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Some Gauss type contiguous relations between Faraut-Korányi hypergeometric functions

Fouzia El Wassouli*, Daoud Oukacha

Department of Mathematics, Faculty of Sciences, University Hassan II, Casablanca, Morocco

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Abstract

In this paper, we give a complete description of the generalized hypergeometric functions, introduced by Faraut and Korányi on the Cartan domain. We establish some Gauss type contiguous relations between these functions on the two Cartan domains of type I_2 and type IV_4 analogous to the classical relations in the one variable case.

Keywords: Gindikin Gamma function, generalized hypergeometric functions, zonal polynomials
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1 Introduction

In [12] Herz has introduced the generalized hypergeometric functions with a matrix argument. These functions have been object of intensive study in multivariate statistical analysis. In fact one can express probability densities occurring naturally. In [3] Constantine has defined these functions in terms of zonal polynomials. Later, in [1, 2, 4, 5] a class of homogeneous invariant polynomials two or more matrix arguments, which generalise the zonal polynomials; many of their basic and integral properties are studied in real cases. Generalized hypergeometric functions associated with arbitrary symmetric cones were considered by J. Faraut and A. Korányi [8]. A more general class of hypergeometric functions was introduced by A. Korányi [13]. In [18] Z. Yan established that the generalized hypergeometric functions ${}_2F_1^{(\mathbf{a})}(\alpha,\beta;\gamma;x_1,...,x_r)$ is the unique solution of the system of the partial differential equations

$$x_1(1-x_i)\frac{\partial^2 F}{\partial x_i^2}\Big\{\gamma-\frac{\mathbf{a}}{2}(r-1)-\Big[\alpha+\beta+1-\frac{\mathbf{a}}{2}(r-1)\Big]x_i+\frac{\mathbf{a}}{2}\sum_{j=1,j\neq i}^{r}\frac{x_i(1-x_i)}{x_i-x_j}\Big\}\frac{\partial F}{\partial x_i}$$
$$-\frac{\mathbf{a}}{2}\sum_{j=1,j\neq i}^{r}\frac{x_j(1-x_j)}{x_i-x_j}\frac{\partial F}{\partial x_j}=\alpha\beta F,\quad i=1,...,r$$

subject to the conditions that

1. F is a symmetric of $x_1,\dots,x_r$ and
2. F is analytic at $x_1=\dots=x_r=0$ and $F(0)=1$.

*Corresponding author
Email addresses: elwassouli@gmail.com (Fouzia El Wassouli), daoudoukacha@gmail.com (Daoud Oukacha)

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$\mathcal{D}$	$V^{\mathbb{C}}$	$\dim V$	$(r,\mathbf{a},\mathbf{b})$
$I_{n,m}(n\leq m)$	$M_{n,m}(\mathbb{C})$	nm	(n, 2, m-n)
$II_n(n\ even)$	$\{z\in M_n(\mathbb{C});\quad {}^tz=-z\}$	$\frac{n}{2}(n-1)$	$(\frac{n}{2},4,0)$
$II_n(n\ odd)$	$\{z\in M_n(\mathbb{C});\quad {}^tz=-z\}$	$n\frac{(n-1)}{2}$	$(\frac{n-1}{2},4,2)$
III_n	$\{z\in M_n(\mathbb{C});\quad {}^tz=z\}$	$\frac{1}{2}n(n+1)$	(n, 1, 0)
IV_n	$\mathbb{C}^n$	n	(2, n-2, 0)
V	$M_{1,2}(\mathbb{O})$	16	(2, 6, 4)
VI	$\{z\in M_{3,3}(\mathbb{O});\quad {}^t\overline{z}=z\}$	27	(3, 8, 0)

Table 1: Irreducible bounded symmetric domains of non-compact type.

Also Z. Yan obtained some analogues of classical results about hypergeometric functions and, in particular he established integral representations of the generalized hypergeometric functions. He obtained the asymptotic behavior of ${}_{p+1}F_p^{(\mathbf{a})}$. As an application, he gave the generalized Rudin-Forelli inequalities in function theory on a bounded symmetric domain, which are due to J. Faraut and A. Korányi for ${}_2F_1^{(\mathbf{a})}(\alpha,\beta;\gamma;t_1,....,t_r)$ with special α,β and γ, see [9].

The aim of this paper is to give some Gauss type contiguous relations between the Faraut-Korányi hypergeometric functions on the two Cartan domains (of rank 2) of type III_2 and type IV_3. The proofs rely on an interesting integral representation for the zonal functions ϕ_m: Namely

$$\phi_m(t_1,t_2)=\frac{\Gamma(\mathbf{a})}{\Gamma(\frac{\mathbf{a}}{2})^2}\int_0^1\left[t_1-(t_1-t_2)y\right]^{m_1-m_2}(t_1t_2)^{m_2}\left[y(1-y)\right]^{\frac{\mathbf{a}}{2}-1}dy. \tag{1.1}$$

The paper is organized as follows. In Section2 we give some notations and preliminaries. In Section3, we get some Gauss type contiguous relations between the Faraut-Korányi hypergeometric functions on the two domain of type III_2 and type IV_3.

2 Notations and preliminaries.

2.1 General setting.

In this section, we review some well known results of Jordan algebra and symmetric domains (referring to [10, 11, 15] for more details of this subject).

Let $\mathcal{D}\subset\mathbb{C}^d$ be a Cartan domain, i.e. $\mathcal{D}$ is an irreducible bounded symmetric domain in the Harish-Chandra realization. This is equivalent to saying that $\mathcal{D}$ is the open unit ball of $\mathbb{C}^d$ with respect to a certain norm $||.||$ such that the group $G:=Aut(\mathcal{D})$ of all biholomorphic automorphisms of $\mathcal{D}$ acts transitively on $\mathcal{D}$. By [15] there exists a triple product $\{.,.,.\}:\quad\mathbb{C}^d\times\mathbb{C}^d\times\mathbb{C}^d\longrightarrow\mathbb{C}^d$, so that $V^{\mathbb{C}}:=(\mathbb{C}^d,||.||,\{.,.,.\})$ is a Jordan-Banach*-triple(JB*-triple). The maximal compact subgroup of G is $K:=\{g\in G;\quad g(0)=0\}=G\cap GL(V^{\mathbb{C}})$, and $\mathcal{D}=G/K$.

We denote by $r,\mathbf{a},\mathbf{b},d$ and p the rank, the characteristic multiplicities, the dimension and the genus of $\mathcal{D}$, respectively

$$d=r+\frac{r(r-1)}{2}\mathbf{a}+r\mathbf{b},\quad p=2+(r-1)\mathbf{a}+\mathbf{b}. \tag{2.1}$$

where $\mathbb{O}$ is the 8-dimensional Cayley algebra. A tripotent $v\in V^{\mathbb{C}}$ is an element satisfying $\{v,v,v\}=v$. The Peirce decomposition associated with the tripotent v is

$$V^{\mathbb{C}}:=V_1^{\mathbb{C}}(v)\oplus V_{\frac{1}{2}}^{\mathbb{C}}(v)\oplus V_0^{\mathbb{C}}(v) \tag{2.2}$$

where $V_\nu^{\mathbb{C}}(v):=\{z\in V^{\mathbb{C}};\{v,v,z\}=\nu z\},\nu=1,\frac{1}{2},0$. The associated Peirce projection $P_\nu(v)$ is the projection whose range is $V_\nu^{\mathbb{C}}(v)$ and whose kernel is the sum of the other two Peirce subspace. The space $V_\nu^{\mathbb{C}}$ are sub-triples of $V^{\mathbb{C}}$, and the rank of the tripotent v is by definition the rank de $V_1^{\mathbb{C}}(v)$. We define $S_j=$ the set of tripotent of rank $j=0,1,...,r$, $S:=S_r$ is the Shilov boundary of $\mathcal{D}$. Let us choose a frame $e_1,e_2,...,e_r$, i.e. a maximal set of tripotents of rank one which are pairwise orthogonal i.e. $\{e_i,e_i,e_j\}=0$ whenever $i\neq j$. The tripotent

$$e=e_1+e_2+...+e_r$$

is maximal (having rank r) and thus $V_0^{\mathbb{C}}(e) = 0$. The stabilizer of e in K. namely

$$L := \{k \in K; \quad k(e) = e\}$$

will play an important role in the sequel. Notice that since K acts transitively on S, we have $S = K/L$. More generally, K acts transitively on the frame, and in particular it is transitive on each of the S_j. The sub-triple $V_1^{\mathbb{C}}(e)$ has the structure of a JB^*-algebra with respect to the product $z \circ w := \frac{1}{2}\{z, e, w\}$ and the in volution $z^* := \{e, z, e\}$, and e is the unit of $V_1^{\mathbb{C}}(e)$.

The real part of $V_1^{\mathbb{C}}(e)$i.e., the subset

$$X = X_1(e) := \{x \in V_1^{\mathbb{C}}(e); \quad x^* = x\}$$

of self-adjoint elements of $V_1^{\mathbb{C}}(e)$ is a Euclidien (or formally-real) Jordan algebra, with determinant ("norme") and trace polynomials

$$\Delta(z) = \det(z) \quad and \quad tr(z) := \langle z, e\rangle \tag{2.3}$$

respectively. Here $\langle z, w\rangle$ denotes the unique K-invariants scalar product on $V^{\mathbb{C}}$ satisfying $\langle e_1, e_1\rangle = 1$. The set

$$\Omega := \{x^2; \quad x \in X, \quad \Delta(z) \neq 0\}$$

is the symmetric cone associated with X. The group L, restricted to X, coincides with the Jordan- algebra automorphisms of X. In particular, it is transitive on the frames of orthogonal minimal idempotents in X whose sum is the unit element e.

For $1 \leq j \leq r$, let $u_j = e_1 + e_2 + ... + e_j$ and let Δ_j denote the determinant polynomial of the Jordan sub-algebra $(V^{\mathbb{C}})^{(j)} := V_1^{\mathbb{C}}(u_j)$ extended to all of $V^{\mathbb{C}}$ via $\Delta_j(z) := \Delta_j(P_j(u_j)z)$. Note that $\Delta_r = \Delta$. The conical function associated with $\mathbf{s} = (s_1, ..., s_r) \in \mathbb{C}^r$ is defined by

$$\Delta_{\mathbf{s}}(x) := \Delta_1(x)^{s_1 - s_2}\Delta_2(x)^{s_2 - s_3}.....\Delta_{r-1}(x)^{s_{r-1} - s_r}\Delta_r(x)^{s_r}, \quad \forall x \in \Omega. \tag{2.4}$$

The Gindikin-Koecher Gamma function associated with the cone Ω is defined for $\mathbf{s} = (s_1, ..., s_r) \in \mathbb{C}^r$ with $\Re[s_j] > (j-1)\frac{\mathbf{a}}{2}$ by the convergent integral

$$\Gamma_{\Omega}^{(\mathbf{a})}(\mathbf{s}) := \int_{\Omega} e^{-tr(x)}\Delta_{\mathbf{s}}(x)\Delta(x)^{\frac{-d_1}{r}} dm(x), \tag{2.5}$$

where $d_1 := \dim_{\mathbb{R}}(X) = \frac{r(r-1)}{2}\mathbf{a} + r$. It is Known that $\Gamma_{\Omega}^{(\mathbf{a})}$ can be expressed as a product of ordinary Gamma functions:

$$\Gamma_{\Omega}^{(\mathbf{a})}(\mathbf{s}) := (2\pi)^{\frac{d_1 - r}{2}} \prod_{j=1}^{r} \Gamma(s_j - (j-1)\frac{\mathbf{a}}{2}). \tag{2.6}$$

When $\mathbf{s} = m$ consists of integers such that $m_1 \geq ... \geq m_r \geq 0$, we write $m \geq 0$. In this case Δ_m is a polynomial. For $m \geq 0$, the K-irreducible component $\mathcal{P}_m$ is the finite linear span $\{\Delta_m \circ k, k \in K\}$. Equipped with the Fischer (or Fock) scalar product

$$\begin{aligned}\langle f, g\rangle : &= f(\partial)\overline{g(\bar{z})} \,|_{z=0} \\ &= \pi^{-d} \int_{\mathbb{C}^d} f(x)\overline{g(x)}e^{-|x|^2} dm(x),\end{aligned}$$

each space $\mathcal{P}_m$ becomes a finite-dimensional Hilbert space of function on $\mathbb{C}^d$, and thus has a reproducing kernel $K^m(x, y)$, holomorphic in x and anti-holomorphic in y. The Faraut-Korányi functions [9, 18] on $\mathcal{D}$ are defined by

$${}_2F_1^{(\mathbf{a})}(\alpha, \beta; \gamma; x) := \sum_{m \geq 0} \frac{(\alpha)_m(\beta)_m}{(\gamma)_m} K^m(x, e). \tag{2.7}$$

Here $(.)_m$ is the generalized Pochhammer symbol

$$(\alpha)_m := \prod_{j=1}^{r} (\alpha - \frac{j-1}{2}\mathbf{a})_{m_j}, \quad \text{where} \quad (\alpha)_k := \alpha(\alpha+1)...(\alpha+k-1). \tag{2.8}$$

Each $\mathcal{P}_m$ contains a unique L-invariant polynomial ϕ_m satisfying the normalization condition $\phi_m(e) = 1$

$$\phi_m(x) = \int_L \Delta_{\mathbf{m}}(k.x)dk.$$

The polynomials ϕ_m are related to the reproducing kernels K^m by the formula we have

$$K^m(x,e) = \frac{d_m}{(\frac{d}{r})_m}\phi_m(x), \tag{2.9}$$

where $d_m := \dim \mathcal{P}_m$. It is known that the last dimension is given by the formula

$$d_m = \frac{(\frac{d}{r})_m}{(q)_m}\pi_m$$

where

$$q := \frac{r-1}{2}\mathbf{a} + 1$$

and

$$\pi_m := \prod_{1\leq i<j\leq r} \frac{m_i - m_j + \frac{j-1}{2}\mathbf{a}}{\frac{j-1}{2}\mathbf{a}} \frac{(\frac{j-i+1}{2}\mathbf{a})_{m_i-m_j}}{(\frac{j-i-1}{2}\mathbf{a}+1)_{m_i-m_j}}.$$

Therefore, the Faraut-Korányi hypergeometric functions on $\mathcal{D}$ can be written as

$$_2F_1^{(\mathbf{a})}(\alpha,\beta;\gamma;x) \quad := \quad \sum_{m\geq 0} \frac{(\alpha)_m(\beta)_m}{(\gamma)_m}\frac{1}{(\frac{d}{r})_m} d_m\phi_m(x). \tag{2.10}$$

Remark 2.1. The Faraut-Korányi hypergeometric functions $_2F_1^{(\mathbf{a})}(\alpha,\beta;\gamma;x)$ is invariant under L acting on x. Therefore, if $x = \sum_{j=1}^{r} t_jc_j$ is the spectral decomposition of x,

$$_2F_1^{(\mathbf{a})}(\alpha,\beta;\gamma;x) ={}_2 F_1^{(\mathbf{a})}(\alpha,\beta;\gamma;t_1,t_2,...,t_r). \tag{2.11}$$

Remark 2.2. Such functions ϕ_m had been introduced by Hua and posed the problem of finding explicit analytic formula of them. In [14], Lassale and Schlosser gave a complex explicit analytic development for ϕ_m, and more generally for the Macdonald polynomial; they have used combinatoric analysis but the explicit expressions obtained are very complicated. Hence Hua's problem remained open except for the case of the Lie ball [7] and the case of the matricial ball where the zonal polynomials are Schur functions. When $V = Sym(m,\mathbb{R})$, they have been much studied in multivariate statistical analysis (see [17]). As a result of work by Debiard [6] and Macdonald [16] it has become clear that in the case of any Euclidean Jordan algebra they are special cases of the so-called Jack polynomials.

2.2 Bounded symmetric domains of type I_2 and type IV_3.

For any matrix a we denote respectively by ${}^ta$ and $\bar{a}$ the transpose and conjugate of a.

Type I_2: Let $\mathcal{D}_2$ be the domain of matrices of order 2 satisfying $I_2 - z\,{}^t\bar{z} > 0$ (positive definite). Let $G = SU(2,2)$ be the group of matrices $SL(4,\mathbb{C})$ which leave invariant the hermitian form on $\mathbb{C}^4$

$$z_1\bar{z}_1 + z_2\bar{z}_2 - z_3\bar{z}_3 - z_4\bar{z}_4.$$

The group G acts on $\mathcal{D}_2$ by

$$g.z = (az+b)(cz+d)^{-1}, \tag{2.12}$$

for every $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in G and every $z \in \mathcal{D}_2$. The action of G on $\mathcal{D}_2$ is transitive. Thus as homogeneous space, we have the identification $\mathcal{D}_2 = G/K$, where K is the stabilizer in G of 0 given by

$$K = S(U(2)\times U(2)) = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}; \qquad a \in U(2),\ d \in U(2)\ et\ \det ad = 1 \right\}.$$

Consider on the space of matrices $V^{\mathbb{C}} = M(2,\mathbb{C})$ the product defined as follows if $z = x \circ y = \frac{1}{2}(xy + yx)$. Then $V^{\mathbb{C}}$ is a Jordan algebra of rank two with an identity element $e = I_2$ and the Koecher's determinant Δ is

$$\Delta(z) = \det z.$$

Let $\left\{c_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, c_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$ be a fixed Jordan frame in $V = Herm(2,\mathbb{C})$. Every $z \in V^{\mathbb{C}}$ can be written as the form

$$z = k.\sum_{j=1}^{2} t_j c_j,$$

with $k \in K$ and $0 \le t_1 \le t_2$. Then the spectral norm of $z \in V^{\mathbb{C}}$ is

$$|z| = \sup_{j\in\{1,2\}} t_j$$

and we can define $\mathcal{D}_2$ in $V^{\mathbb{C}}$ as the open unit ball for the spectral norm

$$\mathcal{D}_2 = \{z \in M(2,\mathbb{C}) / \qquad |z| < 1\}.$$

Let L be the stabilizer of I_2 in K

$$L = \left\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}; \qquad a \in U(2) \ et \ \det a = \pm 1\right\} \simeq U(2).$$

Thus the L-invariant zonal function ϕ_m on $\mathcal{D}_2$ in $M(2,\mathbb{C})$ is given by

$$\phi_m(z) = \int_{U(2)} \Delta_m(k.z)dk; \quad m \in \wedge = \{(m_1,m_2) \in \mathbb{Z}^2 / \ m_1 \ge m_2\},$$

where dk is the normalized Haar measure on $U(2)$ and Δ_m is the conical polynomial defined in [10] by

$$\Delta_m(z) = \Delta_1(z)^{m_1-m_2}\Delta(z)^{m_2}, \qquad \Delta_1(z) = z_{11}, \quad \forall z = (z_{ij})_{1\le i,j\le 2} \in M(2,\mathbb{C}).$$

Type IV_4: The Bounded symmetric domains of type IV_3 (Lie ball) in $\mathbb{C}^4$ is defined by

$$\mathcal{D}_4 = \left\{z =^t (z_1,z_2,z_3,z_4) \in \mathbb{C}^4 / \quad 1 - 2^t\bar{z}z + |^t zz|^2 > 0, \quad |^t zz| < 1\right\},$$

here z is viewing as 4×1 matrix.

The group $SO(4,2)$ consisting of the matrices g in $SL(6,\mathbb{R})$ such that $g^tJg = J$, where $J = \begin{pmatrix} -I_4 & 0_{4,2} \\ 0_{2,4} & I_2 \end{pmatrix}$. The group $SO(4,2)$ acts on $\mathcal{D}_4$ by

$$g[z] = g.z = \left(Az + B\begin{pmatrix} \frac{i}{2}(1 -^t zz) \\ \frac{1}{2}(1 +^t zz) \end{pmatrix}\right)\left((-i,1)\left(Cz + D\begin{pmatrix} \frac{i}{2}(1 -^t zz) \\ \frac{1}{2}(1 +^t zz \end{pmatrix}\right)\right)^{-1}, \tag{2.13}$$

for every $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in G et every $z \in \mathcal{D}_4$. The action of $SO(4,2)$ on $\mathcal{D}_4$ is transitive. In fact, the connected component of the identity, $SO_0(4,2)$ acts transitively. We will let G denote this group. Thus as homogeneous space, we have the identification $\mathcal{D}_4 = G/K$, where K is the stabilizer in G of 0 given by $K = SO(4) \times SO(2)$.

Consider on $V^{\mathbb{C}} = (\mathbb{R}^3 \times \mathbb{R})^{\mathbb{C}} = \mathbb{C}^4$ the product defined as follows if $z = xy$,

$$\begin{cases} z_1 = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 \\ z_j = x_1y_j + x_jy_1, \qquad j \in \{2,3,4\}. \end{cases}$$

Then $V^{\mathbb{C}}$ is a Jordan algebra of rank two with an identity element $e =^t (1,0,0,0)$ and the Koecher's determinant Δ is

$$\Delta(z) = z_1^2 + z_2^2 + z_3^2 + z_4^2.$$

Let $\{\tilde{c}_1 = \frac{1}{2}\,{}^t(1,0,0,1), \tilde{c}_2 = \frac{1}{2}\,{}^t(1,0,0,-1)\}$ be a fixed Jordan frame in $V = \mathbb{R}^3 \times \mathbb{R}$. Every $z \in V^{\mathbb{C}}$ can be written as the form

$$z = k.\sum_{j=1}^{2} t_j c_j,$$

with $k \in K$ and $0 \leq t_1 \leq t_2$. Then the spectral norm of $z \in V^{\mathbb{C}}$ is

$$|z| = \sup_{j\in\{1,2\}} t_j$$

and we can define $\mathcal{D}_4$ in $V^{\mathbb{C}}$ as the open unit ball for the spectral norm

$$\mathcal{D}_4 = \{z \in \mathbb{C}^4 / \qquad |z| < 1\}.$$

Let $\tilde{L}$ be the stabilizer of e in K

$$\tilde{L} = \left\{ \begin{pmatrix} \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & & A & \\ 0 & & & \\ 0 & & & \end{pmatrix} & 0_{4,2} \\ 0_{2,4} & \epsilon I_2 \end{pmatrix}; \qquad \epsilon = \pm 1,\ A \in SO(3) \right\} \simeq SO(3).$$

Thus the $\tilde{L}$-invariant zonal function ϕ_m on the Lie ball in $\mathbb{C}^4$ is given by

$$\phi_m(z) = \int_{SO(3)} \Delta_m(k.z)dk; \quad m \in \wedge = \{(m_1, m_2) \in \mathbb{Z}^2 / \ m_1 \geq m_2\},$$

where dk is the normalized Haar measure on $SO(3)$ and Δ_m is the conical polynomial defined in [10] by

$$\Delta_m(z = (z_1, z_2, z_3, z_4)) = \Delta_1(z)^{m_1 - m_2} \Delta(z)^{m_2}, \qquad \Delta_1(z) = z_1 + z_4.$$

Remark 2.3. If $z_1 = t_1 c_1 + t_2 c_2 = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$ and $z_2 = t_1 \tilde{c}_1 + t_2 \tilde{c}_2 = \frac{1}{2}\,{}^t(t_1 + t_2, 0, 0, t_1 - t_2)$, the zonal function $\phi_m(z_1)$ on domain of type I_2 coincide with the zonal function $\phi_m(z_2)$ on domain of type IV_4.

By using the formula (1.1) and integrating by parts, we obtain

$$\phi_m(t_1, t_2) = 2\frac{t_1^{m_2} t_2^{m_1+1} - t_2^{m_2} t_1^{m_1+1}}{(t_2 - t_1)(m_1 - m_2 + 1)}.$$

Thus the Faraut-Korányi hypergeometric functions on the two Cartan domains of type I_2 and type IV_4 are given by

$$\begin{aligned} & {}_2F_1^{(2)}(\alpha, \beta; \gamma; k_1.z_1) = {}_2F_1^{(2)}(\alpha, \beta; \gamma; k_2.z_2), \quad k_1 \in L, k_2 \in \tilde{L} \\ = \quad & {}_2F_1^{(2)}(\alpha, \beta; \gamma; t_1, t_2) \\ = \quad & 2 \sum_{m_2=0}^{\infty} \sum_{m_1=m_2}^{\infty} \left(\frac{(\alpha)_{m_1} (\alpha - 1)_{m_2} (\beta)_{m_1} (\beta - 1)_{m_2}}{(\gamma)_{m_1} (\gamma - 1)_{m_2} (m_1 + 1)! m_2!} \right) (m_1 - m_2 + 1) \frac{t_1^{m_2} t_2^{m_1+1} - t_2^{m_2} t_1^{m_1+1}}{(t_2 - t_1)}. \end{aligned}$$

3 Gauss type contiguous relations.

In this section, we establish the Gauss type contiguous relations between the Faraut-Korányi hypergeometric functions on the two Cartan domains of type I_2 and type IV_4 generalizing the classical contiguous relation

$$x(1-x)\frac{d}{dx}\,{}_2F_1(\alpha+1, \beta+1; \gamma+1; x) + (\gamma - (\alpha + \beta + 1)x)\,{}_2F_1(\alpha+1, \beta+1; \gamma+1; x) - \gamma\,{}_2F_1(\alpha, \beta; \gamma; x) = 0,$$

where ${}_2F_1(\alpha, \beta; \gamma; x)$ is the classical Gauss hypergeometric Function. Let $H^{(2)}(\alpha, \beta; \gamma; t_1, t_2)$ denote the following function

$$H^{(2)}(\alpha, \beta; \gamma; t_1, t_2) = \frac{2}{t_2 - t_1}\ {}_2F_1^{(2)}(\alpha, \beta; \gamma; t_1, t_2).$$

Theorem 3.1. It holds that

$$\begin{aligned} {}_2F_1^{(2)}(\alpha,\beta;\gamma;t_1,t_2) &= \frac{(\gamma-1)(\gamma-2)}{(\alpha-1)(\alpha-2)(\beta-1)(\beta-2)}\Big[\frac{\partial^2}{\partial t_1\partial t_2}\ {}_2F_1^{(2)}(\alpha-1,\beta-1;\gamma-1;t_1,t_2) \\ &+\frac{1}{(t_2-t_1)}\Big\{\Big(\frac{\partial}{\partial t_2}-\frac{\partial}{\partial t_2}\Big)_2F_1^{(2)}(\alpha-1,\beta-1;\gamma-1;t_1,t_2)\Big\} \\ &-\frac{1}{(t_2-t_1)^2}\ {}_2F_1^{(2)}(\alpha-1,\beta-1;\gamma-1;t_1,t_2)\Big]. \end{aligned}$$

Proof . We use the formula $(a-1)_{k+1}=(a-1)(a)_k$ to rewrite $H^{(2)}(\alpha,\beta;\gamma;t_1,t_2)$ as

$$\begin{aligned} &H^{(2)}(\alpha,\beta;\gamma;t_1,t_2)=\frac{(\gamma-1)(\gamma-2)}{(\alpha-1)(\alpha-2)(\beta-1)(\beta-2)} \\ &\sum_{m_2=0}^{\infty}\sum_{m_1=m_2}^{\infty}\left(\frac{(\alpha-1)_{m_1}(\alpha-2)_{m_2}(\beta-1)_{m_1}(\beta-2)_{m_2}}{(\gamma-1)_{m_1}(\gamma-2)_{m_2}(m_1+1)!m_2!}\right)m_2(m_1+1)(t_1^{m_2-1}t_2^{m_1}-t_2^{m_2-1}t_1^{m_1}). \\ =\ &\frac{(\gamma-1)(\gamma-2)}{(\alpha-1)(\alpha-2)(\beta-1)(\beta-2)}G^{(2)}(\alpha-1,\beta-1;\gamma-1;t_1,t_2). \end{aligned}$$

Since

$$\frac{\partial^2}{\partial t_1\partial t_2}\Big[(t_1^{m_2}t_2^{m_1+1}-t_2^{m_2}t_1^{m_1+1})\Big]=m_2(m_1+1)(t_1^{m_2-1}t_2^{m_1}-t_2^{m_2-1}t_1^{m_1})$$

we have

$$G^{(2)}(\alpha-1,\beta-1;\gamma-1;t_1,t_2) = \frac{\partial}{\partial t_1\partial t_2}H^{(2)}(\alpha-1,\beta-1;\gamma-1;t_1,t_2)$$

From the fact that

$$\begin{aligned} {}_2F_1^{(2)}(\alpha,\beta;\gamma;t_1,t_2) &= \frac{(t_2-t_1)}{2}H^{(2)}(\alpha-1,\beta-1;\gamma-1;t_1,t_2) \\ &= \frac{(t_2-t_1)(\gamma-1)(\gamma-2)}{(\alpha-1)(\alpha-2)(\beta-1)(\beta-2)} \\ &\times \frac{\partial^2}{\partial t_1\partial t_2}\Big(\frac{1}{t_2-t_1}\ {}_2F_1^{(2)}(\alpha-1,\beta-1;\gamma-1;t_1,t_2)\Big) \end{aligned}$$

we get

$$\begin{aligned} {}_2F_1^{(2)}(\alpha,\beta;\gamma;t_1,t_2) &= \frac{(\gamma-1)(\gamma-2)}{(\alpha-1)(\alpha-2)(\beta-1)(\beta-2)}\Big[\frac{\partial^2}{\partial t_1\partial t_2}\ {}_2F_1^{(2)}(\alpha-1,\beta-1;\gamma-1;t_1,t_2) \\ &+\frac{1}{(t_2-t_1)}\Big\{\Big(\frac{\partial}{\partial t_2}-\frac{\partial}{\partial t_2}\Big)\ {}_2F_1^{(2)}(\alpha-1,\beta-1;\gamma-1;t_1,t_2)\Big\} \\ &-\frac{1}{(t_2-t_1)^2}\ {}_2F_1^{(2)}(\alpha-1,\beta-1;\gamma-1;t_1,t_2)\Big]. \end{aligned}$$

□

By applying the Euler type transformation [10, 18]

$${}_2F_1^{(2)}(\alpha-1,\beta-1;\gamma-1;t_1,t_2)=\Big[(1-t_1)(1-t_2)\Big]^{\gamma-\alpha-\beta+1}\ {}_2F_1^{(2)}(\gamma-\alpha,\gamma-\beta;\gamma-1;t_1,t_2)$$

to each term in the Theorem 3.1 and simple calculation we get the following corollary.

Corollary 3.2.

$$
\begin{aligned}
&{}_2F_1^{(2)}(\alpha,\beta;\gamma;t_1,t_2) = \frac{(\gamma-1)(\gamma-2)}{(\alpha-1)(\alpha-2)(\beta-1)(\beta-2)}\Big[(1-t_1)(1-t_2)\Big]^{\gamma-\alpha-\beta} \\
&\Big[\Big\{(\gamma-\alpha-\beta+1)(\gamma-\alpha-\beta) - \frac{(1-t_1)(1-t_2)}{(t_2-t_1)^2}\Big\}\ {}_2F_1^{(2)}\Big(\gamma-\alpha,\gamma-\beta;\gamma-1;t_1,t_2\Big) \\
- &\Big\{\frac{(1-t_1)(1-t_2)}{(t_2-t_1)} + (\gamma-\alpha-\beta+1)(1-t_1)\Big\}\frac{\partial}{\partial t_1}\ {}_2F_1^{(2)}\Big(\gamma-\alpha,\gamma-\beta;\gamma-1;t_1,t_2\Big) \\
+ &\Big\{\frac{(1-t_1)(1-t_2)}{(t_2-t_1)} - (\gamma-\alpha-\beta+1)(1-t_2)\Big\}\frac{\partial}{\partial t_2}\ {}_2F_1^{(2)}\Big(\gamma-\alpha,\gamma-\beta;\gamma-1;t_1,t_2\Big) \\
+ &(1-t_1)(1-t_2)\frac{\partial^2}{\partial t_2\partial t_1}\ {}_2F_1^{(2)}\Big(\gamma-\alpha,\gamma-\beta;\gamma-1;t_1,t_2\Big)\Big]
\end{aligned}
$$

By apply the Pfaff type transformation [10, 18]

$$
{}_2F_1^{(2)}(\alpha-1,\beta-1;\gamma-1;t_1,t_2) = \Big[(1-t_1)(1-t_2)\Big]^{1-\alpha}\ {}_2F_1^{(2)}\Big(\alpha-1,\gamma-\beta;\gamma-1;\frac{t_1}{t_1-1},\frac{t_2}{t_2-1}\Big)
$$

to each term in the Theorem 3.1 and simple calculation we get an other corollary.

Corollary 3.3.

$$
\begin{aligned}
&{}_2F_1^{(2)}(\alpha,\beta;\gamma;t_1,t_2) = \frac{(\gamma-1)(\gamma-2)}{(\alpha-1)(\alpha-2)(\beta-1)(\beta-2)}\Big[(1-t_1)(1-t_2)\Big]^{-\alpha-1} \\
&\Big\{\Big[\alpha(\alpha-1)\Big((1-t_1)(1-t_2)\Big) - \frac{\Big((1-t_1)(1-t_2)\Big)^2}{(t_2-t_1)^2}\Big]\ {}_2F_1^{(2)}\Big(\alpha-1,\gamma-\beta;\gamma-1;\frac{t_1}{t_1-1},\frac{t_2}{t_2-1}\Big) \\
+ &\Big[-\alpha(1-t_2) + \frac{(1-t_2)^2}{t_2-t_1}\Big]\frac{\partial}{\partial t_1}\ {}_2F_1^{(2)}\Big(\alpha-1,\gamma-\beta;\gamma-1;\frac{t_1}{t_1-1},\frac{t_2}{t_2-1}\Big) \\
- &\Big[\alpha(1-t_1) + \frac{(1-t_1)^2}{t_2-t_1}\Big]\frac{\partial}{\partial t_2}\ {}_2F_1^{(2)}\Big(\alpha-1,\gamma-\beta;\gamma-1;\frac{t_1}{t_1-1},\frac{t_2}{t_2-1}\Big) \\
+ &\frac{\partial^2}{\partial t_1\partial t_2}\ {}_2F_1^{(2)}\Big(\alpha-1,\gamma-\beta;\gamma-1;\frac{t_1}{t_1-1},\frac{t_2}{t_2-1}\Big)\Big\}.
\end{aligned}
$$

Remark 3.4. If $(t_1,t_2) = (ue^t, ue^{-t})$ the function ${}_2F_1^{(2)}(\alpha-1,\gamma-\beta;\gamma-1;ue^t,ue^{-t})$ can be written as series involving Gegenbauer polynomials. More precisely, from the formula (1.1), we have

$$
\begin{aligned}
\phi_m(ue^t,ue^{-t}) &= 2u^{m_1+m_2}\int_0^1\Big[\cosh t + (1-2y)\sinh t\Big]^{m_1-m_2}dy \\
&= u^{m_1+m_2}\int_0^\pi\Big[\cosh t + \cos\theta\sinh t\Big]^{m_1-m_2}\sin\theta d\theta \\
&= \frac{\sqrt{\pi}u^{m_1+m_2}}{8\Gamma(\frac{1}{2})}C^1_{m_1-m_2}(\cosh t),
\end{aligned}
$$

where C^1_k denotes the Gegenbauer (ultraspherical) polynomial of degree k

$$
C^1_k(x) = \frac{(-1)^k(k+1)}{2^k(\frac{3}{2})_k}(1-x^2)^{\frac{-1}{2}}\frac{d^k}{dx^k}(1-x^2)^{k+\frac{1}{2}}.
$$

Thus

$$
\begin{aligned}
&{}_2F_1^{(2)}(\alpha,\beta;\gamma;ue^t,ue^{-t}) = \\
&\frac{\sqrt{\pi}}{8\Gamma(\frac{1}{2})}\sum_{m_2=0}^{\infty}\sum_{m_1=m_2}^{\infty}\Bigg(\frac{(\alpha)_{m_1}(\alpha-1)_{m_2}(\beta)_{m_1}(\beta-1)_{m_2}}{(\gamma)_{m_1}(\gamma-1)_{m_2}(m_1+1)!m_2!}\Bigg)(m_1-m_2+1)^2u^{m_1+m_2}C^1_{m_1-m_2}(\cosh t).
\end{aligned}
$$

Theorem 3.5. It holds that

$$
\begin{aligned}
&{}_2F_1^{(2)}(\alpha,\beta;\gamma;ue^t,ue^{-t}) = \frac{(\gamma-1)(\gamma-2)}{4(\alpha-1)(\alpha-2)(\beta-1)(\beta-2)} \\
&\Big[\frac{d^2}{du^2}\ {}_2F_1^{(2)}(\alpha-1,\beta-1;\gamma-1;ue^t,ue^{-t}) + \frac{3}{u}\frac{d}{du}\ {}_2F_1^{(2)}(\alpha-1,\beta-1;\gamma-1;ue^t,ue^{-t}) \\
-\ &\frac{d^2}{dt^2}\ {}_2F_1^{(2)}(\alpha-1,\beta-1;\gamma-1;ue^t,ue^{-t}) - (3-\sinh t)\frac{\cosh t}{\sinh t}\frac{d}{dt}\ {}_2F_1^{(2)}(\alpha-1,\beta-1;\gamma-1;ue^t,ue^{-t})\Big].
\end{aligned}
$$

Proof . We use the formula $(a-1)_{k+1} = (a-1)(a)_k$ to rewrite ${}_2F_1^{(2)}(\alpha,\beta;\gamma;ue^t,ue^{-t})$ as

$$
\begin{aligned}
&{}_2F_1^{(2)}(\alpha,\beta;\gamma;ue^t,ue^{-t}) = \frac{(\gamma-1)(\gamma-2)}{(\alpha-1)(\alpha-2)(\beta-1)(\beta-2)}\frac{\sqrt{\pi}}{8\Gamma(\frac{1}{2})} \\
&\Big[\sum_{m_2=0}^{\infty}\sum_{m_1=m_2}^{\infty}\frac{(\alpha-1)_{m_1}(\alpha-2)_{m_2}(\beta-1)_{m_1}(\beta-2)_{m_2}(m_1-m_2+1)^2}{(\gamma-1)_{m_1}(\gamma-2)_{m_2}(m_1+1)!m_2!}m_2(m_1+1)u^{m_1+m_2-2}C^1_{m_1-m_2}(\cosh t)\Big].
\end{aligned}
$$

Since $m_2(m_1+1) = \frac{1}{4}(m_1+m_2)(m_1+m_2-1) + \frac{3}{4}(m_1+m_2) - \frac{1}{4}(m_1-m_2)(m_1-m_2+2)$, we have

$$
\begin{aligned}
&{}_2F_1^{(2)}(\alpha,\beta;\gamma;ue^t,ue^{-t}) = \frac{(\gamma-1)(\gamma-2)}{4(\alpha-1)(\alpha-2)(\beta-1)(\beta-2)} \\
&\Big[\frac{d^2}{du^2}\ {}_2F_1^{(2)}(\alpha-1,\beta-1;\gamma-1;ue^t,ue^{-t}) + \frac{3}{u}\frac{d}{du}\ {}_2F_1^{(2)}(\alpha-1,\beta-1;\gamma-1;ue^t,ue^{-t}) \\
-\ &\frac{\sqrt{\pi}}{8\Gamma(\frac{1}{2})}\sum_{m_2=0}^{\infty}\sum_{m_1=m_2}^{\infty}\frac{(\alpha-1)_{m_1}(\alpha-2)_{m_2}(\beta-1)_{m_1}(\beta-2)_{m_2}(m_1-m_2+1)^2}{(\gamma-1)_{m_1}(\gamma-2)_{m_2}(m_1+1)!m_2!} \\
&(m_1-m_2)(m_1-m_2+2)u^{m_1+m_2-2}C^1_{m_1-m_2}(\cosh t)\Big].
\end{aligned}
$$

From the fact that $C^1_{m_1-m_2}(x)$ is a solution following Gegenbauer differential equation

$$
(x^2-1)\frac{d^2f(x)}{d^2x} + 3x\frac{df(x)}{dx} = (m_1-m_2)(m_1-m_2+2)f(x)
$$

and

$$
\begin{cases}
\dfrac{df(\cosh t)}{d(\cosh t)} = \dfrac{1}{\sinh t}\dfrac{df(\cosh t)}{dt} \\
\dfrac{d^2f(\cosh t)}{d^2(\cosh t)} = -\dfrac{\cosh t}{\sinh^2 t}\dfrac{df(\cosh t)}{dt} + \dfrac{1}{\sinh^2 t}\dfrac{d^2f(\cosh t)}{dt^2},
\end{cases}
$$

we have

$$
\frac{d^2C^1_{m_1-m_2}(\cosh t)}{dt^2} + (3-\sinh t)\frac{\cosh t}{\sinh t}\frac{dC^1_{m_1-m_2}(\cosh t)}{dt} = (m_1-m_2)(m_1-m_2+2)C^1_{m_1-m_2}(\cosh t),
$$

and

$$
\begin{aligned}
&{}_2F_1^{(2)}(\alpha,\beta;\gamma;ue^t,ue^{-t}) = \frac{(\gamma-1)(\gamma-2)}{4(\alpha-1)(\alpha-2)(\beta-1)(\beta-2)} \\
&\Big[\frac{d^2}{du^2}\ {}_2F_1^{(2)}(\alpha-1,\beta-1;\gamma-1;ue^t,ue^{-t}) + \frac{3}{u}\frac{d}{du}\ {}_2F_1^{(2)}(\alpha-1,\beta-1;\gamma-1;ue^t,ue^{-t}) \\
-\ &\frac{d^2}{dt^2}\ {}_2F_1^{(2)}(\alpha-1,\beta-1;\gamma-1;ue^t,ue^{-t}) - (3-\sinh t)\frac{\cosh t}{\sinh t}\frac{d}{dt}\ {}_2F_1^{(2)}(\alpha-1,\beta-1;\gamma-1;ue^t,ue^{-t})\Big].
\end{aligned}
$$

□

4 Conclusion

The proofs of Theorem 3.1 and Theorem 3.5 rely on interesting integral representation for ϕ_m on domains of rank 2 (Formula(1.1)).It would be nice to have an analogous representation for general rank r in order to generalize our results. Future work we will generalize our results of this paper to exceptional domain $E_{6(-14)}/SO(10)\times SO(2)$.

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