \infty, \\ \|g_j - g\|_{H^1(0,T^*)} \to 0, \text{ as } j \to \infty, \end{cases} \tag{1.7}$$

where T^* is fixed positive constant; A_M, $\tilde{A}_M$ are compact sets depending on a positive constant M; $D^\alpha f$ are partial derivatives with order less than or equal $|\alpha|$, then u_j converges to u in

$$W_1(T) = \left\{u \in C^0([0,T]; H_0^1) \cap C^1([0,T]; L^2) : u' \in L^2(0,T; H_0^1)\right\} \; as \; j \to \infty.$$

Motivated by the above-mentioned inspiring works, in this paper, we consider Prob. (1.1) and we first prove existence, uniqueness of solutions for this problem (Theorem 3.4) by applying the linearization method together with Faedo-Galerkin method and the weak compact method. Next, we consider the continuous dependence of solutions on the nonlinearities of Prob. (1.1). Precisely, if $u = u(\mu)$ and $u_j = u(\mu_j)$ are the solutions of Prob. (1.1) respectively depending on the datum μ and μ_j, such that

$$D_1(\mu_j, \mu) \equiv \sup_{M>0} \|\mu_j - \mu\|_{C^3(A_M)} \to 0, \tag{1.8}$$

where A_M is compact set depending on a positive constant M; then u_j converges to u in

$$W_1(T) = \left\{u \in C^0([0,T]; H_0^1) \cap C^1([0,T]; L^2) : u' \in L^2(0,T; H_0^1)\right\} \; as \, j \to \infty.$$

Finally, we consider the following perturbed problem in a small parameter ε

$$(P_\varepsilon) \begin{cases} u_{tt} - \lambda u_{txx} - \dfrac{\partial^2}{\partial x^2}\left(\mu_\varepsilon[u](x,t)\right) = f(x,t), \; 0 < x < 1, \; 0 < t < T, \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x), \; u_t(x,0) = \tilde{u}_1(x), \end{cases} \tag{1.9}$$

with

$$\mu_\varepsilon[u](x,t) \quad = \quad \mu\left(x, t, u(x,t), \|u(t)\|^2, \|u_x(t)\|^2\right) + \varepsilon \mu_1\left(x, t, u(x,t), \|u(t)\|^2, \|u_x(t)\|^2\right).$$

We shall establish an asymptotic expansion of high order of the solution $u_\varepsilon(x,t)$ of Prob. (P_ε) with respect to a small parameter ε, in which $u_\varepsilon(x,t)$ is approximated by the polynomial of N degree in a small parameter ε (Theorem 5.6).

2 Preliminaries

In this section, we present some notations and materials in order to present main results. Let $\Omega = (0,1)$, $Q_T = (0,1) \times (0,T)$ and we define the scalar product in L^2 by

$$\langle u, v\rangle = \int_0^1 u(x)v(x)dx,$$

and the corresponding norm $\|\cdot\|$, i.e., $\|u\|^2 = \langle u, u\rangle$. Let us denote the standard function spaces by $C^m(\overline{\Omega})$, $L^p = L^p(\Omega)$ and $H^m = H^m(\Omega)$ for $1 \leq p \leq \infty$ and $m \in \mathbb{N}$. Also, we denote that $\|\cdot\|_X$ is a norm in a Banach space X, and $L^p(0,T;X)$, $1 \leq p \leq \infty$, is the Banach space of real functions $u : (0,T) \to X$ measurable with the corresponding norm $\|\cdot\|_{L^p(0,T;X)}$ defined by

$$\|u\|_{L^p(0,T;X)} = \begin{cases} \left(\displaystyle\int_0^T \|u(t)\|_X^p \, dt\right)^{1/p} < \infty, & \text{for } 1 \leq p < \infty, \\ ess\sup\limits_{0<t<T} \|u(t)\|_X \,, & \text{for } p = \infty. \end{cases}$$

On H^1, we use the following norm

$$\|v\|_{H^1} = \left(\|v\|^2 + \|v_x\|^2\right)^{1/2}. \tag{2.1}$$

The following lemma is known.

Lemma 2.1. [16] *The imbeddings $H^1 \hookrightarrow C^0([0,1])$ and $H_0^1 \hookrightarrow C^0([0,1])$ are compact and*

$$\begin{aligned} &\text{(i)} \quad \|v\|_{C^0([0,1])} \leq \sqrt{2}\,\|v\|_{H^1} \ \text{for all } v \in H^1, \\ &\text{(ii)} \quad \|v\|_{C^0([0,1])} \leq \|v_x\| \ \text{for all } v \in H_0^1, \end{aligned} \tag{2.2}$$

where $H_0^1 = \left\{v \in H^1 : v(0) = v(1) = 0\right\}$.

Remark 2.2. By (2.1) and (2.2), it is easy to prove that, on H_0^1, the two norms $v \mapsto \|v\|_{H^1}$ and $v \mapsto \|v_x\|$ are equivalent.

Throughout this paper, we write $u(t)$, $u'(t) = u_t(t) = \dot{u}(t)$, $u''(t) = u_{tt}(t) = \ddot{u}(t)$, $u_x(t) = \bigtriangledown u(t)$, $u_{xx}(t) = \Delta u(t)$, to denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$, $\frac{\partial^2 u}{\partial x^2}(x,t)$, respectively.

With $\mu \in C^k([0,1] \times [0,T^*] \times \mathbb{R} \times \mathbb{R}_+^2)$, $\mu = \mu(x,t,y,z_1,z_2)$, we define $D_1\mu = \frac{\partial \mu}{\partial x}$, $D_2\mu = \frac{\partial \mu}{\partial t}$, $D_3\mu = \frac{\partial \mu}{\partial y}$, $D_{3+i}\mu = \frac{\partial \mu}{\partial z_i}$, $i = 1,2$ and $D^\beta \mu = D_1^{\beta_1} \cdots D_5^{\beta_5}\mu$, $\beta = (\beta_1, \cdots, \beta_5) \in \mathbb{Z}_+^5$, $|\beta| = \beta_1 + \cdots + \beta_5 \leq k$; $D^{(0,\cdots,0)}\mu = \mu$.

Similarly, with $f \in C^1([0,1] \times [0,T^*])$, $f = f(x,t)$, we define $D_1 f = \frac{\partial f}{\partial x}$, $D_2 f = \frac{\partial f}{\partial t}$ and $D^\alpha f = D_1^{\alpha_1} D_2^{\alpha_2} f$; $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$, $|\alpha| = \alpha_1 + \alpha_2 \leq 1$; $D^{(0,0)} f = f$.

3 Local existence and uniqueness

In this section, we consider the local existence and uniqueness of Prob. (1.1). By using the linearization method together with Faedo-Galerkin method, we prove that there exists a recurrent sequence which converges to the weak solution of (1.1). Let $T^* > 0$, we make the following assumptions:

(H_1) $\tilde{u}_0, \tilde{u}_1 \in H^2 \cap H_0^1$;

(H_2) $\mu \in C^3([0,1] \times [0,T^*] \times \mathbb{R} \times \mathbb{R}_+^2)$ and $D_3\mu(x,t,y,z_1,z_2) \geq \mu_* > 0$, for all $(x,t,y,z_1,z_2) \in [0,1] \times [0,T^*] \times \mathbb{R} \times \mathbb{R}_+^2$;

(H_3) $f \in C^1([0,1] \times [0,T^*])$, such that $f(0,t) = f(1,t) = 0$, for all $t \in [0,T^*]$.

A function u is called a weak solution of the initial-boundary value problem (1.1) if

$$u \in W_T = \{u \in L^\infty(0,T;H^2 \cap H_0^1) : u' \in L^\infty(0,T;H^2 \cap H_0^1), u'' \in L^2(0,T;H_0^1) \cap L^\infty(0,T;L^2)\},$$

and u satisfies the variational equation

$$\langle u''(t), v\rangle + \lambda\langle u_x'(t), v_x\rangle + a(t;u(t),v) = \langle f(t), v\rangle, \tag{3.1}$$

for all $v \in H_0^1$, a.e. $t \in (0,T)$, together with the initial conditions

$$u(0) = \tilde{u}_0, \ u'(0) = \tilde{u}_1, \tag{3.2}$$

where

$$\begin{cases} a(t;u(t),v) = \left\langle \dfrac{\partial \mu[u]}{\partial x}(t), v_x \right\rangle = \langle D_1\mu[u](t) + D_3\mu[u](t)u_x(t), v_x\rangle, \\ \mu[u](x,t) = \mu\left(x,t,u(x,t), \|u(t)\|^2, \|u_x(t)\|^2\right), \\ D_j\mu[u](x,t) = D_j\mu\left(x,t,u(x,t), \|u(t)\|^2, \|u_x(t)\|^2\right), \ j = 1,3. \end{cases} \tag{3.3}$$

Let $T^* > 0$ be fixed. For $M > 0$, we put

$$\begin{cases} K_M(\mu) = \|\mu\|_{C^3(A_M)} = \max_{|\beta|\leq 3} \left\|D^\beta \mu\right\|_{C^0(A_M)}, \\ \tilde{K}(f) = \|f\|_{C^1(\bar{Q}_{T^*})} = \max_{|\alpha|\leq 1} \|D^\alpha f\|_{C^0(\bar{Q}_{T^*})}, \\ \|\mu\|_{C^0(A_M)} = \sup_{(x,t,y,z_1,z_2)\in A_M} |\mu(x,t,y,z_1,z_2)|, \\ \|f\|_{C^0(\bar{Q}_{T^*})} = \sup_{(x,t)\in \bar{Q}_{T^*}} |f(x,t)|, \end{cases} \tag{3.4}$$

where $A_M = [0,1] \times [0,T^*] \times [-M,M] \times [0,M^2]^2$, $\bar{Q}_{T^*} = [0,1] \times [0,T^*]$. For any $T \in (0,T^*]$, we consider the set

$$V_T = \{v \in L^\infty(0,T;H^2 \cap H_0^1) : v' \in L^\infty(0,T;H^2 \cap H_0^1),\ v'' \in L^2(0,T;H_0^1)\}, \tag{3.5}$$

then V_T is a Banach space with respect to the norm (see Lions [16])

$$\|v\|_{V_T} = \max\{\|v\|_{L^\infty(0,T;H^2\cap H_0^1)},\ \|v'\|_{L^\infty(0,T;H^2\cap H_0^1)},\ \|v''\|_{L^2(0,T;H_0^1)}\}. \tag{3.6}$$

Also, we define the sets

$$\begin{cases} W(M,T) = \{v \in V_T : \|v\|_{V_T} \leq M\}, \\ W_1(M,T) = \{v \in W(M,T) : v'' \in L^\infty(0,T;L^2)\}. \end{cases} \tag{3.7}$$

In the following, we shall establish a linear recurrent sequence $\{u_m\}$ by choosing the first iteration $u_0 \equiv \tilde{u}_0$, and suppose that

$$u_{m-1} \in W_1(M,T), \tag{3.8}$$

then we shall find u_m in $W_1(M,T)$ satisfing the following problem

$$\begin{cases} \langle u_m''(t), v\rangle + \lambda\langle u_{mx}'(t), v_x\rangle + a_m(t; u_m(t), v) = \langle f(t), v\rangle,\ \forall v \in H_0^1, \\ u_m(0) = \tilde{u}_0,\ u_m'(0) = \tilde{u}_1, \end{cases} \tag{3.9}$$

in which

$$a_m(t;u,v) = \langle D_1\mu[u_{m-1}](t) + D_3\mu[u_{m-1}](t)u_x, v_x\rangle,\ u,v \in H_0^1. \tag{3.10}$$

Note that $a_m(t;u,v)$ can be rewritten in form of

$$a_m(t;u,v) = A_m(t;u,v) + \langle \mu_{1m}(t), v_x\rangle,\ u,v \in H_0^1, \tag{3.11}$$

where

$$\begin{aligned} A_m(t;u,v) &= \langle \mu_{3m}(t)u_x, v_x\rangle,\ u,v \in H_0^1, \\ \mu_{jm}(x,t) &= D_j\mu[u_{m-1}](x,t) \\ &= D_j\mu\left(x,t,u_{m-1}(x,t), \|u_{m-1}(t)\|^2, \|\nabla u_{m-1}(t)\|^2\right),\ j = 1,3. \end{aligned} \tag{3.12}$$

Then, Prob. (3.9) is equivalent to

$$\begin{cases} \langle u_m''(t), v\rangle + \lambda\langle u_{mx}'(t), v_x\rangle + A_m(t; u_m(t), v) = \left\langle \tilde{F}_m(t), v\right\rangle,\ \forall v \in H_0^1, \\ u_m(0) = \tilde{u}_0,\ u_m'(0) = \tilde{u}_1, \end{cases} \tag{3.13}$$

where, $\tilde{F}_m(t) : H_0^1 \to \mathbb{R}$ is a linear continuous functional on H_0^1, which is defined by

$$\left\langle \tilde{F}_m(t), v\right\rangle = \langle f(t), v\rangle - \langle \mu_{1m}(t), v_x\rangle,\ v \in H_0^1,\ 0 \leq t \leq T. \tag{3.14}$$

The existence of u_m is assured by the following theorem.

Theorem 3.1. *Under assumptions $(H_1) - (H_3)$, there exist positive constants M, T such that, for $u_0 \equiv \tilde{u}_0$, there exists a recurrent sequence $\{u_m\} \subset W_1(M,T)$ defined by* (3.8), (3.13) *and* (3.14).

Proof . The proof consists of several steps.

Step 1. The Galerkin approximation. Consider a special orthonormal basis $\{w_j\}$ on $H_0^1 : w_j(x) = \sqrt{2}\sin(j\pi x)$, $j \in \mathbb{N}$, formed by the eigenfunctions of the Laplacian $-\Delta = -\frac{\partial^2}{\partial x^2}$. Put

$$u_m^{(k)}(t) = \sum_{j=1}^{k} c_{mj}^{(k)}(t) w_j, \tag{3.15}$$

where the coefficients $c_{mj}^{(k)}$ satisfy the following system of linear integrodifferential equations

$$\begin{cases} \langle \ddot{u}_m^{(k)}(t), w_j\rangle + \lambda \langle \dot{u}_{mx}^{(k)}(t), w_{jx}\rangle + A_m(t; u_m^{(k)}(t), w_j) = \left\langle \tilde{F}_m(t), w_j \right\rangle, \ 1 \le j \le k, \\ u_m^{(k)}(0) = \tilde{u}_{0k}, \ \dot{u}_m^{(k)}(0) = \tilde{u}_{1k}, \end{cases} \tag{3.16}$$

in which

$$\begin{cases} \tilde{u}_{0k} = \sum_{j=1}^{k} \alpha_j^{(k)} w_j \to \tilde{u}_0 \text{ strongly in } H^2 \cap H_0^1, \\ \tilde{u}_{1k} = \sum_{j=1}^{k} \beta_j^{(k)} w_j \to \tilde{u}_1 \text{ strongly in } H^2 \cap H_0^1. \end{cases} \tag{3.17}$$

The system (3.16) is equivalent to system of linear intergal equations which can be rewritten in the following form

$$c_m^{(k)} = U[c_m^{(k)}], \tag{3.18}$$

where

$$\begin{cases} c_m^{(k)} = (c_{m1}^{(k)}, \cdots, c_{mk}^{(k)}), \\ U[c_m^{(k)}] = (U_1[c_m^{(k)}], \cdots, U_k[c_m^{(k)}]), \\ U_j[c_m^{(k)}](t) = \mathcal{F}_j^{(k)}[c_m^{(k)}](t) + G_j^{(k)}(t), \\ \mathcal{F}_j^{(k)}[c_m^{(k)}](t) = -\sum_{i=1}^{k} \int_0^t d\tau \int_0^\tau e^{-\lambda\lambda_j(\tau - s)} A_{mij}(s) c_{mi}^{(k)}(s) ds, \\ G_j^{(k)}(t) = \alpha_j^{(k)} + \frac{\beta_j^{(k)}}{\lambda\lambda_j}\left(1 - e^{-\lambda\lambda_j t}\right) + \int_0^t d\tau \int_0^\tau e^{-\lambda\lambda_j(\tau - s)} \left\langle \tilde{F}_m(s), w_j \right\rangle ds, \\ \lambda_j = (j\pi)^2, \ A_{mij}(t) = A_m(t; w_i, w_j), \ i, j = 1, \cdots, k, \ 0 \le t \le T. \end{cases} \tag{3.19}$$

Using Banach's contraction principle, it is not difficult to prove that the above fixed point equation admits a unique solution $c_m^{(k)} \in C([0,T]; \mathbb{R}^k)$, so let us omit the details.

Step 2. A priori estimate. Put

$$\begin{aligned} S_m^{(k)}(t) = {} & \left\| \dot{u}_m^{(k)}(t) \right\|^2 + \left\| \dot{u}_{mx}^{(k)}(t) \right\|^2 + \left\| \sqrt{\mu_{3m}(t)} u_{mx}^{(k)}(t) \right\|^2 + \left\| \sqrt{\mu_{3m}(t)} \Delta u_m^{(k)}(t) \right\|^2 + \lambda \left\| \Delta \dot{u}_m^{(k)}(t) \right\|^2 \\ & + 2\lambda \int_0^t \left(\left\| \dot{u}_{mx}^{(k)}(s) \right\|^2 + \left\| \Delta \dot{u}_m^{(k)}(s) \right\|^2 \right) ds + 2 \int_0^t \left\| \ddot{u}_{mx}^{(k)}(s) \right\|^2 ds, \end{aligned} \tag{3.20}$$

then it follows from (3.16) that

$$\begin{aligned} S_m^{(k)}(t) = {} & S_m^{(k)}(0) + 2\left\langle \mu_{3mx}(0)\tilde{u}_{0kx}, \Delta\tilde{u}_{0k} \right\rangle + 2\left\langle \frac{\partial}{\partial x}\left(\mu_{3m}(0)\tilde{u}_{0kx}\right), \Delta\tilde{u}_{1kx} \right\rangle \\ & + \int_0^t ds \int_0^1 \mu_{3m}'(x,s) \left(\left| u_{mx}^{(k)}(x,s) \right|^2 + \left| \Delta u_m^{(k)}(x,s) \right|^2 \right) dx \\ & + 2\int_0^t \left\langle \frac{\partial}{\partial s}\left[\mu_{3mx}(s) u_{mx}^{(k)}(s) \right], \Delta u_m^{(k)}(s) + \Delta \dot{u}_m^{(k)}(s) \right\rangle ds + 2\int_0^t \left\langle \frac{\partial}{\partial s}\left[\mu_{3m}(s) \Delta u_m^{(k)}(s) \right], \Delta \dot{u}_m^{(k)}(s) \right\rangle ds \end{aligned} \tag{3.21}$$

$$
\begin{aligned}
& - 2 \left\langle \mu_{3mx}(t) u_{mx}^{(k)}(t), \Delta u_m^{(k)}(t) \right\rangle - 2 \left\langle \frac{\partial}{\partial x} \left(\mu_{3m}(t) u_{mx}^{(k)}(t) \right), \Delta \dot{u}_m^{(k)}(t) \right\rangle \\
& + 2 \int_0^t \left\langle \tilde{F}_m(s), \dot{u}_m^{(k)}(s) \right\rangle ds + 2 \int_0^t \left\langle \tilde{F}_m(s), -\Delta \dot{u}_m^{(k)}(s) \right\rangle ds + 2 \int_0^t \left\langle \tilde{F}_m(s), -\Delta \ddot{u}_m^{(k)}(s) \right\rangle ds \\
= & S_m^{(k)}(0) + 2 \left\langle \mu_{3mx}(0) \tilde{u}_{0kx}, \Delta \tilde{u}_{0k} \right\rangle + 2 \left\langle \frac{\partial}{\partial x} \left(\mu_{3m}(0) \tilde{u}_{0kx} \right), \Delta \tilde{u}_{1kx} \right\rangle + \sum_{i=1}^{8} J_i.
\end{aligned}
$$

We shall estimate the terms J_i on the right-hand side of (3.21) as follows.

First, we need the following lemma whose proof is easy, hence we omit the details.

Lemma 3.2. *Put* $\bar{\mu}_* = \min\{1, \mu_*, \lambda\}$ *and*

$$
\bar{S}_m^{(k)}(t) = \left\| u_m^{(k)}(t) \right\|_{H^2 \cap H_0^1}^2 + \left\| \dot{u}_m^{(k)}(t) \right\|_{H^2 \cap H_0^1}^2 + \int_0^t \left(\left\| \dot{u}_m^{(k)}(s) \right\|_{H^2 \cap H_0^1}^2 + \left\| \ddot{u}_{mx}^{(k)}(s) \right\|^2 \right) ds. \tag{3.22}
$$

Then, the following estimations are admitted

(i) $S_m^{(k)}(t) \geq \bar{\mu}_* \bar{S}_m^{(k)}(t)$;

(ii) $|\mu'_{im}(x,t)| \leq \left(1 + M + 4M^2\right) K_M(\mu)$, $i = 1, 3$;

(iii) $|\mu_{imx}(x,t)| \leq (1 + 2M) K_M(\mu)$, $i = 1, 3$;

(iv) $|\mu'_{imx}(x,t)| \leq \left(1 + 5M + 6M^2 + 8M^3\right) K_M(\mu)$, $i = 1, 3$;

(v) $\left\| \Delta u_m^{(k)}(t) + \Delta \dot{u}_m^{(k)}(t) \right\| \leq \sqrt{2} \sqrt{\bar{S}_m^{(k)}(t)}$;

(vi) $\left\| \frac{\partial}{\partial x} (\mu_{3m}(t) u_{mx}^{(k)}(t)) \right\| \leq \sqrt{2}(1 + M) K_M(\mu) \sqrt{\bar{S}_m^{(k)}(t)}$;

(vii) $\left\| \frac{\partial}{\partial t} \left[\mu_{3mx}(t) u_{mx}^{(k)}(t) \right] \right\| \leq \left(2 + 7M + 6M^2 + 8M^3\right) K_M(\mu) \sqrt{\bar{S}_m^{(k)}(t)}$;

(viii) $\left\| \frac{\partial}{\partial t} \left[\mu_{3m}(t) \Delta u_m^{(k)}(t) \right] \right\| \leq \left(2 + M + 4M^2\right) K_M(\mu) \sqrt{\bar{S}_m^{(k)}(t)}$;

(ix) $\left\| \frac{\partial^2}{\partial x \partial t} \left(\mu_{3m}(t) u_{mx}^{(k)}(t) \right) \right\| \leq \left(4 + 8M + 10M^2 + 8M^3\right) K_M(\mu) \sqrt{\bar{S}_m^{(k)}(t)}$.

By Lemma 3.2, the terms $J_1 - J_5$ on the right-hand side of (3.21) are estimated as follows.

Using Lemma 3.2 ((ii), (ix), (vi), (vii)), then the terms $J_1 - J_3$ are respectively estimated by

$$
\begin{aligned}
J_1 &= \int_0^t ds \int_0^1 \mu'_{3m}(x,s) \left(\left| u_{mx}^{(k)}(x,s) \right|^2 + \left| \Delta u_m^{(k)}(x,s) \right|^2 \right) dx \\
&\leq \left(1 + M + 4M^2\right) K_M(\mu) \int_0^t \bar{S}_m^{(k)}(s) ds \equiv \bar{C}_1(M) \int_0^t \bar{S}_m^{(k)}(s) ds; \\
J_2 &= 2 \int_0^t \left\langle \frac{\partial}{\partial s} \left[\mu_{3mx}(s) u_{mx}^{(k)}(s) \right], \Delta u_m^{(k)}(s) + \Delta \dot{u}_m^{(k)}(s) \right\rangle ds \\
&\leq 2 \int_0^t \left\| \frac{\partial}{\partial s} \left[\mu_{3mx}(s) u_{mx}^{(k)}(s) \right] \right\| \left\| \Delta u_m^{(k)}(s) + \Delta \dot{u}_m^{(k)}(s) \right\| ds \\
&\leq 2\sqrt{2} \left(2 + 7M + 6M^2 + 8M^3\right) K_M(\mu) \int_0^t \bar{S}_m^{(k)}(s) ds \equiv \bar{C}_2(M) \int_0^t \bar{S}_m^{(k)}(s) ds; \\
J_3 &= 2 \int_0^t \left\langle \frac{\partial}{\partial s} \left[\mu_{3m}(s) \Delta u_m^{(k)}(s) \right], \Delta \dot{u}_m^{(k)}(s) \right\rangle ds \\
&\leq 2 \left(2 + M + 4M^2\right) K_M(\mu) \int_0^t \bar{S}_m^{(k)}(s) ds \equiv \bar{C}_3(M) \int_0^t \bar{S}_m^{(k)}(s) ds.
\end{aligned} \tag{3.23}
$$

In order to estimate the terms J_4 and J_5, we need the following lemma whose proof is easy, hence we omit the details.

Lemma 3.3. *We have the following inequalities*

(i) $\left\|\mu_{3mx}(t)u_{mx}^{(k)}(t)\right\|^2 \leq 2\left\|\mu_{3mx}(0)\tilde{u}_{0kx}\right\|^2 + 2T^*\left(2+7M+6M^2+8M^3\right)^2 K_M^2(\mu)\int_0^t \bar{S}_m^{(k)}(s)ds,$

(ii) $\left\|\frac{\partial}{\partial x}\left(\mu_{3m}(t)u_{mx}^{(k)}(t)\right)\right\|^2 \leq 2\left\|\frac{\partial}{\partial x}\left(\mu_{3m}(0)\tilde{u}_{0kx}\right)\right\|^2 + 2T^*\left(4+8M+10M^2+8M^3\right)^2 K_M^2(\mu)\int_0^t \bar{S}_m^{(k)}(s)ds.$

Using the inequality

$$2ab \leq \beta a^2 + \frac{1}{\beta}b^2,\ \forall a,b\in\mathbb{R},\ \text{with } \beta=\beta_*=\frac{\bar{\mu}_*}{8}, \tag{3.24}$$

and Lemma 3.3 (i), (ii), the terms J_4, J_5 are estimated as follows:

$$\begin{aligned} J_4 &= -2\left\langle \mu_{3mx}(t)u_{mx}^{(k)}(t), \Delta u_m^{(k)}(t)\right\rangle \\ &\leq \beta_*\left\|\Delta u_m^{(k)}(t)\right\|^2 + \frac{1}{\beta_*}\left\|\mu_{3mx}(t)u_{mx}^{(k)}(t)\right\|^2 \\ &\leq \beta_*\bar{S}_m^{(k)}(t) + \frac{2}{\beta_*}\left\|\mu_{3mx}(0)\tilde{u}_{0kx}\right\|^2 + \frac{2}{\beta_*}T^*\left(2+7M+6M^2+8M^3\right)^2 K_M^2(\mu)\int_0^t \bar{S}_m^{(k)}(s)ds \\ &\equiv \beta_*\bar{S}_m^{(k)}(t) + \frac{2}{\beta_*}\left\|\mu_{3mx}(0)\tilde{u}_{0kx}\right\|^2 + \bar{C}_4(M)\int_0^t \bar{S}_m^{(k)}(s)ds; \end{aligned} \tag{3.25}$$

$$\begin{aligned} J_5 &= -2\left\langle \frac{\partial}{\partial x}\left(\mu_{3m}(t)u_{mx}^{(k)}(t)\right), \Delta\dot{u}_m^{(k)}(t)\right\rangle \\ &\leq \beta_*\left\|\Delta\dot{u}_m^{(k)}(t)\right\|^2 + \frac{1}{\beta_*}\left\|\frac{\partial}{\partial x}\left(\mu_{3m}(t)u_{mx}^{(k)}(t)\right)\right\|^2 \\ &\leq \beta_*\bar{S}_m^{(k)}(t) + \frac{2}{\beta_*}\left\|\frac{\partial}{\partial x}\left(\mu_{3m}(0)\tilde{u}_{0kx}\right)\right\|^2 + \frac{2}{\beta_*}T^*\left(4+8M+10M^2+8M^3\right)^2 K_M^2(\mu)\int_0^t \bar{S}_m^{(k)}(s)ds \\ &\equiv \beta_*\bar{S}_m^{(k)}(t) + \frac{2}{\beta_*}\left\|\frac{\partial}{\partial x}\left(\mu_{3m}(0)\tilde{u}_{0kx}\right)\right\|^2 + \bar{C}_5(M)\int_0^t \bar{S}_m^{(k)}(s)ds. \end{aligned}$$

Estimate of J_6. By (3.14), we obtain the following inequality

$$\begin{aligned} \left|\left\langle \tilde{F}_m(t), v\right\rangle\right| &\leq \|f(t)\|\,\|v\| + \|\mu_{1m}(t)\|\,\|v_x\| \\ &\leq \left(\tilde{K}(f)+K_M(\mu)\right)\|v_x\| \equiv \bar{C}_6(M)\|v_x\|,\ v\in H_0^1. \end{aligned} \tag{3.26}$$

Then

$$\begin{aligned} J_6 &= 2\int_0^t \left\langle \tilde{F}_m(\tau), \dot{u}_m^{(k)}(\tau)\right\rangle d\tau \leq 2\bar{C}_6(M)\int_0^t \left\|\dot{u}_{mx}^{(k)}(\tau)\right\| d\tau \\ &\leq 2\bar{C}_6(M)\int_0^t \sqrt{\bar{S}_m^{(k)}(\tau)}d\tau \leq T\bar{C}_6(M) + \bar{C}_6(M)\int_0^t \bar{S}_m^{(k)}(\tau)d\tau. \end{aligned} \tag{3.27}$$

Estimate of J_7. We have

$$\begin{aligned} \left\langle \tilde{F}_m(\tau), -\Delta\dot{u}_m^{(k)}(\tau)\right\rangle &= \left\langle f(\tau), -\Delta\dot{u}_m^{(k)}(\tau)\right\rangle + \left\langle \mu_{1m}(\tau), \Delta\dot{u}_{mx}^{(k)}(\tau)\right\rangle \\ &= \left\langle f(\tau), -\Delta\dot{u}_m^{(k)}(\tau)\right\rangle + \mu_{1m}(1,\tau)\Delta\dot{u}_m^{(k)}(1,\tau) - \mu_{1m}(0,\tau)\Delta\dot{u}_m^{(k)}(0,\tau) - \left\langle \mu_{1mx}(\tau), \Delta\dot{u}_m^{(k)}(\tau)\right\rangle \\ &= \left\langle f(\tau), -\Delta\dot{u}_m^{(k)}(\tau)\right\rangle - \left\langle \mu_{1mx}(\tau), \Delta\dot{u}_m^{(k)}(\tau)\right\rangle. \end{aligned} \tag{3.28}$$

Using the Lemma 3.2 ((iii), (vi)) the term $\left\langle \tilde{F}_m(\tau), -\Delta\dot{u}_m^{(k)}(\tau)\right\rangle$ is estimated as follows:

$$\begin{aligned} \left|\left\langle \tilde{F}_m(\tau), -\Delta\dot{u}_m^{(k)}(\tau)\right\rangle\right| &\leq (\|f(\tau)\| + \|\mu_{1mx}(\tau)\|) \left\|\Delta\dot{u}_m^{(k)}(\tau)\right\| \\ &\leq \left(\tilde{K}(f) + (1+2M)K_M(\mu)\right) \left\|\Delta\dot{u}_m^{(k)}(\tau)\right\| \\ &\leq \left[\tilde{K}(f) + (1+2M)K_M(\mu)\right] \sqrt{\bar{S}_m^{(k)}(\tau)} \\ &\equiv \bar{C}_7(M)\sqrt{\bar{S}_m^{(k)}(\tau)}. \end{aligned} \tag{3.29}$$

Therefore

$$\begin{aligned} J_7 &= 2\int_0^t \left\langle \tilde{F}_m(\tau), -\Delta\dot{u}_m^{(k)}(\tau)\right\rangle d\tau \leq 2\bar{C}_7(M)\int_0^t \sqrt{\bar{S}_m^{(k)}(\tau)}d\tau \\ &\leq T\bar{C}_7(M) + \bar{C}_7(M)\int_0^t \bar{S}_m^{(k)}(\tau)d\tau. \end{aligned} \tag{3.30}$$

Estimate of J_8. Similarly with (3.28), we have

$$\begin{aligned} \left\langle \tilde{F}_m(\tau), -\Delta\ddot{u}_m^{(k)}(\tau)\right\rangle &= -\left\langle f(\tau), \Delta\ddot{u}_m^{(k)}(\tau)\right\rangle + \left\langle \mu_{1m}(\tau), \Delta\ddot{u}_{mx}^{(k)}(\tau)\right\rangle \\ &= \left\langle f_x(\tau), \ddot{u}_{mx}^{(k)}(\tau)\right\rangle + \mu_{1m}(1,\tau)\Delta\ddot{u}_m^{(k)}(1,\tau) - \mu_{1m}(0,\tau)\Delta\ddot{u}_m^{(k)}(0,\tau) - \left\langle \mu_{1mx}(\tau), \Delta\ddot{u}_m^{(k)}(\tau)\right\rangle \\ &= \left\langle f_x(\tau), \ddot{u}_{mx}^{(k)}(\tau)\right\rangle - \left\langle \mu_{1mx}(\tau), \Delta\ddot{u}_m^{(k)}(\tau)\right\rangle. \end{aligned} \tag{3.31}$$

Then, we rewrite J_8 as follows

$$\begin{aligned} J_8 &= 2\int_0^t \left\langle \tilde{F}_m(\tau), -\Delta\ddot{u}_m^{(k)}(\tau)\right\rangle d\tau \\ &= 2\int_0^t \left\langle f_x(\tau), \ddot{u}_{mx}^{(k)}(\tau)\right\rangle d\tau - 2\int_0^t \left\langle \mu_{1mx}(\tau), \Delta\ddot{u}_m^{(k)}(\tau)\right\rangle d\tau \\ &\equiv J_8^{(1)} + J_8^{(2)}. \end{aligned} \tag{3.32}$$

Estimate of $J_8^{(1)}$. Note that $\|f_x(\tau)\| \leq \tilde{K}(f)$, then $J_8^{(1)}$ is estimated as follows:

$$\begin{aligned} J_8^{(1)} &= 2\int_0^t \left\langle f_x(\tau), \ddot{u}_{mx}^{(k)}(\tau)\right\rangle d\tau \leq 2\int_0^t \|f_x(\tau)\| \left\|\ddot{u}_{mx}^{(k)}(\tau)\right\| d\tau \\ &\leq \frac{1}{\beta_*}\int_0^t \|f_x(\tau)\|^2 d\tau + \beta_*\int_0^t \left\|\ddot{u}_{mx}^{(k)}(\tau)\right\|^2 d\tau \\ &\leq \frac{1}{\beta_*}T\tilde{K}^2(f) + \beta_*\bar{S}_m^{(k)}(t) \equiv TC_8^{(1*)}(M) + \beta_*\bar{S}_m^{(k)}(t). \end{aligned} \tag{3.33}$$

Estimate of $J_8^{(2)}$. Using integration by parts, we rewrite $J_8^{(2)}$ as follows

$$\begin{aligned} J_8^{(2)} &= -2\int_0^t \left\langle \mu_{1mx}(\tau), \Delta\ddot{u}_m^{(k)}(\tau)\right\rangle d\tau \\ &= 2\left\langle \mu_{1mx}(0), \Delta\tilde{u}_{1k}\right\rangle - 2\left\langle \mu_{1mx}(t), \Delta\dot{u}_m^{(k)}(t)\right\rangle + 2\int_0^t \left\langle \mu_{1mx\tau}(\tau), \Delta\dot{u}_m^{(k)}(\tau)\right\rangle d\tau \\ &= 2\left\langle \mu_{1mx}(0), \Delta\tilde{u}_{1k}\right\rangle + J_8^{(2b)} + J_8^{(2a)}. \end{aligned} \tag{3.34}$$

Using the inequality (3.24) and Lemma 3.2 (iv) the terms $J_8^{(2a)}$, $J_8^{(2b)}$ are estimated as follows:

$$\begin{aligned} J_8^{(2a)} &= 2\int_0^t \left\langle \mu_{1mx\tau}(\tau), \Delta\dot{u}_m^{(k)}(\tau)\right\rangle d\tau \leq 2\int_0^t \|\mu_{1mx\tau}(\tau)\| \left\|\Delta\dot{u}_m^{(k)}(\tau)\right\| d\tau \\ &\leq 2\left(1+5M+6M^2+8M^3\right) K_M(\mu)\int_0^t \sqrt{\bar{S}_m^{(k)}(\tau)}d\tau \\ &\equiv 2\bar{C}_8(M)\int_0^t \sqrt{\bar{S}_m^{(k)}(\tau)}d\tau \leq T\bar{C}_8(M) + \bar{C}_8(M)\int_0^t \bar{S}_m^{(k)}(\tau)d\tau; \end{aligned} \tag{3.35}$$

$$\begin{aligned} J_8^{(2b)} &= -2\left\langle \mu_{1mx}(t), \Delta\dot{u}_m^{(k)}(t)\right\rangle \leq 2\|\mu_{1mx}(t)\| \sqrt{\bar{S}_m^{(k)}(t)} \\ &\leq \beta_*\bar{S}_m^{(k)}(t) + \frac{1}{\beta_*}\|\mu_{1mx}(t)\|^2 \\ &\leq \beta_*\bar{S}_m^{(k)}(t) + \frac{2}{\beta_*}\left[\|\mu_{1mx}(0)\|^2 + \left(\int_0^t \|\mu'_{1mx}(s)\| ds\right)^2\right] \\ &\leq \beta_*\bar{S}_m^{(k)}(t) + \frac{2}{\beta_*}\|\mu_{1mx}(0)\|^2 + \frac{2T}{\beta_*}T^*\left(\bar{C}_8(M)\right)^2. \end{aligned}$$

Then

$$\begin{aligned} J_8^{(2)} &= 2\left\langle \mu_{1mx}(0), \Delta\tilde{u}_{1k}\right\rangle + J_8^{(2a)} + J_8^{(2b)} \\ &\leq 2\left\langle \mu_{1mx}(0), \Delta\tilde{u}_{1k}\right\rangle + T\bar{C}_8(M) + \bar{C}_8(M)\int_0^t \bar{S}_m^{(k)}(\tau)d\tau + \beta_*\bar{S}_m^{(k)}(t) + \frac{2}{\beta_*}\|\mu_{1mx}(0)\|^2 + \frac{2T}{\beta_*}T^*\left(\bar{C}_8(M)\right)^2 \\ &\leq 2\left\langle \mu_{1mx}(0), \Delta\tilde{u}_{1k}\right\rangle + T\bar{C}_8(M) + \bar{C}_8(M)\int_0^t \bar{S}_m^{(k)}(\tau)d\tau + \beta_*\bar{S}_m^{(k)}(t) + \frac{2}{\beta_*}\|\mu_{1mx}(0)\|^2 + \frac{2T}{\beta_*}T^*\left(\bar{C}_8(M)\right)^2 \\ &= 2\left\langle \mu_{1mx}(0), \Delta\tilde{u}_{1k}\right\rangle + \frac{2}{\beta_*}\|\mu_{1mx}(0)\|^2 + \beta_*\bar{S}_m^{(k)}(t) + T\left[1 + \frac{2}{\beta_*}T^*\bar{C}_8(M)\right]\bar{C}_8(M) + \bar{C}_8(M)\int_0^t \bar{S}_m^{(k)}(\tau)d\tau \\ &= 2\left\langle \mu_{1mx}(0), \Delta\tilde{u}_{1k}\right\rangle + \frac{2}{\beta_*}\|\mu_{1mx}(0)\|^2 + \beta_*\bar{S}_m^{(k)}(t) + TC_8^{(2*)}(M) + \bar{C}_8(M)\int_0^t \bar{S}_m^{(k)}(\tau)d\tau. \end{aligned} \tag{3.36}$$

Thus, J_8 is estimated by

$$\begin{aligned} J_8 \leq & TC_8^{(1*)}(M) + \beta_*\bar{S}_m^{(k)}(t) + 2\left\langle \mu_{1mx}(0), \Delta\tilde{u}_{1k}\right\rangle + \frac{2}{\beta_*}\|\mu_{1mx}(0)\|^2 + \beta_*\bar{S}_m^{(k)}(t) + TC_8^{(2*)}(M) + \bar{C}_8(M)\int_0^t \bar{S}_m^{(k)}(\tau)d\tau \\ =& 2\beta_*\bar{S}_m^{(k)}(t) + 2\left\langle \mu_{1mx}(0), \Delta\tilde{u}_{1k}\right\rangle + \frac{2}{\beta_*}\|\mu_{1mx}(0)\|^2 + T\left(C_8^{(1*)}(M) + C_8^{(2*)}(M)\right) + \bar{C}_8(M)\int_0^t \bar{S}_m^{(k)}(\tau)d\tau \\ \equiv& 2\beta_*\bar{S}_m^{(k)}(t) + 2\left\langle \mu_{1mx}(0), \Delta\tilde{u}_{1k}\right\rangle + \frac{2}{\beta_*}\|\mu_{1mx}(0)\|^2 + TC_8^{(*)}(M) + \bar{C}_8(M)\int_0^t \bar{S}_m^{(k)}(s)ds. \end{aligned} \tag{3.37}$$

Combining (3.23), (3.25), (3.27), (3.30), (3.37), it implies from (3.21) and Lemma 3.2 (i) that

$$\bar{S}_m^{(k)}(t) \leq \bar{S}_{0m}^{(k)} + T\bar{D}_1(M) + \bar{D}_2(M)\int_0^t \bar{S}_m^{(k)}(s)ds, \tag{3.38}$$

where

$$\begin{cases} \bar{S}_{0m}^{(k)} = \dfrac{2}{\bar{\mu}_*}S_m^{(k)}(0) + \dfrac{4}{\bar{\mu}_*}\left(\left\langle \mu_{3mx}(0)\tilde{u}_{0kx}, \Delta\tilde{u}_{0k}\right\rangle + \left\langle \mu_{1mx}(0), \Delta\tilde{u}_{1k}\right\rangle\right) + \dfrac{4}{\bar{\mu}_*}\left\langle \dfrac{\partial}{\partial x}\left(\mu_{3m}(0)\tilde{u}_{0kx}\right), \Delta\tilde{u}_{1kx}\right\rangle \\ \qquad + \dfrac{32}{\bar{\mu}_*^2}\left(\|\mu_{1mx}(0)\|^2 + \|\mu_{3mx}(0)\tilde{u}_{0kx}\|^2 + \left\|\dfrac{\partial}{\partial x}\left(\mu_{3m}(0)\tilde{u}_{0kx}\right)\right\|^2\right), \\ \bar{D}_1(M) = \dfrac{2}{\bar{\mu}_*}\left(\bar{C}_6(M) + \bar{C}_7(M) + C_8^{(*)}(M)\right), \\ \bar{D}_2(M) = \dfrac{2}{\bar{\mu}_*}\displaystyle\sum_{i=1}^{8}\bar{C}_i(M). \end{cases} \tag{3.39}$$

The convergences given by (3.17) show that there exists a positive constant M independent of k and m such that

$$\bar{S}_{0m}^{(k)} \leq \frac{M^2}{2}, \text{ for all } m,\ k \in \mathbb{N}. \tag{3.40}$$

Then, T can be chosen small enough such that

$$\left(\frac{M^2}{2} + T\bar{D}_1(M)\right) e^{T\bar{D}_2(M)} \leq M^2, \tag{3.41}$$

and

$$k_T = 3\sqrt{T\bar{D}_1^*(M)}\exp(T\bar{D}_2^*(M)) < 1, \tag{3.42}$$

where

$$\begin{cases} \bar{D}_1^*(M) = \dfrac{8}{\bar{\mu}_*^2}(1+4M)^2(1+M^2)K_M^2(\mu), \\ \bar{D}_2^*(M) = \dfrac{1}{\bar{\mu}_*}(1+M+4M^2)K_M(\mu). \end{cases} \tag{3.43}$$

It follows from (3.38), (3.40) and (3.41) that

$$\bar{S}_m^{(k)}(t) \leq M^2 e^{-T\bar{D}_2(M)} + \bar{D}_2(M)\int_0^t \bar{S}_m^{(k)}(s)ds. \tag{3.44}$$

By using Gronwall's Lemma, we deduce from (3.44) that

$$\bar{S}_m^{(k)}(t) \leq M^2 e^{-T\bar{D}_2(M)} e^{t\bar{D}_2(M)} \leq M^2, \tag{3.45}$$

for all $t \in [0,T]$, for all $m,\ k \in \mathbb{N}$. Therefore, we have

$$u_m^{(k)} \in W_1(M,T), \text{ for all } m \text{ and } k \in \mathbb{N}. \tag{3.46}$$

Step 3. Limiting process. By (3.46), there exists a subsequence of $\{u_m^{(k)}\}$ with the same symbol, such that

$$\begin{cases} u_m^{(k)} \to u_m & \text{in} \quad L^\infty(0,T;H^2\cap H_0^1) \text{ weak*}, \\ \dot{u}_m^{(k)} \to u_m' & \text{in} \quad L^\infty(0,T;H^2\cap H_0^1) \text{ weak*}, \\ \ddot{u}_m^{(k)} \to u_m'' & \text{in} \quad L^2(0,T;H_0^1) \text{ weak}, \\ u_m \in W(M,T). \end{cases} \tag{3.47}$$

Passing to limit in (3.16), we have u_m satisfying (3.13) and (3.14) in $L^2(0,T)$. On the other hand, we deduce from $(3.13)_1$ and $(3.47)_4$ that

$$\begin{aligned} u_m'' &= \lambda u_{mxx}' + \frac{\partial}{\partial x}\left(\mu_{1m}(t) + \mu_{3m}(t)u_{mx}(t)\right) - \int_0^t \frac{\partial}{\partial x}\left(\bar{\mu}_{1m}(s,t) + \bar{\mu}_{4m}(s,t)u_{mx}(s)\right)ds + F_m \\ &\equiv \hat{F}_m \in L^\infty(0,T;L^2). \end{aligned}$$

Thus, $u_m \in W_1(M,T)$. Theorem 3.1 is proved. □

By using Theorem 3.1 and the compact imbedding theorems, we shall prove the existence and uniqueness of weak local solutions to Prob. (1.1). We first introduce the Banach space (see Lions [16]) as follows

$$W_1(T) = \left\{u \in C^0([0,T];H_0^1) \cap C^1([0,T];L^2) : u' \in L^2(0,T;H_0^1)\right\}, \tag{3.48}$$

with respect to the norm $\|u\|_{W_1(T)} = \|u\|_{C^0([0,T];H_0^1)} + \|u'\|_{C^0(0,T;L^2)} + \|u'\|_{L^2(0,T;H_0^1)}$.

Then we have the following theorem.

Theorem 3.4. *Suppose that the assumptions $(H_1)-(H_3)$ hold. Then the recurrent sequence $\{u_m\}$ defined by* (3.8), (3.13) *and* (3.14) *strongly converges to u in $W_1(T)$. Furthermore, u is a unique weak solution of Prob.* (1.1) *and $u \in W_1(M,T)$. On the other hand, the following estimation is valid*

$$\|u_m - u\|_{W_1(T)} \leq C_T k_T^m, \text{ for all } m \in \mathbb{N}, \tag{3.49}$$

where $k_T \in [0,1)$ is defined as in (3.42) *and C_T is a constant depending only on T, f, μ, $\tilde{u}_0$, $\tilde{u}_1$.*

Proof of Theorem 3.4. First, we prove the local existence of Prob. (1.1). We begin by proving that $\{u_m\}$ (in Theorem 3.1) is a Cauchy sequence in $W_1(T)$. Let $w_m = u_{m+1} - u_m$. Then w_m satisfies the variational problem

$$\begin{cases} \langle w_m''(t), v\rangle + \lambda\langle w_{mx}'(t), v_x\rangle + B_m(t,v) = 0,\ \forall v \in H_0^1, \\ w_m(0) = w_m'(0) = 0, \end{cases} \tag{3.50}$$

where

$$B_m(t,v) = a_{m+1}(t; u_{m+1}(t), v) - a_m(t; u_m(t), v),\ \ v \in H_0^1. \tag{3.51}$$

Taking $v = w_m'(t)$ in $(3.50)_1$ and then integrating in t, we get

$$\begin{aligned} \bar{\mu}_* \bar{S}_m(t) &\leq S_m(t) = \int_0^t ds \int_0^1 \mu_{3m+1}'(x,s) w_{mx}^2(x,s)dx \\ &\quad -2\int_0^t \langle [\mu_{3m+1}(s) - \mu_{3m}(s)]\, u_{mx}(s), w_{mx}'(s)\rangle\, ds \\ &\quad -2\int_0^t \langle \mu_{1m+1}(s) - \mu_{1m}(s), w_{mx}'(s)\rangle\, ds \\ &= \sum\nolimits_{j=1}^3 \bar{I}_j, \end{aligned} \tag{3.52}$$

where $\bar{\mu}_* = \min\{1, \mu_*, \lambda\}$ and

$$\begin{aligned} S_m(t) &= \|w_m'(t)\|^2 + \left\|\sqrt{\mu_{3m+1}(t)} w_{mx}(t)\right\|^2 + 2\lambda \int_0^t \|w_{mx}'(s)\|^2\, ds, \\ \bar{S}_m(t) &= \|w_m'(t)\|^2 + \|w_{mx}(t)\|^2 + \int_0^t \|w_{mx}'(s)\|^2\, ds. \end{aligned} \tag{3.53}$$

Next, the integrals on right-hand side of (3.52) are estimated as follows.

By the following inequalities

$$\begin{aligned} \left|\mu_{3m+1}'(x,s)\right| &\leq \left(1 + M + 4M^2\right) K_M(\mu), \\ |\mu_{im+1}(x,t) - \mu_{im}(x,t)| &\leq (1+4M)K_M(\mu)\,\|w_{m-1}\|_{W_1(T)},\ \ i = 1,3, \end{aligned} \tag{3.54}$$

the terms $\bar{I}_1$, $\bar{I}_2$, $\bar{I}_3$ are estimated by

$$\begin{aligned} \bar{I}_1 &= \int_0^t ds \int_0^1 \mu_{3m+1}'(x,s) w_{mx}^2(x,s)dx \leq (1+M+4M^2)K_M(\mu)\int_0^t \bar{S}_m(s)ds, \\ \bar{I}_2 &= -2\int_0^t \langle [\mu_{3m+1}(s) - \mu_{3m}(s)]\, u_{mx}(s), w_{mx}'(s)\rangle\, ds \\ &\leq 2(1+4M)MK_M(\mu)\,\|w_{m-1}\|_{W_1(T)} \int_0^t \|w_{mx}'(s)\|\, ds \\ &\leq \gamma\int_0^t \|w_{mx}'(s)\|^2\, ds + \frac{1}{\gamma}T(1+4M)^2M^2K_M^2(\mu)\,\|w_{m-1}\|_{W_1(T)}^2 \\ &\leq \gamma\bar{S}_m(t) + \frac{1}{\gamma}T(1+4M)^2M^2K_M^2(\mu)\,\|w_{m-1}\|_{W_1(T)}^2\, ; \\ \bar{I}_3 &= -2\int_0^t \langle \mu_{1m+1}(s) - \mu_{1m}(s), w_{mx}'(s)\rangle\, ds \\ &\leq 2(1+4M)K_M(\mu)\,\|w_{m-1}\|_{W_1(T)} \int_0^t \|w_{mx}'(s)\|\, ds \\ &\leq \gamma\int_0^t \|w_{mx}'(s)\|^2\, ds + \frac{1}{\gamma}T(1+4M)^2K_M^2(\mu)\,\|w_{m-1}\|_{W_1(T)}^2 \\ &\leq \gamma\bar{S}_m(t) + \frac{1}{\gamma}T(1+4M)^2K_M^2(\mu)\,\|w_{m-1}\|_{W_1(T)}^2\, . \end{aligned} \tag{3.55}$$

Combining the estimations (3.55), we deduce from (3.52) that

$$\bar{S}_m(t) \leq T\bar{D}_1^*(M) \|w_{m-1}\|_{W_1(T)}^2 + 2\bar{D}_2^*(M) \int_0^t \bar{S}_m(s)ds, \tag{3.56}$$

where $\bar{D}_1^*(M)$, $\bar{D}_2^*(M)$ are defined by (3.39).

Using Gronwall's lemma, we get from (3.56) that

$$\bar{S}_m(t) \leq T\bar{D}_1^*(M) \|w_{m-1}\|_{W_1(T)}^2 \exp\left(2T\bar{D}_2^*(M)\right), \tag{3.57}$$

hence, it leads to

$$\|w_m\|_{W_1(T)} \leq k_T \|w_{m-1}\|_{W_1(T)}, \ \forall m \in \mathbb{N}, \tag{3.58}$$

where the constant $k_T < 1$ is defined as in (3.42), which implies that

$$\|u_{m+p} - u_m\|_{W_1(T)} \leq \frac{M}{1-k_T} k_T^m, \ \forall m, \ p \in \mathbb{N}. \tag{3.59}$$

It follows that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Then there exists $u \in W_1(T)$ such that

$$u_m \to u \text{ strongly in } W_1(T). \tag{3.60}$$

Note that $u_m \in W(M,T)$, then there exists a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ such that

$$\begin{cases} u_{m_j} \to u & \text{in} \quad L^\infty(0,T;H^2 \cap H_0^1) \text{ weak*}, \\ u'_{m_j} \to u' & \text{in} \quad L^\infty(0,T;H^2 \cap H_0^1) \text{ weak*}, \\ u''_{m_j} \to u'' & \text{in} \quad L^2(0,T;H_0^1) \text{ weak}, \\ u \in W(M,T). \end{cases} \tag{3.61}$$

On the other hand, using the equality

$$\begin{aligned} a_m(t;u_m(t),v) - a(t;u(t),v) &= \langle \mu_{3m}(t)u_{mx}(t) - D_3\mu[u](t)u_x(t) + \mu_{1m}(t) - D_1\mu[u](t), v_x\rangle \\ &= \langle \mu_{3m}(t)\left[u_{mx}(t) - u_x(t)\right] + \left[\mu_{3m}(t) - D_3\mu[u](t)\right]u_x(t), v_x\rangle + \langle \mu_{1m}(t) - D_1\mu[u](t), v_x\rangle, \end{aligned} \tag{3.62}$$

and the inequality

$$|\mu_{im}(x,t) - D_i\mu[u](x,t)| \leq (1+4M)K_M(\mu) \|w_{m-1}\|_{W_1(T)}, \ i = 1,3, \tag{3.63}$$

we get

$$|a_m(t;u_m(t),v) - a(t;u(t),v)| \quad \leq \quad K_M(\mu)\left[\|u_m - u\|_{W_1(T)} + (1+4M)(1+M)\|u_{m-1} - u\|_{W_1(T)}\right]\|v_x\|. \tag{3.64}$$

Hence

$$a_m(t;u_m(t),v) \to a(t;u(t),v) \text{ in } L^\infty(0,T) \text{ weak*, for all } v \in H_0^1. \tag{3.65}$$

Passing to limit in (3.9) and (3.10) as $m = m_j \to \infty$, it implies from (3.60), (3.61) and (3.65) that there exists $u \in W(M,T)$ satisfying (3.1), (3.2).

On the other hand, we derive from (3.1) and $(3.61)_4$ that

$$\begin{aligned} u'' &= \lambda u'_{xx} + \frac{\partial}{\partial x}\left[D_1\mu[u](t) + D_3\mu[u](t)u_x(t)\right] + f(t) \\ &\equiv \hat{f} \in L^\infty(0,T;L^2). \end{aligned}$$

Thus $u \in W_1(M,T)$. The proof of existence is completed. Finally, we need to prove the uniqueness of solutions. Let u_1, $u_2 \in W_1(M,T)$ be two weak solutions of Prob. (1.1). Then $u = u_1 - u_2$ satisfies the variational problem

$$\begin{cases} \langle u''(t), v\rangle + \lambda\langle u'_x(t), v_x\rangle + B(t,v) = 0, \forall v \in H_0^1, \\ u(0) = u'(0) = 0, \end{cases} \tag{3.66}$$

where

$$B(t,v) = \langle D_3\mu[u_1](t)u_x(t) + [D_3\mu[u_1](t) - D_3\mu[u_2](t)]\, u_{2x}(t), v_x\rangle \\ +\langle D_1\mu[u_1](t) - D_1\mu[u_2](t), v_x\rangle,\ v \in H_0^1. \tag{3.67}$$

Taking $v = u'(t)$ in $(3.66)_1$ and integrating in time from 0 to t, we get

$$\bar{\mu}_*\bar{Z}(t) \leq \int_0^t dr \int_0^r \frac{\partial}{\partial r}\left(D_3\mu[u_1]\right)(x,r)u_x^2(x,r)dx - 2\int_0^t \langle [D_3\mu[u_1](s) - D_3\mu[u_2](s)]\, u_{2x}(s), u_x'(s)\rangle ds \\ -2\int_0^t \langle D_1\mu[u_1](s) - D_1\mu[u_2](s), u_x'(s)\rangle ds, \tag{3.68}$$

where $\bar{\mu}_* = \min\{1, \mu_*, \lambda\}$ and $\bar{Z}(t) = \|u'(t)\|^2 + \|u_x(t)\|^2 + \int_0^t \|u_x'(s)\|^2 ds$. Through similar calculations in Theorem 3.2, we obtain from (3.68) that

$$(\bar{\mu}_* - 2\delta)\bar{Z}(t) \leq \left[1 + M + 4M^2 + \frac{1}{\delta}(1+4M)^2(1+M^2)K_M(\mu)\right] K_M(\mu) \int_0^t \bar{Z}(s)ds, \tag{3.69}$$

for all $\delta > 0$. Then, by choosing $\delta > 0$ such that $\bar{\mu}_* - 2\delta > 0$ and using Gronwall lemma, we deduce from (3.69) that $\bar{Z}(t) \equiv 0$, i.e., $u = u_1 - u_2 = 0$. Therefore, uniqueness is proved. The proof of Theorem 3.4 is done. □

4 Continuous dependence

In this section, we assume that $\lambda > 0$ and $\tilde{u}_0, \tilde{u}_1 \in H^2 \cap H_0^1$, $f \in C^1([0,1] \times [0,T^*])$. By Theorem 3.4, Prob. (1.1) admits a unique solution u depending on the datum μ :

$$u = u(\mu),$$

where μ satisfy the assumptions (H_2). First, we note that if the datum μ, μ_j satisfy (H_2) and in addition, the following condition is fulfilled

$$D_1(\mu_j, \mu) \equiv \sup_{M>0} \|\mu_j - \mu\|_{C^3(A_M)} \to 0, \tag{4.1}$$

as $j \to \infty$, then there exists $j_0 \in \mathbb{N}$ (independent of M) such that

$$\|\mu_j\|_{C^3(A_M)} \leq 1 + \|\mu\|_{C^3(A_M)},\ \forall M > 0,\ \forall j \geq j_0.$$

By setting the constant $K_M(\mu)$, we deduce from the above estimation that

$$K_M(\mu_j) \leq 1 + K_M(\mu),\ \forall M > 0,\ \forall j \geq j_0. \tag{4.2}$$

Therefore, the Galerkin approximation sequence $\{u_m^{(k)}\}$ corresponding to $\mu = \mu_j$, $j \geq j_0$ also satisfies the priori estimates as in Theorem 3.1 and

$$u_m^{(k)} \in W_1(M,T),\ \text{for all } m \text{ and } k \in \mathbb{N}, \tag{4.3}$$

where M, T are constants independent of j. Indeed, in the process, we can choose the positive constants M and T as in (3.41) - (3.42) with replacing $K_M(\mu)$, $D_1^2\mu$, $D_1D_3\mu$, $D_3^2\mu$ by $1 + K_M(\mu)$, $1 + \left|D_1^2\mu\right|$, $1 + |D_1D_3\mu|$, $1 + \left|D_3^2\mu\right|$, respectively.

Hence, the limitation u_j of $\{u_m^{(k)}\}$, as $k \to +\infty$ and $m \to +\infty$ later, is the unique weak solution of Prob. (1.1) corresponding to $\mu = \mu_j$, $j \geq j_0$ satisfying

$$u_j \in W_1(M,T),\ \text{for all } j \geq j_0. \tag{4.4}$$

Moreover, by the same argument used in Theorem 3.4, we can prove that the limitation u of $\{u_j\}$ as $j \to +\infty$, is the unique weak solution of Prob. (1.1) and $u \in W_1(M,T)$.

Consequently, we have the following theorem.

Theorem 4.1. *For any $\lambda > 0$, $\tilde{u}_0$, $\tilde{u}_1 \in H^2 \cap H_0^1$, $f \in C^1([0,1] \times [0,T^*])$, suppose that (H_2) and the condition (4.1) hold. Then, there exists a positive constant T such that the solution of Prob. (1.1) is continuous dependence on the datum μ, i.e., if μ and μ_j satisfy (H_2) and (4.1), then*

$$u_j = u(\mu_j) \to u \text{ strongly in } W_1(T), \text{ as } j \to \infty. \tag{4.5}$$

Moreover, we have the estimation

$$\|u_j - u\|_{W_1(T)} \le C_T D_1(\mu_j, \mu), \ \forall j \ge j_0, \tag{4.6}$$

where C_T is a constant only depending on T, f, μ, $\tilde{u}_0$ and $\tilde{u}_1$.

Proof of Theorem 4.1. Setting

$$\begin{aligned} D_i\mu_j[u_j](x,t) &= D_i\mu_j\left(x,t,u_j(x,t), \|u_j(t)\|^2, \|u_{jx}(t)\|^2\right), \\ D_i\mu[u](x,t) &= D_i\mu\left(x,t,u(x,t), \|u(t)\|^2, \|u_x(t)\|^2\right), \ i=1,3, \ j \in \mathbb{N}, \end{aligned} \tag{4.7}$$

then $w_j = u_j - u$, satisfies the variational problem

$$\begin{cases} \langle w_j''(t), v\rangle + \lambda\langle w_{jx}'(t), v_x\rangle + a_j(t;u_j(t),v) - a(t;u(t),v) = 0, \ \forall v \in H_0^1, \\ w_j(0) = w_j'(0) = 0, \end{cases} \tag{4.8}$$

where

$$\begin{aligned} a_j(t;u_j(t),v) &= \langle D_3\mu_j[u_j](t)u_{jx}(t), v_x\rangle + \langle D_1\mu_j[u_j](t), v_x\rangle, \\ a(t;u(t),v) &= \langle D_3\mu[u](t)u_x(t), v_x\rangle + \langle D_1\mu[u](t), v_x\rangle. \end{aligned} \tag{4.9}$$

On the other hand, by the following equalities

$$\begin{aligned} a_j(t;u_j(t),v) - a(t;u(t),v) &= \langle D_3\mu_j[u_j](t)w_{jx}(t), v_x\rangle + \langle [D_3\mu_j[u_j](t) - D_3\mu[u](t)]\, u_x(t), v_x\rangle \\ &\quad + \langle D_1\mu_j[u_j](t) - D_1\mu[u](t), v_x\rangle, \end{aligned} \tag{4.10}$$

we rewrite (4.8) by

$$\begin{cases} \langle w_j''(t), v\rangle + \lambda\langle w_{jx}'(t), v_x\rangle + \langle D_3\mu_j[u_j](t)w_{jx}(t), v_x\rangle + \langle [D_3\mu_j[u_j](t) - D_3\mu[u](t)]\, u_x(t), v_x\rangle \\ \qquad + \langle D_1\mu_j[u_j](t) - D_1\mu[u](t), v_x\rangle = 0, \ \forall v \in H_0^1, \\ w_j(0) = w_j'(0) = 0. \end{cases} \tag{4.11}$$

Taking $v = w_j'(t)$ in $(4.11)_1$ and then integrating in t, we get

$$\begin{aligned} \bar{\mu}_* \bar{S}_j(t) &\le S_j(t) = \int_0^t ds \int_0^1 \frac{\partial}{\partial s}\left(D_3\mu_j[u_j](x,s)\right) w_{jx}^2(x,s)dx \\ &\quad -2\int_0^t \left\langle [D_3\mu_j[u_j](s) - D_3\mu[u](s)]\, u_x(s), w_{jx}'(s)\right\rangle ds - 2\int_0^t \left\langle D_1\mu_j[u_j](s) - D_1\mu[u](s), w_{jx}'(s)\right\rangle ds \\ &\equiv \sum_{k=1}^3 J_k, \end{aligned} \tag{4.12}$$

where $\bar{\mu}_* = \min\{1, \mu_*, \lambda\}$ and

$$\begin{aligned} S_j(t) &= \left\|w_j'(t)\right\|^2 + \left\|\sqrt{D_3\mu_j[u_j](t)}w_{jx}(t)\right\|^2 + 2\lambda\int_0^t \left\|w_{jx}'(s)\right\|^2 ds, \\ \bar{S}_j(t) &= \left\|w_j'(t)\right\|^2 + \|w_{jx}(t)\|^2 + \int_0^t \left\|w_{jx}'(s)\right\|^2 ds. \end{aligned} \tag{4.13}$$

We estimate the terms I_j on the right-hand side of (4.12) as follows.

Estimate of I_1. By the estimation

$$\begin{aligned}\left|\frac{\partial}{\partial t}\left(D_3\mu_j[u_j](x,t)\right)\right| &\le K_M(\mu_j)\left[1+\left|u_j'(x,t)\right|+2\|u_j(t)\|\left\|u_j'(t)\right\|+2\|u_{jx}(t)\|\left\|u_{jx}'(t)\right\|\right]\\ &\le (1+K_M(\mu))\left[1+\left|u_j'(x,t)\right|+2\|u_j(t)\|\left\|u_j'(t)\right\|+2\|u_{jx}(t)\|\left\|u_{jx}'(t)\right\|\right]\\ &\le (1+K_M(\mu))\left[1+\left\|u_{jx}'(t)\right\|+4\|u_{jx}(t)\|\left\|u_{jx}'(t)\right\|\right]\\ &\le (1+K_M(\mu))\left(1+M+4M^2\right),\end{aligned}$$

we have

$$\begin{aligned}I_1 &= \int_0^t ds\int_0^1 \frac{\partial}{\partial s}\left(D_3\mu_j[u_j](x,s)\right)w_{jx}^2(x,s)dx \qquad (4.14)\\ &\le (1+K_M(\mu))\left(1+M+4M^2\right)\int_0^t \|w_{jx}(s)\|^2 ds\\ &\le (1+K_M(\mu))\left(1+M+4M^2\right)\int_0^t \bar{S}_j(s)ds.\end{aligned}$$

Estimate of I_2. By the estimation

$$\begin{aligned}|D_3\mu_j[u_j](x,t)-D_3\mu[u](x,t)| &\le |D_3\mu_j[u_j](x,t)-D_3\mu[u_j](x,t)|+|D_3\mu[u_j](x,t)-D_3\mu[u](x,t)|\\ &\le D_1(\mu_j,\mu)+K_M(\mu)(1+4M)\|w_{jx}(t)\|\\ &\le D_1(\mu_j,\mu)+K_M(\mu)(1+4M)\sqrt{\bar{S}_j(t)},\end{aligned}$$

we obtain

$$\begin{aligned}I_2 &= -2\int_0^t \left\langle [D_3\mu_j[u_j](s)-D_3\mu[u](s)]\,u_x(s), w_{jx}'(s)\right\rangle ds \qquad (4.15)\\ &\le 2\int_0^t \left(D_1(\mu_j,\mu)+K_M(\mu)(1+4M)\sqrt{\bar{S}_j(s)}\right)M\left\|w_{jx}'(s)\right\|ds\\ &= 2M\int_0^t \left(D_1(\mu_j,\mu)\left\|w_{jx}'(s)\right\|+K_M(\mu)(1+4M)\sqrt{\bar{S}_j(s)}\left\|w_{jx}'(s)\right\|\right)ds\\ &= 2MD_1(\mu_j,\mu)\int_0^t \left\|w_{jx}'(s)\right\|ds+2MK_M(\mu)(1+4M)\int_0^t\sqrt{\bar{S}_j(s)}\left\|w_{jx}'(s)\right\|ds\\ &\le 2\sqrt{T}MD_1(\mu_j,\mu)\left(\int_0^t\left\|w_{jx}'(s)\right\|^2ds\right)^{1/2}+2MK_M(\mu)(1+4M)\left(\int_0^t\bar{S}_j(s)ds\right)^{1/2}\left(\int_0^t\left\|w_{jx}'(s)\right\|^2ds\right)^{1/2}\\ &\le 2\sqrt{T}MD_1(\mu_j,\mu)\sqrt{\bar{S}_j(t)}+2MK_M(\mu)(1+4M)\left(\int_0^t\bar{S}_j(s)ds\right)^{1/2}\sqrt{\bar{S}_j(t)}\\ &\le 2\delta\bar{S}_j(t)+\frac{1}{\delta}T^*M^2D_1^2(\mu_j,\mu)+\frac{1}{\delta}M^2K_M^2(\mu)(1+4M)^2\int_0^t\bar{S}_j(s)ds.\end{aligned}$$

Estimate of I_3. Similarly to I_2, we have also

$$\begin{aligned}I_3 &= -2\int_0^t\left\langle D_1\mu_j[u_j](s)-D_1\mu[u](s), w_{jx}'(s)\right\rangle ds \qquad (4.16)\\ &\le 2\delta\bar{S}_j(t)+\frac{1}{\delta}T^*D_1^2(\mu_j,\mu)+\frac{1}{\delta}K_M^2(\mu)(1+4M)^2\int_0^t\bar{S}_j(s)ds.\end{aligned}$$

Finally, by choosing $\delta=\frac{\bar{\mu}_*}{8}$, we get from (4.12), (4.14)-(4.16) that

$$\bar{S}_j(t)\le\alpha(M)E_j+\beta(M)\int_0^t\bar{S}_j(s)ds, \qquad (4.17)$$

where

$$\begin{aligned} E_j &= D_1^2(\mu_j,\mu), \\ \alpha(M) &= \frac{16}{\bar{\mu}_*^2}T^*(1+M^2), \\ \beta(M) &= \frac{2}{\bar{\mu}_*}\left(1+K_M(\mu)\right)\left(1+M+4M^2\right)+\frac{16}{\bar{\mu}_*^2}(1+M^2)\left(1+4M\right)^2K_M^2(\mu). \end{aligned} \tag{4.18}$$

Using Gronwall's lemma, we have

$$\bar{S}_j(t) \le \alpha(M)E_j\exp(T\beta(M)). \tag{4.19}$$

This derive that

$$\|u_j-u\|_{W_1(T)} \le 3\sqrt{\alpha(M)E_j\exp(T\beta(M))} \le C_T D_1(\mu_j,\mu),\ \forall j \ge j_0. \tag{4.20}$$

Theorem 4.1 is proved. □

Remark 4.2. We give here an example, in which the condition (4.1) is satisfied.

Considering $\{\mu_j\}$ defined by

$$\mu_j(x,t,y,z_1,z_2)=\mu(x,t,y,z_1,z_2)+\frac{xt}{j}\text{arctg}(y),$$

$(x,t,y,z_1,z_2)\in[0,1]\times[0,T^*]\times\mathbb{R}\times\mathbb{R}_+^2$, where $\mu\in C^3([0,1]\times[0,T^*]\times\mathbb{R}\times\mathbb{R}_+^2)$ satisfies (H_2).

It is easy to check that $\mu_j\in C^3([0,1]\times[0,T^*]\times\mathbb{R}\times\mathbb{R}_+^2)$ also satisfies (H_2) and

$$\begin{aligned} D_1(\mu_j,\mu) &\equiv \sup_{M>0}\|\mu_j-\mu\|_{C^3(A_M)} \\ &= \sup_{M>0}\left(\max_{|\beta|\le 3}\left\|D^\beta\mu_j-D^\beta\mu\right\|_{C^0(A_M)}\right) \\ &\le \frac{1}{j}\max\left\{\frac{\pi}{2},2T^*\right\}\to 0,\ \text{as } j\to\infty. \end{aligned}$$

5 Asymptotic expansion of the solution with respect to a small parameter

In this section, let (H_1), (H_3) hold. We also make the following assumptions:

(H_2') $\mu_1\in C^3([0,1]\times[0,T^*]\times\mathbb{R}\times\mathbb{R}_+^2)$
and $D_3\mu_1(x,t,y,z_1,z_2)\ge 0$, for all $(x,t,y,z_1,z_2)\in[0,1]\times[0,T^*]\times\mathbb{R}\times\mathbb{R}_+^2$.

We consider the following perturbed problem, where ε is a small parameter, with $0<\varepsilon<1$:

$$(P_\varepsilon)\begin{cases} u_{tt}-\lambda u_{txx}-\dfrac{\partial^2}{\partial x^2}\left(\mu_\varepsilon[u](x,t)\right)=f\left(x,t\right),\ 0<x<1,\ 0<t<T, \\ u\left(0,t\right)=u\left(1,t\right)=0, \\ u\left(x,0\right)=\tilde{u}_0\left(x\right),\ u_t\left(x,0\right)=\tilde{u}_1\left(x\right), \end{cases}$$

where

$$\mu_\varepsilon[u](x,t) = \mu\left(x,t,u(x,t),\|u(t)\|^2,\|u_x(t)\|^2\right)+\varepsilon\mu_1\left(x,t,u(x,t),\|u(t)\|^2,\|u_x(t)\|^2\right).$$

By the assumptions (H_1), (H_2), (H_2'), (H_3) and Theorem 3.4, Prob. (P_ε) has a unique weak solution u depending on $\varepsilon : u=u_\varepsilon$. When $\varepsilon=0$, (P_ε) is denoted by $(\tilde{P}_0)$. We shall study the asymptotic expansion of the solution u_ε of Prob. (P_ε) with respect to a small parameter ε. We use the following notations. For a multi-index $\alpha=(\alpha_1,\cdots,\alpha_N)\in\mathbb{Z}_+^N$, and $x=(x_1,\cdots,x_N)\in\mathbb{R}^N$, we put

$$\begin{cases} |\alpha|=\alpha_1+\cdots+\alpha_N,\ \alpha!=\alpha_1!\cdots\alpha_N!, \\ x^\alpha=x_1^{\alpha_1}\cdots x_N^{\alpha_N}, \\ \alpha,\ \beta\in\mathbb{Z}_+^N,\ \alpha\le\beta \Longleftrightarrow \alpha_i\le\beta_i\quad\forall i=1,\cdots,N. \end{cases}$$

First, we shall need the following lemma.

Lemma 5.1. *Let* m, $N \in \mathbb{N}$, $x = (x_1, \cdots, x_N) \in \mathbb{R}^N$, *and* $\varepsilon \in \mathbb{R}$. *Then*

$$\left(\sum_{i=1}^{N} x_i \varepsilon^i\right)^m = \sum_{k=m}^{mN} P_N^{[m]}[x]_k \varepsilon^k, \tag{5.1}$$

where the coefficients $P_N^{[m]}[x]_k$, $m \le k \le mN$ *depending on* $x = (x_1, \cdots, x_N)$ *are defined by the formula*

$$P_N^{[m]}[x]_k = \begin{cases} x_k, & 1 \le k \le N,\ m = 1, \\ \sum\limits_{\alpha \in A_k^{[m]}(N)} \frac{m!}{\alpha!} x^\alpha, & m \le k \le mN,\ m \ge 2, \end{cases} \tag{5.2}$$

with $A_k^{[m]}(N) = \left\{\alpha \in \mathbb{Z}_+^N : |\alpha| = m,\ \sum_{i=1}^N i\alpha_i = k\right\}$.

The proof of this lemma is easy, hence we omit the details. □

The lemma below shows that the coefficient $P_N^{[m]}[x]_k$ of ε^k in the formula (5.1) depends only on $x_1, \cdots, x_{k-1}$.

Lemma 5.2. *Let* m, $N \in \mathbb{N} \setminus \{1\}$, $x = (x_1, \cdots, x_N) \in \mathbb{R}^N$, *and* $\varepsilon \in \mathbb{R}$, $|\varepsilon| \le 1$. *Then*

$$\left(\sum_{i=1}^{N} x_i \varepsilon^i\right)^m = \sum_{k=m}^{N} \bar{P}_k^{[m]}[x_1, \cdots, x_{k-1}] \varepsilon^k + \varepsilon^{N+1} R_N^{[m]}[x, \varepsilon], \tag{5.3}$$

where the coefficients $\bar{P}_k^{[m]}[x_1, \cdots, x_{k-1}]$, $m \le k \le N$ *depending on* $(x_1, \cdots, x_{k-1})$ *are defined by the formula*

$$\bar{P}_k^{[m]}[x_1, \cdots, x_{k-1}] = \sum_{\alpha \in \bar{A}_k^{[m]}} \frac{m!}{\alpha_1! \cdots \alpha_{k-1}!} x_1^{\alpha_1} \cdots x_{k-1}^{\alpha_{k-1}},\ m \le k \le N, \tag{5.4}$$

with

$$\bar{A}_k^{[m]} = \left\{\alpha \in \mathbb{Z}_+^{k-1} : \alpha_1 + \cdots + \alpha_{k-1} = m,\ \sum_{i=1}^{k-1} i\alpha_i = k\right\}, \tag{5.5}$$

and

$$\begin{aligned} \left|R_N^{[m]}[x, \varepsilon]\right| &\le (N \|x\|_{\mathbb{R}^N})^m \sum_{k=0}^{(m-1)N-1} |\varepsilon|^k \le (m-1)N^{m+1} \|x\|_{\mathbb{R}^N}^m, \\ \|x\|_{\mathbb{R}^N} &= \max_{1 \le i \le N} |x_i|. \end{aligned} \tag{5.6}$$

Proof of Lemma 5.2**.** We rewrite the formula (5.1) as follows

$$\left(\sum_{i=1}^{N} x_i \varepsilon^i\right)^m = \sum_{k=m}^{N} P_N^{[m]}[x]_k \varepsilon^k + \sum_{k=N+1}^{mN} P_N^{[m]}[x]_k \varepsilon^k, \tag{5.7}$$

With $2 \le m \le k \le N$, $\alpha \in A_k^{[m]}(N)$, we deduce that $\alpha_i = 0$, $\forall i \in \{k, k+1, \cdots, N\}$. Indeed, if there exists $i_0 \in \{k, k+1, \cdots, N\}$ such that $\alpha_{i_0} \ge 1$, then

$$k = \sum_{i=1}^{N} i\alpha_i \ge i_0 \alpha_{i_0} \ge i_0 \ge k,$$

which leads to

$$\sum_{i=1}^{N} i\alpha_i = i_0 \alpha_{i_0} = i_0 = k.$$

So

$$i_0 = k,\ \alpha_{i_0} = 1,\ \alpha_j = 0,\ \forall j \neq i_0.$$

It follows that $2 \leq m = |\alpha| = \sum_{i=1}^{N} \alpha_i = \alpha_{i_0} = 1$. This is a contradiction. Therefore

$$\alpha_i = 0,\ \forall i \in \{k, k+1, \cdots, N\}.$$

Then, for $2 \leq m \leq k \leq N$, the set of multi-indices $A_k^{[m]}(N)$ is replaced by

$$\bar{A}_k^{[m]} = \left\{ \alpha \in \mathbb{Z}_+^{k-1} : \alpha_1 + \cdots + \alpha_{k-1} = m,\ \sum_{i=1}^{k-1} i\alpha_i = k \right\}, \tag{5.8}$$

and therefore

$$P_N^{[m]}[x]_k = \sum_{\alpha \in \bar{A}_k^{[m]}} \frac{m!}{\alpha_1! \cdots \alpha_{k-1}!} x_1^{\alpha_1} \cdots x_{k-1}^{\alpha_{k-1}} \equiv \bar{P}_k^{[m]}[x_1, \cdots, x_{k-1}],\ m \leq k \leq N. \tag{5.9}$$

On the other hand

$$\begin{aligned} \left| \varepsilon^{N+1} R_N^{[m]}[x, \varepsilon] \right| &= \left| \sum_{k=N+1}^{mN} P_N^{[m]}[x]_k \varepsilon^k \right| \\ &\leq |\varepsilon|^{N+1} \sum_{k=N+1}^{mN} \left| P_N^{[m]}[x]_k \right| |\varepsilon|^{k-N-1} \\ &\leq |\varepsilon|^{N+1} \sum_{k=N+1}^{mN} \left| P_N^{[m]}[x]_k \right|, \end{aligned} \tag{5.10}$$

and

$$\begin{aligned} \left| P_N^{[1]}[x]_k \right| &= \left| \sum_{\alpha \in A_k^{[m]}(N)} \frac{m!}{\alpha!} x^\alpha \right| \leq \sum_{\alpha \in A_k^{[m]}(N)} \frac{m!}{\alpha!} |x^\alpha| \\ &\leq \sum_{\alpha \in A_k^{[m]}(N)} \frac{m!}{\alpha!} \|x\|_{\mathbb{R}^N}^{|\alpha|} = \sum_{\alpha \in A_k^{[m]}(N)} \frac{m!}{\alpha!} \|x\|_{\mathbb{R}^N}^{m} \\ &\leq \sum_{\alpha \in \mathbb{Z}_+^N,\ |\alpha| = m} \frac{m!}{\alpha!} \|x\|_{\mathbb{R}^N}^{m} = m! \|x\|_{\mathbb{R}^N}^{m} \sum_{\alpha \in \mathbb{Z}_+^N,\ |\alpha| = m} \frac{1}{\alpha!}. \end{aligned}$$

We notice that $\sum_{\alpha \in \mathbb{Z}_+^N,\ |\alpha| = m} \frac{1}{\alpha!} = \frac{N^m}{m!}$, hence

$$\left| P_N^{[1]}[x]_k \right| \leq m! \|x\|_{\mathbb{R}^N}^{m} \sum_{\alpha \in \mathbb{Z}_+^N,\ |\alpha| = m} \frac{1}{\alpha!} = N^m \|x\|_{\mathbb{R}^N}^{m}. \tag{5.11}$$

This leads to

$$\begin{aligned} \left| \varepsilon^{N+1} R_N^{[m]}[x, \varepsilon] \right| &\leq |\varepsilon|^{N+1} \sum_{k=N+1}^{mN} \left| P_N^{[m]}[x]_k \right| \\ &\leq |\varepsilon|^{N+1} (mN - N) N^m \|x\|_{\mathbb{R}^N}^{m} \\ &= |\varepsilon|^{N+1} (m-1) N^{m+1} \|x\|_{\mathbb{R}^N}^{m}. \end{aligned} \tag{5.12}$$

From (5.7), (5.9), (5.12), we deduce that (5.3) is true. Lemma 5.2 is proved. □

By the same proof with Lemma 5.2 we have the following Lemma

Lemma 5.3. *Let* $m,\, N \in \mathbb{N} \setminus \{1\}$, $x = (x_1, \cdots, x_{2N}) \in \mathbb{R}^{2N}$, *and* $\varepsilon \in \mathbb{R}$, $|\varepsilon| \leq 1$. *Then*

$$\left(\sum_{i=1}^{2N} x_i \varepsilon^i\right)^m = \sum_{k=m}^{N} \bar{P}_k^{[m]}[x_1, \cdots, x_{k-1}]\varepsilon^k + \varepsilon^{N+1} R_{2N}^{[m]}[x, \varepsilon], \tag{5.13}$$

where the coefficients $\bar{P}_k^{[m]}[x_1, \cdots, x_{k-1}]$, $m \leq k \leq N$ *depending on* $(x_1, \cdots, x_{k-1})$ *are defined by the formula* (5.4), (5.5) *and*

$$\begin{aligned} \left|R_{2N}^{[m]}[x, \varepsilon]\right| &\leq (2m-1)2^m N^{m+1} \|x\|_{\mathbb{R}^{2N}}^m, \\ \|x\|_{\mathbb{R}^{2N}} &= \max_{1 \leq i \leq 2N} |x_i|. \end{aligned} \tag{5.14}$$

Now, we assume that

$(H_2^{(N)})$ $\mu,\, \mu_1 \in C^{N+3}([0,1] \times [0,T^*] \times \mathbb{R} \times \mathbb{R}_+^2)$,
$D_3\mu(x,t,y,z_1,z_2) \geq \mu_* > 0$, $D_3\mu_1(x,t,y,z_1,z_2) \geq 0$, for all $(x,t,z_1,z_2,z_3) \in [0,1] \times [0,T^*] \times \mathbb{R} \times \mathbb{R}_+^2$.

We also use the notations

$$\begin{aligned} \mu_\varepsilon[u](x,t) &= \mu[u](x,t) + \varepsilon\mu_1[u](x,t), \\ \mu[u](x,t) &= \mu\left(x, t, u(x,t), \|u(t)\|^2, \|u_x(t)\|^2\right), \\ \mu_1[u](x,t) &= \mu_1\left(x, t, u(x,t), \|u(t)\|^2, \|u_x(t)\|^2\right). \end{aligned}$$

Let $u_0 \in W_1(M,T)$ be a unique weak solution of problem $(\tilde{P}_0)$ (as in Theorem 3.5) corresponding to $\varepsilon = 0$, *i.e.*,

$$(\tilde{P}_0) \begin{cases} u_0'' - \lambda\Delta u_0' - \dfrac{\partial}{\partial x}\left(D_3\mu[u_0](t)u_{0x}\right) = F_0,\ 0 < x < 1,\ 0 < t < T, \\ u_0(0,t) = u_0(1,t) = 0, \\ u_0(x,0) = \tilde{u}_0(x),\ u_0'(x,0) = \tilde{u}_1(x). \end{cases}$$

Considering the sequence of weak solutions u_r, $1 \leq r \leq N$, of the following problems:

$$(\tilde{P}_r) \begin{cases} u_r'' - \lambda\Delta u_r' - \dfrac{\partial}{\partial x}\left(D_3\mu[u_0](t)\, u_{rx}(t)\right) = F_r,\ 0 < x < 1,\ 0 < t < T, \\ u_r(0,t) = u_r(1,t) = 0, \\ u_r(x,0) = u_r'(x,0) = 0, \\ u_r \in W_1(M,T), \end{cases}$$

where F_r, $1 \leq r \leq N$, are defined by the recurrence formulas

$$F_r = \begin{cases} f + \dfrac{\partial}{\partial x}\left(D_1\mu[u_0](t)\right), & r = 0, \\ \dfrac{\partial}{\partial x}\left(\bar{\Psi}_1 + u_{0x}\Psi_1\right), & r = 1, \\ \dfrac{\partial}{\partial x}\left(\bar{\Psi}_r + u_{0x}\Psi_r + \displaystyle\sum_{1 \leq i,j \leq N,\ i+j=r} u_{jx}\Psi_i\right), & 2 \leq r \leq N, \end{cases} \tag{5.15}$$

with

$$\begin{aligned} \bar{\Psi}_r &= \rho_r^{[1]}[N, D_1\mu] + \rho_{r-1}^{[1]}[N-1, D_1\mu_1], \\ \Psi_r &= \rho_r^{[1]}[N, D_3\mu] + \rho_{r-1}^{[1]}[N-1, D_3\mu_1],\ 1 \leq r \leq N, \end{aligned} \tag{5.16}$$

defined by the formulas

$$\rho_r^{[1]}[N, D_1\mu] = \begin{cases} D_1\mu[u_0], & r = 0, \\ \displaystyle\sum_{m \in \mathbb{Z}_+^3,\ 1 \leq |m| \leq r} \frac{1}{m!} D^m D_1\mu[u_0]\tilde{\Phi}_r[m, N, \vec{u}, \vec{\sigma}^{(1)}, \vec{\sigma}^{(2)}], & 1 \leq r \leq N, \end{cases} \tag{5.17}$$

in which

$$\tilde{\Phi}_r[m,N,\vec{u},\vec{\sigma}^{(1)},\vec{\sigma}^{(2)}] = \sum_{(k_1,k_2,k_3)\in\bar{A}_r(m,N)} \bar{P}_{k_1}^{[m_1]}[u_1,\cdots,u_{k_1-1}]\bar{P}_{k_2}^{[m_2]}[\sigma_1^{(1)},\cdots,\sigma_{k_2-1}^{(1)}]\bar{P}_{k_3}^{[m_3]}[\sigma_1^{(2)},\cdots,\sigma_{k_3-1}^{(2)}], \quad (5.18)$$

and

$$\bar{A}_r(m,N) = \{(k_1,k_2,k_3)\in\mathbb{Z}_+^3 : m_1\le k_1\le N, m_2\le k_2\le N, m_3\le k_3\le N, k_1+k_2+k_3=r\}, \quad (5.19)$$

$m=(m_1,\cdots,m_3)\in\mathbb{Z}_+^3$, $|m|=m_1+\cdots+m_3$, $m!=m_1!\cdots m_3!$, $D^m=D_3^{m_1}D_4^{m_2}D_5^{m_3}$, and $\vec{\sigma}^{(1)}=(\sigma_1^{(1)},\cdots,\sigma_{2N}^{(1)})$, $\vec{\sigma}^{(2)}=(\sigma_1^{(2)},\cdots,\sigma_{2N}^{(2)})$, are defined by

$$\sigma_i^{(1)} = \begin{cases} 2\langle u_0,u_1\rangle, & i=1,\\ 2\langle u_0,u_i\rangle+\sum_{j=1}^{i}\langle u_j,u_{i-j}\rangle, & 2\le i\le N,\\ \sum_{j=1}^{i}\langle u_j,u_{i-j}\rangle, & N+1\le i\le 2N,\end{cases} \quad (5.20)$$

$$\sigma_i^{(2)} = \begin{cases} 2\langle \nabla u_0,\nabla u_1\rangle, & i=1,\\ 2\langle \nabla u_0,\nabla u_i\rangle+\sum_{j=1}^{i}\langle \nabla u_j,\nabla u_{i-j}\rangle, & 2\le i\le N,\\ \sum_{j=1}^{i}\langle \nabla u_j,\nabla u_{i-j}\rangle, & N+1\le i\le 2N.\end{cases}$$

Then, we have the following lemma.

Lemma 5.4. *Let* $\rho_r^{[1]}[N,D_1\mu]=\rho_r^{[1]}[N,D_1\mu;u_0,\vec{u},\vec{\sigma}^{(1)},\vec{\sigma}^{(2)}]$, $0\le r\le N$, *be the functions defined by formulas* (5.17)-(5.20). *Let* $h=\sum_{r=0}^{N}u_r\varepsilon^r$. *Then we have*

$$D_1\mu[h] = D_1\mu[u_0]+\sum_{r=1}^{N}\rho_r^{[1]}[N,D_1\mu]\varepsilon^r+\varepsilon^{N+1}\bar{R}_N^{[1]}[D_1\mu,\varepsilon], \quad (5.21)$$

$$D_3\mu[h] = D_3\mu[u_0]+\sum_{r=1}^{N}\rho_r^{[1]}[N,D_3\mu]\varepsilon^r+\varepsilon^{N+1}\bar{R}_N^{[1]}[D_3\mu,\varepsilon],$$

with $\left\|\bar{R}_N^{[1]}[D_1\mu,\varepsilon]\right\|_{L^\infty(0,T;L^2)}+\left\|\bar{R}_N^{[1]}[D_3\mu,\varepsilon]\right\|_{L^\infty(0,T;L^2)}\le C$, *where* C *is a constant depending only on* N, T, f, $D_1\mu$, $D_3\mu$, u_r, $0\le r\le N$.

Proof of Lemma 5.4. (i) In the case of $N=1$, the proof of (5.21) is easy, hence we omit the details. We only prove the case of $N\ge 2$. Let $h=u_0+\sum_{i=1}^{N}u_i\varepsilon^i\equiv u_0+h_1$. We rewrite as below

$$\begin{aligned} D_1\mu[h] &= D_1\mu[u_0+h_1]=D_1\mu\left(x,t,h(x,t),\|h(t)\|^2,\|\nabla h(t)\|^2\right) \quad (5.22)\\ &= D_1\mu(x,t,u_0+h_1,\|u_0+h_1\|^2,\|\nabla u_0+\nabla h_1\|^2)\\ &= f(x,t,u_0+h_1,\|u_0\|^2+\xi_2,\|\nabla u_0\|^2+\xi_3),\end{aligned}$$

where $\xi_2=\|u_0+h_1\|^2-\|u_0\|^2$, $\xi_3=\|\nabla u_0+\nabla h_1\|^2-\|\nabla u_0\|^2$.

By using Taylor's expansion of the function $D_1\mu[u_0+h_1]$ around the point $[u_0]=(x,t,u_0,\|u_0\|^2,\|\nabla u_0\|^2)$ up to order $N+1$, we obtain

$$\begin{aligned} D_1\mu[u_0+h_1] &= D_1\mu[u_0]+D_3D_1\mu[u_0]h_1+D_4D_1\mu[u_0]\xi_2+D_5D_1\mu[u_0]\xi_3 \quad (5.23)\\ &\quad+\sum_{\substack{m=(m_1,\cdots,m_3)\in\mathbb{Z}_+^3,\\ 2\le|m|\le N}}\frac{1}{m!}D^mD_1\mu[u_0]h_1^{m_1}\xi_2^{m_2}\xi_3^{m_3}+R_N^{(1)}[D_1\mu,h_1,\xi_2,\xi_3],\end{aligned}$$

where

$$\begin{aligned} R_N^{(1)}[D_1\mu, h_1, \xi_2, \xi_3] &= \sum_{m\in\mathbb{Z}_+^3,\ |m|=N+1} \frac{N+1}{m!}\left(\int_0^1 (1-\theta)^N D^m D_1\hat{\mu}(\theta)d\theta\right) h_1^{m_1}\xi_2^{m_2}\xi_3^{m_3} \\ &= \varepsilon^{N+1} R_N^{(2)}[D_1\mu, h_1, \xi_2, \xi_3, \varepsilon], \\ D^m D_1\hat{\mu}(\theta) &= D^m D_1\mu(x, t, u_0+\theta h_1, \|u_0\|^2+\theta\xi_2, \|\nabla u_0\|^2+\theta\xi_3). \end{aligned} \tag{5.24}$$

By the formula (5.3), it follows that

$$h_1^{m_1} = \left(\sum_{i=1}^N u_i\varepsilon^i\right)^{m_1} = \sum_{k=m_1}^N \bar{P}_k^{[m_1]}[h_1]\varepsilon^k + \varepsilon^{N+1}R_N^{[m_1]}[\vec{u}, \varepsilon], \tag{5.25}$$

and

$$\left|R_N^{[m_1]}[\vec{u}, \varepsilon]\right| \le (m_1-1)N^{m_1+1}\|\vec{u}\|_{\mathbb{R}^N}^{m_1}, \ \ |\varepsilon| \le 1, \tag{5.26}$$

where $\vec{u} = (u_1, \cdots, u_N)$, $\|\vec{u}\|_{\mathbb{R}^N} = \max_{1\le i\le N}|u_i|$, and in the formula (5.25) we shorten $\bar{P}_k^{[m_1]}[h_1]$ instead of $\bar{P}_k^{[m_1]}[u_1, \cdots, u_{k-1}]$.

On the other hand,

$$\begin{aligned} \xi_2 &= \|u_0+h_1\|^2 - \|u_0\|^2 = 2\langle u_0, h_1\rangle + \|h_1\|^2 \equiv \sum_{i=1}^{2N}\sigma_i^{(1)}\varepsilon^i, \\ \xi_3 &= \|\nabla u_0+\nabla h_1\|^2 - \|\nabla u_0\|^2 = 2\langle \nabla u_0, \nabla h_1\rangle + \|\nabla h_1\|^2 \equiv \sum_{i=1}^{2N}\sigma_i^{(2)}\varepsilon^i, \end{aligned} \tag{5.27}$$

with $\sigma_i^{(1)}$, $\sigma_i^{(2)}$, $1 \le i \le 2N$ are defined by (5.20). By the formula (5.13), it follows from (5.27) that

$$\begin{aligned} \xi_2^{m_2} &= \left(\sum_{i=1}^{2N}\sigma_i^{(1)}\varepsilon^i\right)^{m_2} = \sum_{k=m_2}^N \bar{P}_k^{[m_2]}[\xi_2]\varepsilon^k + \varepsilon^{N+1}R_{2N}^{[m_2]}[\vec{\sigma}^{(1)}, \varepsilon], \\ \xi_3^{m_3} &= \left(\sum_{i=1}^{2N}\sigma_i^{(2)}\varepsilon^i\right)^{m_3} = \sum_{k=m_3}^N \bar{P}_k^{[m_3]}[\xi_3]\varepsilon^k + \varepsilon^{N+1}R_{2N}^{[m_3]}[\vec{\sigma}^{(2)}, \varepsilon], \end{aligned} \tag{5.28}$$

where

$$\begin{aligned} \bar{P}_k^{[m_2]}[\xi_2] &= \bar{P}_k^{[m_2]}[\sigma_1^{(1)}, \cdots, \sigma_{k-1}^{(1)}] = \sum_{\alpha\in\bar{A}_k^{[m_2]}} \frac{m_2!}{\alpha_1!\cdots\alpha_{k-1}!}\left(\sigma_1^{(1)}\right)^{\alpha_1}\cdots\left(\sigma_{k-1}^{(1)}\right)^{\alpha_{k-1}}, \\ \bar{P}_k^{[m_3]}[\xi_3] &= \bar{P}_k^{[m_3]}[\sigma_1^{(2)}, \cdots, \sigma_{k-1}^{(2)}] = \sum_{\alpha\in\bar{A}_k^{[m_3]}} \frac{m_3!}{\alpha_1!\cdots\alpha_{k-1}!}\left(\sigma_1^{(2)}\right)^{\alpha_1}\cdots\left(\sigma_{k-1}^{(2)}\right)^{\alpha_{k-1}}, \\ \bar{A}_k^{[m_s]} &= \left\{\alpha\in\mathbb{Z}_+^k : \alpha_1+\cdots+\alpha_{k-1} = m_s, \ \ \sum_{i=1}^{k-1} i\alpha_i = k\right\}, \ s = 2, 3, \end{aligned}$$

and

$$\begin{aligned} \left|R_{2N}^{[m_2]}[\vec{\sigma}^{(1)}, \varepsilon]\right| &\le (2m_2-1)2^{m_2}N^{m_2+1}\left\|\vec{\sigma}^{(1)}\right\|_{\mathbb{R}^{2N}}^{m_2}, \ \ |\varepsilon| \le 1, \\ \left|R_{2N}^{[m_3]}[\vec{\sigma}^{(2)}, \varepsilon]\right| &\le (2m_3-1)2^{m_3}N^{m_3+1}\left\|\vec{\sigma}^{(2)}\right\|_{\mathbb{R}^{2N}}^{m_3}, \ \ |\varepsilon| \le 1, \\ \vec{\sigma}^{(r)} &= (\sigma_1^{(r)}, \cdots, \sigma_{2N}^{(r)}), \\ \left\|\vec{\sigma}^{(r)}\right\|_{\mathbb{R}^{2N}} &= \max_{1\le i\le 2N}\left|\sigma_i^{(r)}\right|, \ r = 1, 2. \end{aligned} \tag{5.29}$$

Therefore, it follows from (5.25), (5.28), that

$$
\begin{aligned}
h_1^{m_1}\xi_2^{m_2}\xi_3^{m_3} &= \left(\sum_{k_1=m_1}^{N} \bar{P}_{k_1}^{[m_1]}[h_1]\varepsilon^{k_1}\right)\left(\sum_{k_2=m_2}^{N} \bar{P}_{k_2}^{[m_2]}[\xi_2]\varepsilon^{k_2}\right)\left(\sum_{k_3=m_3}^{N} \bar{P}_{k_3}^{[m_3]}[\xi_3]\varepsilon^{k_3}\right) + \varepsilon^{N+1}R_N^{[m]}[\vec{u},\vec{\sigma}^{(1)},\vec{\sigma}^{(2)},\varepsilon] \\
&= \sum_{k_1=m_1}^{N}\sum_{k_2=m_2}^{N}\sum_{k_3=m_3}^{N} \bar{P}_{k_1}^{[m_1]}[h_1]\bar{P}_{k_2}^{[m_2]}[\xi_2]\bar{P}_{k_3}^{[m_3]}[\xi_3]\varepsilon^{k_1+k_2+k_3} + \varepsilon^{N+1}R_N^{[m]}[\vec{u},\vec{\sigma}^{(1)},\vec{\sigma}^{(2)},\varepsilon] \\
&= \sum_{r=|m|}^{3N}\left(\sum_{(k_1,k_2,k_3)\in\bar{A}_r(m,N)} \bar{P}_{k_1}^{[m_1]}[h_1]\bar{P}_{k_2}^{[m_2]}[\xi_2]\bar{P}_{k_3}^{[m_3]}[\xi_3]\right)\varepsilon^{r} + \varepsilon^{N+1}R_N^{[m]}[\vec{u},\vec{\sigma}^{(1)},\vec{\sigma}^{(2)},\varepsilon] \\
&= \sum_{r=|m|}^{3N} \tilde{\Phi}_r[m,N,\vec{u},\vec{\sigma}^{(1)},\vec{\sigma}^{(2)}]\varepsilon^r + \varepsilon^{N+1}R_N^{[m]}[\vec{u},\vec{\sigma}^{(1)},\vec{\sigma}^{(2)},\varepsilon] \\
&= \sum_{r=|m|}^{N} \tilde{\Phi}_r[m,N,\vec{u},\vec{\sigma}^{(1)},\vec{\sigma}^{(2)}]\varepsilon^r + \varepsilon^{N+1}\left(\sum_{r=N+1}^{3N} \tilde{\Phi}_r[m,N,\vec{u},\vec{\sigma}^{(1)},\vec{\sigma}^{(2)}]\varepsilon^{r-N-1} + R_N^{[m]}[\vec{u},\vec{\sigma}^{(1)},\vec{\sigma}^{(2)},\varepsilon]\right) \\
&= \sum_{r=|m|}^{N} \tilde{\Phi}_r[m,N,\vec{u},\vec{\sigma}^{(1)},\vec{\sigma}^{(2)}]\varepsilon^r + \varepsilon^{N+1}R_N^{[*]}[m,\vec{u},\vec{\sigma}^{(1)},\vec{\sigma}^{(2)},\varepsilon], \qquad (5.30)
\end{aligned}
$$

where

$$
R_N^{[*]}[m,\vec{u},\vec{\sigma}^{(1)},\vec{\sigma}^{(2)},\varepsilon] = \sum_{r=N+1}^{3N} \tilde{\Phi}_r[m,N,\vec{u},\vec{\sigma}^{(1)},\vec{\sigma}^{(2)}]\varepsilon^{r-N-1} + R_N^{[m]}[\vec{u},\vec{\sigma}^{(1)},\vec{\sigma}^{(2)},\varepsilon], \qquad (5.31)
$$

$$
\begin{aligned}
R_N^{[m]}[\vec{u},\vec{\sigma}^{(1)},\vec{\sigma}^{(2)},\varepsilon] =& R_N^{[m_1]}[\vec{u},\varepsilon]\left(\sum_{k=m_2}^{N} \bar{P}_k^{[m_2]}[\xi_2]\varepsilon^k\right)\left(\sum_{k=m_3}^{N} \bar{P}_k^{[m_3]}[\xi_3]\varepsilon^k\right) \\
&+ R_{2N}^{[m_2]}[\vec{\sigma}^{(1)},\varepsilon]\left(\sum_{i=1}^{N} u_i\varepsilon^i\right)^{m_1}\left(\sum_{k=m_3}^{N} \bar{P}_k^{[m_3]}[\xi_3]\varepsilon^k\right) + R_{2N}^{[m_3]}[\vec{\sigma}^{(2)},\varepsilon]\left(\sum_{i=1}^{N} u_i\varepsilon^i\right)^{m_1}\left(\sum_{i=1}^{2N} \sigma_i^{(1)}\varepsilon^i\right)^{m_2}
\end{aligned}
$$

$m=(m_1,\cdots,m_3)\in\mathbb{Z}_+^3$, $|m|=m_1+\cdots+m_3$, $m!=m_1!\cdots m_3!$, $D^m=D_3^{m_1}D_4^{m_2}D_5^{m_3}$, where $\tilde{\Phi}_r[m,N,\vec{u},\vec{\sigma}^{(1)},\vec{\sigma}^{(2)}]$ and $\bar{A}_r(m,N)$ are defined by (5.18) - (5.20). Hence, we deduce from (5.23), (5.24), (5.30) that

$$
\begin{aligned}
D_1\mu[u_0+h_1] =& D_1\mu[u_0] + \sum_{\substack{1\le|m|\le N \\ m=(m_1,\cdots,m_3)\in\mathbb{Z}_+^3}} \frac{1}{m!}D^m D_1\mu[u_0]h_1^{m_1}\xi_2^{m_2}\xi_3^{m_3} + \varepsilon^{N+1}R_N^{(2)}[D_1\mu,h_1,\xi_2,\xi_3,\varepsilon] \\
=& D_1\mu[u_0] + \sum_{m\in\mathbb{Z}_+^3,\ 1\le|m|\le N} \frac{1}{m!}D^m D_1\mu[u_0]\sum_{r=|m|}^{N} \tilde{\Phi}_r[m,N,\vec{u},\vec{\sigma}^{(1)},\vec{\sigma}^{(2)}]\varepsilon^r \\
&+ \varepsilon^{N+1}\sum_{m\in\mathbb{Z}_+^3,\ 1\le|m|\le N} \frac{1}{m!}D^m D_1\mu[u_0]R_N^{[*]}[m,\vec{u},\vec{\sigma}^{(1)},\vec{\sigma}^{(2)},\varepsilon] + \varepsilon^{N+1}R_N^{(2)}[D_1\mu,h_1,\xi_2,\xi_3,\varepsilon] \\
=& D_1\mu[u_0] + \sum_{m\in\mathbb{Z}_+^3,\ 1\le|m|\le N} \frac{1}{m!}D^m D_1\mu[u_0]\sum_{r=|m|}^{N} \tilde{\Phi}_r[m,N,\vec{u},\vec{\sigma}^{(1)},\vec{\sigma}^{(2)}]\varepsilon^r \\
&+ \varepsilon^{N+1}\left[\sum_{m\in\mathbb{Z}_+^3,\ 1\le|m|\le N} \frac{1}{m!}D^m D_1\mu[u_0]R_N^{[*]}[m,\vec{u},\vec{\sigma}^{(1)},\vec{\sigma}^{(2)},\varepsilon] + R_N^{(2)}[D_1\mu,h_1,\xi_2,\xi_3,\varepsilon]\right] \\
=& D_1\mu[u_0] + \sum_{r=1}^{N}\left(\sum_{1\le|m|\le r} \frac{1}{m!}D^m D_1\mu[u_0]\tilde{\Phi}_r[m,N,\vec{u},\vec{\sigma}^{(1)},\vec{\sigma}^{(2)}]\right)\varepsilon^r \\
&+ \varepsilon^{N+1}\left[\sum_{m\in\mathbb{Z}_+^3,\ 1\le|m|\le N} \frac{1}{m!}D^m D_1\mu[u_0]R_N^{[*]}[m,\vec{u},\vec{\sigma}^{(1)},\vec{\sigma}^{(2)},\varepsilon] + R_N^{(2)}[D_1\mu,h_1,\xi_2,\xi_3,\varepsilon]\right]
\end{aligned}
$$

$$=D_1\mu[u_0]+\sum_{r=1}^{N}\left(\sum_{1\le|m|\le r}\frac{1}{m!}D^m D_1\mu[u_0]\tilde{\Phi}_r[m,N,\vec{u},\vec{\sigma}^{(1)},\vec{\sigma}^{(2)}]\right)\varepsilon^r+\varepsilon^{N+1}\bar{R}_N^{[1]}[D_1\mu,\varepsilon]$$

$$=\sum_{r=0}^{N}\rho_r^{[1]}[N,D_1\mu]\varepsilon^r+\varepsilon^{N+1}\bar{R}_N^{[1]}[D_1\mu,\varepsilon], \tag{5.32}$$

where

$$\bar{R}_N^{[1]}[D_1\mu,\varepsilon] = \sum_{m\in\mathbb{Z}_+^3,\ 1\le|m|\le N}\frac{1}{m!}D^m D_1\mu[u_0]R_N^{[*]}[m,\vec{u},\vec{\sigma}^{(1)},\vec{\sigma}^{(2)},\varepsilon]+R_N^{(2)}[D_1\mu,h_1,\xi_2,\xi_3,\varepsilon], \tag{5.33}$$

with $\rho_r^{[1]}[N,D_1\mu]=\rho_r^{[1]}[N,D_1\mu;u_0,\vec{u},\vec{\sigma}^{(1)},\vec{\sigma}^{(2)}]$, $0\le r\le N$, are defined by (5.17)-(5.20), and

$$\left\|\bar{R}_N^{[1]}[D_1\mu,\varepsilon]\right\|_{L^\infty(0,T;L^2)}\le C, \tag{5.34}$$

where C is a constant depending only on N, T, $D_1\mu$, u_r, $r=0,1,\cdots,N$. Hence, the formula $(5.21)_1$ is proved.

(ii) In the case of $D_3\mu[h](x,t)$, applying the formulas (5.17)-(5.20) with $D_1\mu=D_3\mu$, we obtain formulas

$$D_3\mu[h]=\sum_{r=0}^{N}\rho_r^{[1]}[N,D_3\mu]\varepsilon^r+\varepsilon^{N+1}\bar{R}_N^{[1]}[D_3\mu,\varepsilon], \tag{5.35}$$

and $\left\|\bar{R}_N^{[1]}[D_3\mu,\varepsilon]\right\|_{L^\infty(0,T;L^2)}\le C$, where C is a constant depending only on N, T, $D_3\mu$, u_r, $r=0,1,\cdots,N$.

Therefore the formula $(5.21)_2$ is proved. This completes the proof of the lemma 5.4. □

Remark 5.5. Lemma 5.4 is a generalization of a formula contained in [17] (formula (4.44), p. 263) and it is useful to obtain the following Lemma 5.5. These Lemmas are the key to obtain the asymptotic expansion of the weak solution $u=u_\varepsilon$ of order $N+1$ in a small parameter ε.

Let $u=u_\varepsilon\in W_1(M,T)$ be a unique weak solution of the problem (P_ε). Then $v=u-\sum_{r=0}^{N}u_r\varepsilon^r\equiv u-h$ satisfies the problem

$$\begin{cases} v''-\lambda\Delta v'-\dfrac{\partial}{\partial x}\left[D_3\mu_\varepsilon[v+h](t)v_x\right]=\dfrac{\partial}{\partial x}\left[\left(D_3\mu_\varepsilon[v+h](t)-D_3\mu_\varepsilon[h](t)\right)h_x\right]+\dfrac{\partial}{\partial x}\left[D_1\mu_\varepsilon[v+h](t)-D_1\mu_\varepsilon[h](t)\right]\\ +E_\varepsilon(x,t),\qquad 0<x<1,\ 0<t<T,\\ v(0,t)=v(1,t)=0,\\ v(x,0)=v'(x,0)=0, \end{cases} \tag{5.36}$$

where

$$E_\varepsilon(x,t)=f+\frac{\partial}{\partial x}\left[D_1\mu_\varepsilon[h](t)-D_1\mu[u_0](t)\right]+\frac{\partial}{\partial x}\left[\left(D_3\mu_\varepsilon[h](t)-D_3\mu[u_0](t)\right)h_x\right]+\frac{\partial}{\partial x}\left[D_1\mu[u_0](t)\right]-\sum_{r=0}^{N}F_r\varepsilon^r. \tag{5.37}$$

Lemma 5.6. *Under the assumptions* (H_1), $(H_2^{(N)})$ *and* (H_3), *there exists a constant* $\bar{C}_*$ *such that*

$$\|E_\varepsilon\|_{L^\infty(0,T;L^2)}\le\bar{C}_*\varepsilon^{N+1}, \tag{5.38}$$

where $\bar{C}_*$ *is a constant depending only on* N, T, μ, μ_1, u_r, $0\le r\le N$.

Proof . In the case of $N=1$, the proof is easy. The details are omitted. We only consider $N\ge2$.

By using formulas $(5.21)_1$ for the functions $D_1\mu[h]$ and $D_1\mu_1[h]$

$$D_1\mu[h]-D_1\mu[u_0] = \sum_{r=1}^{N}\rho_r^{[1]}[N,D_1\mu]\varepsilon^r+\varepsilon^{N+1}\bar{R}_N^{[1]}[D_1\mu,\varepsilon], \tag{5.39}$$

$$D_1\mu_1[h] = \sum_{r=0}^{N-1}\rho_r^{[1]}[N-1,D_1\mu_1]\varepsilon^r+\varepsilon^N\bar{R}_{N-1}^{[1]}[D_1\mu_1,\varepsilon].$$

We rewrite $\varepsilon D_1\mu_1[h]$ as follows

$$\varepsilon D_1\mu_1[h] = \sum_{r=1}^{N} \rho_{r-1}^{[1]}[N-1, D_1\mu_1]\varepsilon^r + \varepsilon^{N+1}\bar{R}_{N-1}^{[1]}[D_1\mu_1, \varepsilon]. \tag{5.40}$$

Hence, we deduce from $(5.39)_1$ and (5.40), that

$$\begin{aligned} D_1\mu_\varepsilon[h](t) - D_1\mu[u_0](t) =& D_1\mu[h](t) - D_1\mu[u_0](t) + \varepsilon D_1\mu_1[h] \\ =& \sum_{r=1}^{N} \rho_r^{[1]}[N, D_1\mu]\varepsilon^r + \varepsilon^{N+1}\bar{R}_N^{[1]}[D_1\mu, \varepsilon] + \sum_{r=1}^{N} \rho_{r-1}^{[1]}[N-1, D_1\mu_1]\varepsilon^r + \varepsilon^{N+1}\bar{R}_{N-1}^{[1]}[D_1\mu_1, \varepsilon] \\ =& \sum_{r=1}^{N} \left(\rho_r^{[1]}[N, D_1\mu] + \rho_{r-1}^{[1]}[N-1, D_1\mu_1]\right)\varepsilon^r + \varepsilon^{N+1}\left(\bar{R}_N^{[1]}[D_1\mu, \varepsilon] + \bar{R}_{N-1}^{[1]}[D_1\mu_1, \varepsilon]\right) \\ =& \sum_{r=1}^{N} \left(\rho_r^{[1]}[N, D_1\mu] + \rho_{r-1}^{[1]}[N-1, D_1\mu_1]\right)\varepsilon^r + \varepsilon^{N+1}\bar{R}_N^{(1)}[D_1\mu, D_1\mu_1, \varepsilon], \end{aligned} \tag{5.41}$$

where

$$\bar{R}_N^{(1)}[D_1\mu, D_1\mu_1, \varepsilon] = \bar{R}_N^{[1]}[D_1\mu, \varepsilon] + \bar{R}_{N-1}^{[1]}[D_1\mu_1, \varepsilon] \tag{5.42}$$

is bounded in $L^\infty(0,T;L^2)$ by a constant depending only on N, T, $D_1\mu$, $D_1\mu_1$, u_r, $0 \le r \le N$. This implies

$$\begin{aligned} \frac{\partial}{\partial x}\left[\ D_1\mu_\varepsilon[h](t) - D_1\mu[u_0](t)\ \right] &= \sum_{r=1}^{N} \frac{\partial}{\partial x}\left(\rho_r^{[1]}[N, D_1\mu] + \rho_{r-1}^{[1]}[N-1, D_1\mu_1]\right)\varepsilon^r + \varepsilon^{N+1}\frac{\partial}{\partial x}\bar{R}_N^{(1)}[D_1\mu, D_1\mu_1, \varepsilon] \\ &= \sum_{r=1}^{N} \frac{\partial}{\partial x}\bar{\Psi}_r\varepsilon^r + \varepsilon^{N+1}\frac{\partial}{\partial x}\bar{R}_N^{(1)}[D_1\mu, D_1\mu_1, \varepsilon], \end{aligned} \tag{5.43}$$

where $\bar{\Psi}_r$, $1 \le r \le N$ are defined by $(5.16)_1$. Similarly,

$$\begin{aligned} D_3\mu_\varepsilon[h](t) - D_3\mu[u_0](t) &= \sum_{r=1}^{N} \left(\rho_r^{[1]}[N, D_3\mu] + \rho_{r-1}^{[1]}[N-1, D_3\mu_1]\right)\varepsilon^r + \varepsilon^{N+1}\bar{R}_N^{(1)}[D_3\mu, D_3\mu_1, \varepsilon] \\ &= \sum_{r=1}^{N} \Psi_r\varepsilon^r + \varepsilon^{N+1}\bar{R}_N^{(1)}[D_3\mu, D_3\mu_1, \varepsilon], \end{aligned} \tag{5.44}$$

where Ψ_r, $1 \le r \le N$ are defined by $(5.16)_2$. Hence

$$\begin{aligned} \left(D_3\mu_\varepsilon[h](t) - D_3\mu[u_0](t)\right)h_x =& \left(\sum_{r=1}^{N} \Psi_r\varepsilon^r\right)\left(\sum_{r=0}^{N} u_{rx}\varepsilon^r\right) + \varepsilon^{N+1}h_x\bar{R}_N^{(1)}[D_3\mu, D_3\mu_1, \varepsilon] \\ =& \sum_{r=1}^{N} \left(u_{0x}\Psi_r\right)\varepsilon^r + \left(\sum_{r=1}^{N} \Psi_r\varepsilon^r\right)\left(\sum_{r=1}^{N} u_{rx}\varepsilon^r\right) + \varepsilon^{N+1}h_x\bar{R}_N^{(1)}[D_3\mu, D_3\mu_1, \varepsilon] \\ =& \sum_{r=1}^{N} \left(u_{0x}\Psi_r\right)\varepsilon^r + \sum_{r=2}^{2N}\left(\sum_{1\le i,j\le N,\ i+j=r} u_{jx}\Psi_i\right)\varepsilon^r + \varepsilon^{N+1}h_x\bar{R}_N^{(1)}[D_3\mu, D_3\mu_1, \varepsilon] \\ =& \sum_{r=1}^{N} \left(u_{0x}\Psi_r\right)\varepsilon^r + \sum_{r=2}^{N}\left(\sum_{1\le i,j\le N,\ i+j=r} u_{jx}\Psi_i\right)\varepsilon^r + \sum_{r=N+1}^{2N}\left(\sum_{1\le i,j\le N,\ i+j=r} u_{jx}\Psi_i\right)\varepsilon^r \\ &+ \varepsilon^{N+1}h_x\bar{R}_N^{(1)}[D_3\mu, D_3\mu_1, \varepsilon] \end{aligned}$$

$$
\begin{aligned}
&= (u_{0x}\Psi_1)\,\varepsilon + \sum_{r=2}^{N}\left(u_{0x}\Psi_r + \sum_{1\le i,j\le N,\ i+j=r} u_{jx}\Psi_i\right)\varepsilon^r \\
&\quad + \varepsilon^{N+1}\left[\sum_{r=N+1}^{2N}\left(\sum_{1\le i,j\le N,\ i+j=r} u_{jx}\Psi_i\right)\varepsilon^{r-N-1} + h_x \bar{R}_N^{(1)}[D_3\mu, D_3\mu_1,\varepsilon]\right] \\
&= (u_{0x}\Psi_1)\,\varepsilon + \sum_{r=2}^{N}\left(u_{0x}\Psi_r + \sum_{1\le i,j\le N,\ i+j=r} u_{jx}\Psi_i\right)\varepsilon^r \\
&\quad + \varepsilon^{N+1}\bar{R}_N^{(2)}[D_3\mu, D_3\mu_1,\Psi_1,\cdots,\Psi_N,u_0,\cdots,u_N,\varepsilon],
\end{aligned}
$$

where

$$
\bar{R}_N^{(2)}[D_3\mu, D_3\mu_1,\Psi_1,\cdots,\Psi_N,u_0,\cdots,u_N,\varepsilon] = \sum_{r=N+1}^{2N}\left(\sum_{1\le i,j\le N,\ i+j=r} u_{jx}\Psi_i\right)\varepsilon^{r-N-1} + h_x\bar{R}_N^{(1)}[D_3\mu, D_3\mu_1,\varepsilon]. \tag{5.45}
$$

We deduce that

$$
\begin{aligned}
\frac{\partial}{\partial x}\left[(D_3\mu_\varepsilon[h](t) - D_3\mu[u_0](t))\,h_x\right] \;=\; & \frac{\partial}{\partial x}(u_{0x}\Psi_1)\,\varepsilon + \sum_{r=2}^{N}\frac{\partial}{\partial x}\left(u_{0x}\Psi_r + \sum_{1\le i,j\le N,\ i+j=r} u_{jx}\Psi_i\right)\varepsilon^r \\
& + \varepsilon^{N+1}\frac{\partial}{\partial x}\bar{R}_N^{(2)}[D_3\mu, D_3\mu_1,\Psi_1,\cdots,\Psi_N,u_0,\cdots,u_N,\varepsilon].
\end{aligned} \tag{5.46}
$$

Combining (5.15), (5.37), (5.43), and (5.46), we then obtain

$$
\begin{aligned}
E_\varepsilon(x,t) =& f + \frac{\partial}{\partial x}\left[D_1\mu_\varepsilon[h](t) - D_1\mu[u_0](t)\right] + \frac{\partial}{\partial x}\left[(D_3\mu_\varepsilon[h](t) - D_3\mu[u_0](t))\,h_x\right] + \frac{\partial}{\partial x}\left[D_1\mu[u_0](t)\right] - \sum_{r=0}^{N} F_r\varepsilon^r \\
=& f + \sum_{r=1}^{N}\frac{\partial}{\partial x}\left(\bar{\Psi}_r\right)\varepsilon^r + \varepsilon^{N+1}\frac{\partial}{\partial x}\bar{R}_N^{(1)}[D_1\mu, D_1\mu_1,\varepsilon] \\
& + \frac{\partial}{\partial x}(u_{0x}\Psi_1)\,\varepsilon + \sum_{r=2}^{N}\frac{\partial}{\partial x}\left(u_{0x}\Psi_r + \sum_{1\le i,j\le N,\ i+j=r} u_{jx}\Psi_i\right)\varepsilon^r \\
& + \varepsilon^{N+1}\frac{\partial}{\partial x}\bar{R}_N^{(2)}[D_3\mu, D_3\mu_1,\Psi_1,\cdots,\Psi_N,u_0,\cdots,u_N,\varepsilon] + \frac{\partial}{\partial x}\left[D_1\mu[u_0](t)\right] - \sum_{r=0}^{N} F_r\varepsilon^r \\
=& \left(f + \frac{\partial}{\partial x}\left[D_1\mu[u_0](t)\right] - F_0\right) + \left(\frac{\partial}{\partial x}\left[\bar{\Psi}_1 + u_{0x}\Psi_1\right] - F_1\right)\varepsilon \\
& + \sum_{r=2}^{N}\left[\frac{\partial}{\partial x}\left(\bar{\Psi}_r + u_{0x}\Psi_r + \sum_{1\le i,j\le N,\ i+j=r} u_{jx}\Psi_i\right) - F_r\right]\varepsilon^r \\
& + \varepsilon^{N+1}\frac{\partial}{\partial x}\left[\bar{R}_N^{(1)}[D_1\mu, D_1\mu_1,\varepsilon] + \bar{R}_N^{(2)}[D_3\mu, D_3\mu_1,\Psi_1,\cdots,\Psi_N,u_0,\cdots,u_N,\varepsilon]\right] \\
=& \varepsilon^{N+1}\frac{\partial}{\partial x}\bar{R}_N^{(3)}[\mu,\mu_1,\Psi_1,\cdots,\Psi_N,u_0,\cdots,u_N,\varepsilon],
\end{aligned}
$$

where

$$
\bar{R}_N^{(3)}[\mu,\mu_1,\Psi_1,\cdots,\Psi_N,u_0,\cdots,u_N,\varepsilon] = \bar{R}_N^{(1)}[D_1\mu, D_1\mu_1,\varepsilon] + \bar{R}_N^{(2)}[D_3\mu, D_3\mu_1,\Psi_1,\cdots,\Psi_N,u_0,\cdots,u_N,\varepsilon]. \tag{5.47}
$$

By the functions $u_r \in W_1(M,T)$, $0 \le r \le N$, we obtain from (5.42), (5.45) and (5.47) that

$$
\|E_\varepsilon\|_{L^\infty(0,T;L^2)} \le \bar{C}_*\varepsilon^{N+1}, \tag{5.48}
$$

where $\bar{C}_*$ is a constant depending only on N, T, μ, μ_1, u_r, $0 \leq r \leq N$.

This completes thee proof. □

Now, we estimate $v = u - \sum_{r=0}^{N} u_r \varepsilon^r$. By multiplying the two sides of (5.36) by v', we verify without difficulty that

$$\begin{aligned}
\lambda_* \bar{S}(t) + \lambda \int_0^t \|v_x'(s)\|^2 ds \leq & S(t) + \lambda \int_0^t \|v_x'(s)\|^2 ds \\
= & \int_0^t ds \int_0^1 b_\varepsilon'(x,s) v_x^2(x,s) dx + 2\int_0^t \langle E_\varepsilon(s), v'(s)\rangle ds \\
& - 2\int_0^t \langle D_1\mu_\varepsilon[v+h](s) - D_1\mu_\varepsilon[h](s), v_x'(s)\rangle ds \\
& - 2\int_0^t \langle (D_3\mu_\varepsilon[v+h](s) - D_3\mu_\varepsilon[h](s))\, h_x(s), v_x'(s)\rangle ds \\
= & J_1 + J_2 + J_3 + J_4,
\end{aligned} \tag{5.49}$$

where $\lambda_* = \min\{1, \lambda, \mu_*\}$ and

$$\begin{cases}
S(t) = \|v'(t)\|^2 + \left\|\sqrt{b_\varepsilon(t)} v_x(t)\right\|^2 + \lambda \int_0^t \|v_x'(s)\|^2 ds, \\
\bar{S}(t) = \|v'(t)\|^2 + \|v_x(t)\|^2 + \int_0^t \|v_x'(s)\|^2 ds, \\
b_\varepsilon(x,t) = D_3\mu_\varepsilon[v+h](t).
\end{cases} \tag{5.50}$$

We estimate the integrals on the right-hand side of (5.49) as follows.

Estimating J_1. Note that

$$\begin{aligned}
b_\varepsilon'(x,t) = & D_2D_3\mu_\varepsilon[v+h](t) + D_3^3\mu_\varepsilon[v+h](t)\,(v'(t)+h'(t)) + 2D_4D_3\mu_\varepsilon[v+h](t)\langle v(t)+h(t), v'(t)+h'(t)\rangle \\
& + 2D_5D_3\mu_\varepsilon[v+h](t)\langle v_x(t)+h_x(t), v_x'(t)+h_x'(t)\rangle,
\end{aligned}$$

and

$$\|v_x(t) + h_x(t)\| \leq \|v_x(t)\| + \sum_{r=0}^{N} \varepsilon^r \|u_{rx}(t)\| \leq (N+2)M = M_*.$$

Hence

$$\begin{aligned}
|b_\varepsilon'(x,t)| \leq & K_{M_*}(\mu,\mu_1)\,[1 + \|v_x'(t)+h_x'(t)\| + 4\|v_x(t)+h_x(t)\|\,\|v_x'(t)+h_x'(t)\|] \\
\leq & K_{M_*}(\mu,\mu_1)\left(1 + M_* + 4M_*^2\right) \equiv \tilde{b}_{M_*},
\end{aligned} \tag{5.51}$$

where $K_{M_*}(\mu,\mu_1) = K_{M_*}(\mu) + K_{M_*}(\mu_1)$. We deduce from (5.51), the term J_1 is estimated as follows

$$J_1 = \int_0^t ds \int_0^1 b_\varepsilon'(x,s) v_x^2(x,s) dx \leq \tilde{b}_{M_*} \int_0^t \bar{S}(s) ds. \tag{5.52}$$

Estimating J_2. By (5.38), (5.50), we obtain

$$J_2 = 2\int_0^t \langle E_\varepsilon(s), v'(s)\rangle ds \leq T\bar{C}_*^2 \varepsilon^{2N+2} + \int_0^t \bar{S}(s) ds. \tag{5.53}$$

Estimating J_3. We have

$$\begin{aligned}
|D_1\mu_\varepsilon[v+h](x,t) - D_1\mu_\varepsilon[h](x,t)| \leq & K_{M_*}(\mu,\mu_1)\left[|v(x,t)| + \left|\|v+h\|^2 - \|h\|^2\right| + \left|\|v_x+h_x\|^2 - \|h_x\|^2\right|\,\right] \\
\leq & K_{M_*}(\mu,\mu_1)\,(1+4M_*)\,\|v_x(t)\| \leq K_{M_*}(\mu,\mu_1)\,(1+4M_*)\sqrt{\bar{S}(t)}.
\end{aligned} \tag{5.54}$$

Using the inequality $2ab \leq \beta a^2 + \frac{1}{\beta}b^2$, $\forall a, b \in \mathbb{R}$, $\forall \beta > 0$, with $\beta = \frac{\lambda}{2}$, and (5.50), (5.54), the term J_3 is estimated as follows

$$\begin{aligned} J_3 &= -2\int_0^t \langle D_1\mu_\varepsilon[v+h](s) - D_1\mu_\varepsilon[h](s), v_x'(s)\rangle ds \\ &\leq 2K_{M_*}(\mu,\mu_1)(1+4M_*)\int_0^t \sqrt{\bar{S}(s)}\,\|v_x'(s)\|\,ds \\ &\leq \frac{2}{\lambda}K_{M_*}^2(\mu,\mu_1)(1+4M_*)^2\int_0^t \bar{S}(s)ds + \frac{\lambda}{2}\int_0^t \|v_x'(s)\|^2 ds. \end{aligned} \tag{5.55}$$

Estimating J_4. Similarly

$$\begin{aligned} J_4 &= -2\int_0^t \langle (D_3\mu_\varepsilon[v+h](s) - D_3\mu_\varepsilon[h](s))\,h_x(s), v_x'(s)\rangle ds \\ &\leq 2K_{M_*}(\mu,\mu_1)(1+4M_*)\,M_*\int_0^t \sqrt{\bar{S}(s)}\,\|v_x'(s)\|\,ds \\ &\leq \frac{2}{\lambda}K_{M_*}^2(\mu,\mu_1)(1+4M_*)^2 M_*^2\int_0^t \bar{S}(s)ds + \frac{\lambda}{2}\int_0^t \|v_x'(s)\|^2 ds. \end{aligned} \tag{5.56}$$

Combining (5.49), (5.50), (5.52), (5.53), (5.55) and (5.56), we then obtain

$$\bar{S}(t) \leq T\zeta_1(M)\varepsilon^{2N+2} + \zeta_2(M)\int_0^t \bar{S}(s)ds, \tag{5.57}$$

where $\zeta_1(M) = \frac{1}{\lambda_*}\bar{C}_*^2$, $\zeta_2(M) = \frac{1}{\lambda_*}\left[1 + \tilde{b}_{M_*} + \frac{2}{\lambda}K_{M_*}^2(\mu,\mu_1)(1+4M_*)^2\left(1+M_*^2\right)\right]$. Using Gronwall's lemma, we get from (5.57) that

$$\bar{S}(t) \leq T\zeta_1(M)\varepsilon^{2N+2}\exp\left(T\zeta_2(M)\right) \equiv T\bar{D}_{M_*}^2\varepsilon^{2N+2}, \tag{5.58}$$

hence, it leads to

$$\|v\|_{W_1(T)} \leq 3\sqrt{T\zeta_1(M)\exp\left(T\zeta_2(M)\right)}\varepsilon^{N+1} = C_T\varepsilon^{N+1},$$

or

$$\left\| u - \sum_{r=0}^N u_r\varepsilon^r \right\|_{W_1(T)} \leq C_T\varepsilon^{N+1}. \tag{5.59}$$

Finally, we have the following theorem.

Theorem 5.7. *Let* (H_1), $(H_2^{(N)})$ *and* (H_3) *hold. Then there exist constants* $M > 0$ *and* $T > 0$ *such that, for every* $\varepsilon \in (0,1)$, *Problem* (P_ε) *has a unique weak solution* $u_\varepsilon \in W_1(M,T)$ *satisfying the asymptotic estimation up to order* $N+1$ *as in* (5.59), *where the functions* u_r, $r = 0, 1, \cdots, N$ *are the weak solutions of the problems* $(\tilde{P}_r)$, $r = 0, 1, \cdots, N$, *respectively, and* C_T *is a constant depending only on* N, T, f, μ, μ_1, u_r, $r = 0, 1, \cdots, N$.

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