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ON THE STUDY OF HILBERT-TYPE INEQUALITIES WITH MULTI-PARAMETERS: A SURVEY

B. YANG[1*] AND TH. M. RASSIAS[2]

ABSTRACT. In this paper, we provide a short account of the study of Hilbert-type inequalities during the past almost 100 years by introducing multi-parameters and using the method of weight coefficients. A basic theorem of Hilbert-type inequalities with the homogeneous kernel of $-\lambda-$degree and parameters is proved.

1. INTRODUCTION: HILBERT'S INEQUALITY WITH NO PARAMETER

In 1908, H. Weyl published the following well known Hilbert's inequality: If $\{a_n\}, \{b_n\}$ are real sequences, such that $0 < \sum_{n=1}^{\infty} a_n^2 < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, then[1]

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \pi(\sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} b_n^2)^{\frac{1}{2}} \tag{1.1}$$

where the constant factor π is the best possible. The integral analogue of (1.1) states as follows: If $f(x), g(x)$ are measurable functions, such that

$$0 < \int_0^{\infty} f^2(x)dx < \infty,\ 0 < \int_0^{\infty} g^2(x)dx < \infty,$$

then[2]

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}dxdy < \pi(\int_0^{\infty} f^2(x)dx\int_0^{\infty} g^2(x)dx)^{\frac{1}{2}}, \tag{1.2}$$

where the constant π is still the best possible.

The operator expression of (1.1) can be stated as follows: If

$$l^2 := \{a = \{a_m\}_{m=1}^{\infty} ||\, ||a||_2 := (\sum_{m=1}^{\infty} a_m^2)^{\frac{1}{2}} < \infty\}$$

is a space of real sequences, $T : l^2 \to l^2$ is a linear operator, such that for $a = \{a_m\} \in l^2$, there exists a unique $c = \{c_n\} \in l^2$, satisfying

$$c_n := (Ta)(n) = \sum_{m=1}^{\infty}\frac{a_m}{m+n}(n \in \mathbf{N}), \tag{1.3}$$

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*: Corresponding author.

then for $b = \{b_n\} \in l^2$, the inner product of Ta and b is defined as follows:

$$(Ta, b) = (c, b) = \sum_{n=1}^{\infty}(\sum_{m=1}^{\infty} \frac{a_m}{m+n})b_n = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{m+n}. \tag{1.4}$$

Indicating the norm of a as $||a||_2 = (\sum_{n=1}^{\infty} a_n^2)^{\frac{1}{2}}$, we may rewrite (1.1) as follows: $(Ta, b) < \pi||a||_2||b||_2$, where $||a||_2, ||b||_2 > 0$. The equivalent form is stated as follows: $||Ta||_2 < \pi||a||_2$, which makes $c \in l^2$ and $||T||_2 \le \pi$. It can be written in an equivalent form to (1.1) as follows:

$$\sum_{n=1}^{\infty}(\sum_{m=1}^{\infty} \frac{a_m}{m+n})^2 < \pi^2 \sum_{n=1}^{\infty} a_n^2, \tag{1.5}$$

where the constant factor π^2 is the best possible. Hence the Hilbert's operator T defined by (1.3) is bounded and $||T|| = \pi$ (cf. [3]).

Similarly we may define Hilbert's integral operator $\widetilde{T}$ as follows: If $L^2(0, \infty)$ is a real space of functions, $\widetilde{T} : L^2(0, \infty) \to L^2(0, \infty)$, such that for any $f \in L^2(0, \infty)$, there exists only $h = \widetilde{T}f \in L^2(0, \infty)$, satisfying

$$(\widetilde{T}f)(y) = h(y) := \int_0^{\infty} \frac{f(x)}{x+y} dx \ (y \in (0, \infty)). \tag{1.6}$$

Then for $g \in L^2(0, \infty)$, the inner product of $\widetilde{T}f$ and g is defined as follows:

$$(\widetilde{T}f, g) = \int_0^{\infty}(\int_0^{\infty} \frac{f(x)}{x+y} dx) g(y) dy = \int_0^{\infty}\int_0^{\infty} \frac{f(x)g(y)}{x+y} dx dy. \tag{1.7}$$

Defining the norm of f by $||f||_2 = (\int_0^{\infty} f^2(x)dx)^{\frac{1}{2}}$, we may rewrite (1.2) as follows: $(\widetilde{T}f, g) < \pi||f||_2||g||_2$, and prove that[4] $||\widetilde{T}f||_2 < \pi||f||_2$ and $||\widetilde{T}|| = \pi$. Hence we can get the equivalent form of (1.2) as follows

$$\int_0^{\infty}(\int_0^{\infty} \frac{f(x)}{x+y} dx)^2 dy < \pi^2 \int_0^{\infty} f^2(x) dx, \tag{1.8}$$

where the constant factor π^2 is still the best possible.

Hilbert's inequalities (1.1) and (1.2) are important in mathematical analysis and its applications. In the past 100 years, a large number of mathematicians have investigated the subject of Hilbert's inequalities as well as Hilbert-type inequalities in a very broad context and proved a variety of several inequalities. In the present paper, we provide an overview of the study of Hilbert-type inequalities including (1.1) and (1.2) depending upon parameters.

2. Hilbert's inequality with one pair of conjugate exponents

In 1925, G. H. Hardy provided the best extensions of (1.1) and (1.5) by introducing one pair of conjugate exponents $(p, q)(\frac{1}{p} + \frac{1}{q} = 1)$ as[5]: If $p > 1, a_n, b_n \ge 0, n \in \mathbf{N}$, such that $0 < \sum_{n=1}^{\infty} a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\frac{\pi}{p})}(\sum_{n=1}^{\infty} a_n^p)^{\frac{1}{p}}(\sum_{n=1}^{\infty} b_n^q)^{\frac{1}{q}}, \tag{2.1}$$

$$\sum_{n=1}^{\infty}(\sum_{m=1}^{\infty}\frac{a_m}{m+n})^p < [\frac{\pi}{\sin(\frac{\pi}{p})}]^p\sum_{n=1}^{\infty}a_n^p, \tag{2.2}$$

where the constant factors $\pi/\sin(\frac{\pi}{p})$ and $[\pi/\sin(\frac{\pi}{p})]^p$ are the best possible, and inequalities (2.1) and (2.2) are equivalent. For $p = q = 2$, inequality (2.1) reduces to (1.1), and (2.2) reduces to (1.5). We name inequality (2.1) Hardy-Hilbert's inequality. The integral analogues of (2.1) and (2.2) are indicated as follows[6]: If $f(x), g(x)$ are non-negative measurable functions, such that $0 < \int_0^{\infty} f^p(x)dx < \infty$ and $0 < \int_0^{\infty} g^q(x)dx < \infty$, then

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\frac{\pi}{p})}(\int_0^{\infty} f^p(x)dx)^{\frac{1}{p}}(\int_0^{\infty} g^q(x)dx)^{\frac{1}{q}}, \tag{2.3}$$

$$\int_0^{\infty}(\int_0^{\infty}\frac{f(x)}{x+y}dx)^p dy < [\frac{\pi}{\sin(\frac{\pi}{p})}]^p\int_0^{\infty} f^p(x)dx, \tag{2.4}$$

where the constant factors $\pi/\sin(\frac{\pi}{p})$ and $[\pi/\sin(\frac{\pi}{p})]^p$ are the best possible, and inequalities (2.3) and (2.4) are equivalent. For $p = q = 2$, inequality (2.3) reduces to (1.2), and (2.4) reduces to (1.8). Inequality (2.3) is called Hardy-Hilbert's integral inequality. In 1934, Hardy, Littlewood and Polya [6] gave some important applications of (2.1)-(2.4). For the general homogeneous kernel of $-1-$degree, a basic theorem was given as follows (see [6], Theorem 319, Theorem 318 and Theorem 336):

Theorem 2.1. *If $p > 1, \frac{1}{p}+\frac{1}{q} = 1, k(x,y)$ is a homogeneous function of $-1-$degree, satisfying $k(ux,uy) = u^{-1}k(x,y)(u,x,y > 0)$, $k = \int_0^{\infty} k(u,1)u^{-1/p}du$ is a finite number, and $f(x), g(x)$ are non-negative measurable functions, then the following equivalent inequalities hold:*

$$\int_0^{\infty}\int_0^{\infty} k(x,y)f(x)g(y)dxdy \le k(\int_0^{\infty} f^p(x)dx)^{\frac{1}{p}}(\int_0^{\infty} g^q(x)dx)^{\frac{1}{q}}, \tag{2.5}$$

$$\int_0^{\infty}(\int_0^{\infty} k(x,y)f(x)dx)^p dy \le k^p\int_0^{\infty} f^p(x)dx, \tag{2.6}$$

where the constant factors k and k^p are the best possible.

If both $k(u,1)u^{-1/p}$ and $k(1,u)u^{-1/q}$ are decreasing functions, one still has the following equivalent inequalities with the best constant factors:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} k(m,n)a_m b_n \le k(\sum_{n=1}^{\infty} a_n^p)^{\frac{1}{p}}(\sum_{n=1}^{\infty} b_n^q)^{\frac{1}{q}}, \tag{2.7}$$

$$\sum_{n=1}^{\infty}(\sum_{m=1}^{\infty} k(m,n)a_m)^p \le k^p\sum_{n=1}^{\infty} a_n^p. \tag{2.8}$$

For $0 < p < 1$, if $k = \int_0^{\infty} k(u,1)u^{-1/p}du$ is finite, then one has the reverse forms of (2.5) and (2.6).

Remark 2.2. (a) Hardy, Littlewood and Polya [6] had not proved that the constant factors in (2.5), (2.6), (2.7) and (2.8) are the best possible. This is still an essential open problem. (b) Another open problem is to prove the reverse and reverse-type forms of inequalities (2.5)-(2.8).

For some particular types of functions of $k(x,y)$ (adding as well some conditions upon (2.3) and (2.5)), we obtain some classical Hilbert-type inequalities as follows:

(i) For $k(x,y)=\frac{1}{x+y}$, since $k=\pi/\sin(\frac{\pi}{p})$, (2.5) and (2.6) reduce to (2.3) and (2.4), (2.7) and (2.8) reduce to (2.1) and (2.2);

(ii) for $k(x,y)=\frac{1}{\max\{x,y\}}$, since $k=pq$, (2.5) -(2.8) reduce to

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}}dxdy < pq(\int_0^\infty f^p(x)dx)^{\frac{1}{p}}(\int_0^\infty g^q(x)dx)^{\frac{1}{q}}, \tag{2.9}$$

$$\int_0^\infty(\int_0^\infty \frac{f(x)}{\max\{x,y\}}dx)^p dy < (pq)^p\int_0^\infty f^p(x)dx, \tag{2.10}$$

$$\sum_{n=1}^\infty\sum_{m=1}^\infty \frac{a_m b_n}{\max\{m,n\}} < pq(\sum_{n=1}^\infty a_n^p)^{\frac{1}{p}}(\sum_{n=1}^\infty b_n^q)^{\frac{1}{q}}, \tag{2.11}$$

$$\sum_{n=1}^\infty(\sum_{m=1}^\infty \frac{a_m}{\max\{m,n\}})^p < (pq)^p\sum_{n=1}^\infty a_n^p; \tag{2.12}$$

(iii) for $k(x,y)=\frac{\ln(x/y)}{x-y}$, since $k=[\pi/\sin(\frac{\pi}{p})]^2$, (2.5) - (2.8) reduce to

$$\int_0^\infty\int_0^\infty \frac{\ln(x/y)f(x)g(y)}{x-y}dxdy < [\pi/\sin(\frac{\pi}{p})]^2(\int_0^\infty f^p(x)dx)^{\frac{1}{p}}(\int_0^\infty g^q(x)dx)^{\frac{1}{q}}, \tag{2.13}$$

$$\int_0^\infty(\int_0^\infty \frac{\ln(x/y)f(x)}{x-y}dx)^p dy < [\frac{\pi}{\sin(\frac{\pi}{p})}]^{2p}\int_0^\infty f^p(x)dx, \tag{2.14}$$

$$\sum_{n=1}^\infty\sum_{m=1}^\infty \frac{\ln(m/n)a_m b_n}{m-n} < [\frac{\pi}{\sin(\frac{\pi}{p})}]^2(\sum_{n=1}^\infty a_n^p)^{\frac{1}{p}}(\sum_{n=1}^\infty b_n^q)^{\frac{1}{q}}, \tag{2.15}$$

$$\sum_{n=1}^\infty(\sum_{m=1}^\infty \frac{\ln(m/n)a_m}{m-n})^p < [\frac{\pi}{\sin(\frac{\pi}{p})}]^{2p}\sum_{n=1}^\infty a_n^p. \tag{2.16}$$

Hardy, Littlewood and Polya [6] proved various multiple generalizations of (2.5) and (2.7). In [6], it was introduced a pair of non-conjugate exponents (p,q) and the following was proved. If $p>1, \frac{1}{p}+\frac{1}{q}\geq 1, 0<\lambda=2-(\frac{1}{p}+\frac{1}{q})\leq 1$, then

$$\sum_{n=1}^\infty\sum_{m=1}^\infty \frac{a_m b_n}{(m+n)^\lambda} \leq K(p,q)(\sum_{n=1}^\infty a_n^p)^{\frac{1}{p}}(\sum_{n=1}^\infty b_n^q)^{\frac{1}{q}}, \tag{2.17}$$

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy \leq K(p,q)(\int_0^\infty f^p(x)dx)^{\frac{1}{p}}(\int_0^\infty g^q(x)dx)^{\frac{1}{q}}, \tag{2.18}$$

where $K(p,q)$ is the best value only for $\lambda=1$. In 1937, Levin [7] proved an extension of (2.18) for the kernel

$$k(x,y)=\frac{1}{|x+y|^\lambda}\ (x,y\in(-\infty,\infty)).$$

In 1951, Bonsall [8] considered the forms of (2.17) and (2.18) in the case of general kernel (but the constants are not the best possible unless (p,q) is a pair of conjugate exponents). In 1991, Mitrinovic, Pecaric and Fink [9] provided an extensive analysis of these inequalities in their book.

3. Hilbert's inequality with an independent parameter

In the period 1979–2002, a number of mathematicians investigated (1.1) and (1.2) in several ways as follows:

(a) Hu [10] provided an improvement of (1.2) (for $f = g$) as follows

$$\int_0^\infty \int_0^\infty \frac{f(x)f(y)}{x+y}dxdy < \pi[(\int_0^\infty f^2(x)dx - \frac{1}{4}(\int_0^\infty f^2(x)\cos\sqrt{x}dx)^2]^{\frac{1}{2}}, \tag{3.1}$$

and then he obtained some interesting estimates (see [11]).

(b) In 1998, B.G.Pachpatte provided a proof of the following inequality that is similar to (1.2) by using Jensen's inequality[12]:

$$\int_0^a \int_0^b \frac{f(x)g(y)}{x+y}dxdy < \frac{1}{2}\sqrt{ab}(\int_0^a (a-x)f'^2(x)dx \int_0^b (b-x)g'^2(x)dx)^{\frac{1}{2}}. \tag{3.2}$$

For additional results the readers are referred to [13].

(c) In 1999, Gao [14] obtained the following improvement of (1.2) by applying techniques from Algebra and Mathematical Analysis:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \pi\sqrt{1-R}(\int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx)^{\frac{1}{2}}, \tag{3.3}$$

where $R = \frac{1}{\pi}(\frac{u}{||g||} - \frac{v}{||f||})^2, u = \sqrt{\frac{2}{\pi}}(g, e), v = \sqrt{2\pi}(f, e^{-s}), e(t) = \int_0^\infty \frac{e^{-s}}{s+t}ds.$

(d) In 2002, K. Zhang [15] gave an improvement of (1.2) by using operator theory as follows:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy$$

$$\leq \frac{\pi}{\sqrt{2}}[\int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx + (\int_0^\infty f(x)g(x)dx)^2]^{\frac{1}{2}}. \tag{3.4}$$

(e) In 1991, Xu et al. [16] proved the following inequality:

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{m+n} < (\sum_{n=1}^\infty [\pi - \frac{\theta}{\sqrt{n}}]a_n^2 \sum_{n=1}^\infty [\pi - \frac{\theta}{\sqrt{n}}]b_n^2)^{\frac{1}{2}}, \tag{3.5}$$

where $\theta = 1.1213^+$. The proof is as follows: By Cauchy's inequality,

$$\begin{aligned} \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{m+n} &= \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{1}{m+n}[(\frac{m}{n})^{\frac{1}{4}}a_m][(\frac{n}{m})^{\frac{1}{4}}b_n] \\ &\leq (\sum_{n=1}^\infty \omega(n)a_n^2 \sum_{n=1}^\infty \omega(n)b_n^2)^{\frac{1}{2}}, \end{aligned} \tag{3.6}$$

where the weight coefficient $\omega(n)$ is defined by

$$\omega(n) := \sum_{m=1}^\infty \frac{1}{m+n}(\frac{m}{n})^{\frac{1}{2}}, n \in \mathbf{N}.$$

Setting $\omega(n) = \pi - \frac{\theta(n)}{\sqrt{n}}$, it follows that

$$\theta(n) = [\pi - \sum_{m=1}^{\infty} \frac{1}{m+n} (\frac{m}{n})^{\frac{1}{2}}] n^{1/2}.$$

By applying methods from Mathematical Analysis, we obtain $\theta(n) > \theta = 1.1213^+$. Following the same method, Xu at al [17] proved a strengthened version of inequality (2.1) in the following form:

$$\omega(n) < \pi - \frac{\theta}{\sqrt{n}} (n \in \mathbf{N}, \theta = 1.1213^+),$$

and by (3.6), it follows that inequality (3.5) is valid.

By the same way, Xu et al.[17] also gave a strengthened version of (2.1) as follows:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \{\sum_{n=1}^{\infty} [\frac{\pi}{\sin(\frac{\pi}{p})} - \frac{p-1}{n^{1/p} + n^{-1/q}}] a_n^p\}^{\frac{1}{p}}$$

$$\times \{\sum_{n=1}^{\infty} [\frac{\pi}{\sin(\frac{\pi}{p})} - \frac{q-1}{n^{1/q} + n^{-1/p}}] b_n^q\}^{\frac{1}{q}},$$

In the year 1997-1998, Yang and Gao [18], [19] applying methods from [16], [17] proved a strengthened version of (2.1) in the following form:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \{\sum_{n=1}^{\infty} [\frac{\pi}{\sin(\frac{\pi}{p})} - \frac{1-\gamma}{n^{1/p}}] a_n^p\}^{\frac{1}{p}} \{\sum_{n=1}^{\infty} [\frac{\pi}{\sin(\frac{\pi}{p})} - \frac{1-\gamma}{n^{1/q}}] b_n^q\}^{\frac{1}{q}},$$

where $1-\gamma = 0.42278433^+$ (γ is the Euler constant). In 1998, again by applying the approach of weight coefficient, Yang [20] first introduced an independent parameter $0 < \lambda \le 1$ and proved an extension of inequality (1.2) as follows:

$$\int_0^{\infty}\int_0^{\infty} \frac{f(x)g(y)}{(x+y)^{\lambda}} dx dy < B(\frac{\lambda}{2}, \frac{\lambda}{2}) (\int_0^{\infty} x^{1-\lambda} f^2(x) dx \int_0^{\infty} x^{1-\lambda} g^2(x) dx)^{\frac{1}{2}}, \quad (3.7)$$

where $B(u,v)(u,v>0)$ is the Beta function. For $\lambda = 1$, (3.7) reduces to (1.2).

In 1999, Kuang [21] proved another extension of (1.2) in the form :

$$\int_0^{\infty}\int_0^{\infty} \frac{f(x)g(y)}{x^{\lambda} + y^{\lambda}} dx dy < \frac{\pi}{\lambda \sin(\frac{\pi}{2\lambda})} (\int_0^{\infty} x^{1-\lambda} f^2(x) dx \int_0^{\infty} x^{1-\lambda} g^2(x) dx)^{\frac{1}{2}}. \quad (3.8)$$

Since then several mathematicians proved a number of best extensions of (1.1), (1.2), (2.1) and (2.3) (cf. [22]-[31]). For example,

(a) in 2001, Yang [32] proved an extension of (1.1) in the following form : For $0 < \lambda \le 4$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{(m+n)^{\lambda}} < B(\frac{\lambda}{2}, \frac{\lambda}{2}) (\sum_{n=1}^{\infty} n^{1-\lambda} a_n^2 \sum_{n=1}^{\infty} n^{1-\lambda} b_n^2)^{\frac{1}{2}}, \quad (3.9)$$

where $B(\frac{\lambda}{2}, \frac{\lambda}{2})$ is the best possible; Hong [33] also proved a multiple extension of (1.2).

(b) In 2002, Yang [34] obtained another extension of (1.1) as follows:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^\lambda+n^\lambda}<\frac{\pi}{\lambda\sin(\frac{\pi}{p})}\{\sum_{n=1}^{\infty}n^{(p-1)(1-\lambda)}a_n^p\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty}n^{(q-1)(1-\lambda)}b_n^q\}^{\frac{1}{q}}, \tag{3.10}$$

where the constant factor $\frac{\pi}{\lambda\sin(\frac{\pi}{p})}$ $(0<\lambda\le 2)$ is the best possible.

(c) In 2003, Yang and Rassias [35] studied the way of weight coefficient and introduced independent parameters to prove certain improvements as well as best extensions of Hilbert-type inequalities.

4. Hilbert's inequality with two pairs of conjugate exponents

In 2004, Yang [36] proved the dual form of (2.3) in the following form:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n}<\frac{\pi}{\sin(\frac{\pi}{p})}(\sum_{n=1}^{\infty}n^{p-2}a_n^p)^{\frac{1}{p}}(\sum_{n=1}^{\infty}n^{q-2}b_n^q)^{\frac{1}{q}}, \tag{4.1}$$

where $\pi/\sin(\frac{\pi}{p})$ is the best possible. For $p=q=2$, both (4.1) and (2.1) reduce to (1.1). This means that there are two different best extensions of inequality (1.1).

Yang [37] proved an extension of inequality (2.3) by introducing two pairs of conjugate exponents and an independent parameter as follows: If $p,r>1, \frac{1}{p}+\frac{1}{q}=1, \frac{1}{r}+\frac{1}{s}=1, \lambda>0, f,g\ge 0$ such that $0<||f||_{p,\omega}:=[\int_0^\infty x^{p(1-\frac{\lambda}{r})-1}f^p(x)dx]^{\frac{1}{p}}<\infty$ and $0<||g||_{q,\varpi}:=[\int_0^\infty x^{q(1-\frac{\lambda}{s})-1}g^q(x)dx]^{\frac{1}{q}}<\infty$, then

$$\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{x^\lambda+y^\lambda}dxdy<\frac{\pi}{\lambda\sin(\frac{\pi}{r})}||f||_{p,\omega}||g||_{q,\varpi}, \tag{4.2}$$

where the constant factor $\pi/[\lambda\sin(\frac{\pi}{r})]$ is the best possible. In 2005, Yang et al. [38] gave a best multiple extension of (2.3) and deduced the following particular result:

$$\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{(x+y)^\lambda}dxdy<B(\frac{\lambda}{r},\frac{\lambda}{s})||f||_{p,\omega}||g||_{q,\varpi}, \tag{4.3}$$

where the constant factor $B(\frac{\lambda}{r},\frac{\lambda}{s})$ is the best possible.

For $\lambda=1, r=q, s=p$, both (4.2) and (4.3) reduce to (2.3); For $\lambda=1, r=p, s=q$, both (4.2) and (4.3) reduce to the dual form of (2.3) as:

$$\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{x+y}dxdy<\frac{\pi}{\sin(\frac{\pi}{p})}(\int_0^\infty x^{p-2}f^p(x)dx)^{\frac{1}{p}}(\int_0^\infty x^{q-2}g^q(x)dx)^{\frac{1}{q}}. \tag{4.4}$$

In 2005, Yang [39] gave an extension of (3.9) and (3.10) with two pairs of conjugate exponents and two parameters $\alpha,\lambda>0(\alpha\lambda\le\min\{r,s\})$ as

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m^\alpha+n^\alpha)^\lambda}<\frac{1}{\alpha}B(\frac{\lambda}{r},\frac{\lambda}{s})\{\sum_{n=1}^{\infty}n^{p(1-\frac{\alpha\lambda}{r})-1}a_n^p\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty}n^{q(1-\frac{\alpha\lambda}{s})-1}b_n^q\}^{\frac{1}{q}}, \tag{4.5}$$

where the constant factor $\frac{1}{\alpha}B(\frac{\lambda}{r},\frac{\lambda}{s})$ is the best possible. In [39] a reverse type inequality of (4.5) was obtained as follows: for $0<p<1$,

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m^\alpha+n^\alpha)^\lambda}>\frac{1}{\alpha}B(\frac{\lambda}{r},\frac{\lambda}{s})$$
$$\times\{\sum_{n=1}^{\infty}[1-\theta_p(n)]n^{p(1-\frac{\alpha\lambda}{r})-1}a_n^p\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty}n^{q(1-\frac{\alpha\lambda}{s})-1}b_n^q\}^{\frac{1}{q}}, \tag{4.6}$$

where the constant $\frac{1}{\alpha}B(\frac{\lambda}{r},\frac{\lambda}{s})$ is the best possible $(0<\theta_p(n)=O(\frac{1}{n^{(\lambda\alpha)/s}})<1)$.

In 2006, Hong [40] gave a best extension of (4.2) and (4.3) with the kernel

$$k(x,y)=\frac{1}{(||x||_\alpha^\beta+||y||_\alpha^\beta)^\lambda}(\alpha,\beta,\lambda>0;x,y\in\mathbf{R}_+^n).$$

Brnetic et al. [41] considered an extension of (4.3) with the general homogeneous kernel $k(x,y)$. A number of mathematicians have also considered an operator formulation of (2.5) and (2.6).

Suppose that $k(x,y)\geq 0$ is symmetric with $k(y,x)=k(x,y)$, and $k_0(p):=\int_0^\infty k(x,y)(\frac{x}{y})^{\frac{1}{r}}dy$ $(r=p,q;x>0)$ is a finite number independent of x. Define Hilbert-type integral operator $T:L^r(0,\infty)\to L^r(0,\infty)$ $(r=p,q)$ as follows: for $f(\geq 0)\in L^p(0,\infty)$, there exists a unique $h=Tf\in L^p(0,\infty)$, such that

$$(Tf)(y)=h(y):=\int_0^\infty k(x,y)f(x)dx, y\in(0,\infty). \tag{4.7}$$

Then we may define the formal inner product of Tf and g as follows

$$(Tf,g)=h(y):=\int_0^\infty\int_0^\infty k(x,y)f(x)g(y)dxdy. \tag{4.8}$$

Yang [42] proved that T is bounded and $||T||\leq k_0(p)$; if for $\varepsilon>0$ small enough, the integral $\int_0^\infty k(x,y)(\frac{x}{y})^{\frac{1+\varepsilon}{r}}dy=k_\varepsilon(p)$ is also a finite number independent of $x>0$, and $k_\varepsilon(p)=k_0(p)+o(1)(\varepsilon\to 0^+)$, then $||T||=k_0(p)$; if $||T||>0, f(\geq 0)\in L^p(0,\infty), g(\geq 0)\in L^q(0,\infty)$ and $||f||_p=(\int_0^\infty f^p(x)dx)^{\frac{1}{p}}, ||g||_q>0$, then one has two equivalent inequalities as follows:

$$(Tf,g)<||T||\cdot||f||_p\cdot||g||_q,\ ||Tf||_p<||T||\cdot||f||_p, \tag{4.9}$$

where the constant factor $||T||$ is the best possible.

Note. In particular, for $k(x,y)$ being $-1-$degree homogeneous, inequalities (4.9) reduce to (2.7)-(2.8)(in the symmetric kernel).

In 2006-2007, Yang et al. [43]-[48] also considered (4.9) in the cases of inner product spaces, the dispersed space $l^r(r=p,q)$ and the multiple integral operator T. In 2008, Yang et al. [49],[50] considered some conditions which make sure the Hilbert-type integral operators are bounded.

5. A basic theorem of Hilbert-type inequality with parameters

If $k_\lambda(x,y)$ is a measurable function, satisfying for $\lambda,u,x,y>0$,

$$k_\lambda(ux,uy)=u^{-\lambda}k_\lambda(x,y),$$

then we call $k_\lambda(x,y)$ the homogeneous function of -λ-degree. In 2009, Yang [51] obtained the following result.

Theorem 5.1. *Suppose that* $p, r > 1, \frac{1}{p}+\frac{1}{q}=1, \frac{1}{r}+\frac{1}{s}=1, \lambda>0, k_\lambda(x,y)(\geq 0)$ *is a homogeneous function of* $-\lambda$*-degree, and* $k_\lambda(r) := \int_0^\infty k(u,1)u^{\frac{\lambda}{r}-1}du$ *is a positive number for all* (r,s)*. If* $f(x), g(x)$ *are non-negative measurable functions,*

$$0 < ||f||_{p,\omega} = [\int_0^\infty x^{p(1-\frac{\lambda}{r})-1} f^p(x)dx]^{\frac{1}{p}} < \infty$$

and

$$0 < ||g||_{q,\varpi} = [\int_0^\infty x^{q(1-\frac{\lambda}{s})-1} g^q(x)dx]^{\frac{1}{q}} < \infty,$$

then we have the equivalent inequalities as:

$$I_\lambda := \int_0^\infty \int_0^\infty k_\lambda(x,y) f(x)g(y)dxdy < k_\lambda(r)||f||_{p,\omega}||g||_{q,\varpi}, \tag{5.1}$$

$$J_\lambda := \int_0^\infty y^{\frac{p\lambda}{s}-1}(\int_0^\infty k_\lambda(x,y)f(x)dx)^p dy < k_\lambda^p(r)||f||_{p,\omega}^p, \tag{5.2}$$

where the constant factors $k_\lambda(r)$ *and* $k_\lambda^p(r)$ *are the best possible.*

If both $k(u,1)u^{(\lambda/s)-1}$ and $k(1,u)u^{(\lambda/r)-1}$ are decreasing functions, which are strictly decreasing respectively in a subinterval, $a_n, b_n \geq 0$, such that

$$0 < ||a||_{p,\omega} := [\sum_{n=1}^\infty n^{p(1-\frac{\lambda}{r})-1} a_n^p]^{\frac{1}{p}} < \infty$$

and

$$0 < ||b||_{q,\varpi} := [\sum_{n=1}^\infty n^{q(1-\frac{\lambda}{s})-1} b_n^q]^{\frac{1}{q}} < \infty,$$

then we have the following equivalent inequalities with the best constant factors:

$$\widetilde{I}_\lambda := \sum_{n=1}^\infty \sum_{m=1}^\infty k_\lambda(m,n)a_m b_n < k_\lambda(r)||a||_{p,\omega}||b||_{q,\varpi}, \tag{5.3}$$

$$\widetilde{J}_\lambda := \sum_{n=1}^\infty n^{\frac{p\lambda}{s}-1}(\sum_{m=1}^\infty k_\lambda(m,n)a_m)^p < k_\lambda^p(r)||a||_{p,\omega}^p. \tag{5.4}$$

For $0 < p < 1$, we have the reverse forms of (5.1) and (5.2). If

$$k_\lambda(u,1) = O(\frac{1}{u^\alpha})(\alpha > \frac{\lambda}{r}; u \to \infty),$$

and in a subinterval containing 1, $k_\lambda(u,1) > 0$, then we have three equivalent reverse forms of (5.3) and (5.4) as

$$\widetilde{I}_\lambda > k_\lambda(r)\{\sum_{n=1}^\infty [1-\widetilde{\theta}_\lambda(r,n)]n^{p(1-\frac{\lambda}{r})-1}a_n^p\}^{\frac{1}{p}}||b||_{q,\varpi}, \tag{5.5}$$

$$\widetilde{J}_\lambda > k_\lambda^p(r)\sum_{n=1}^\infty [1-\widetilde{\theta}_\lambda(r,n)]n^{p(1-\frac{\lambda}{r})-1}a_n^p, \tag{5.6}$$

$$\sum_{m=1}^\infty \frac{m^{\frac{q\lambda}{r}-1}}{[1-\widetilde{\theta}_\lambda(r,m)]^{q-1}}(\sum_{n=1}^\infty k_\lambda(m,n)b_n)^q < k_\lambda^q(r)||b||_{q,\varpi}^q, \tag{5.7}$$

where the constant factors are the best possible ($\widetilde{\theta}_\lambda(r,n) = O(\frac{1}{n^{\lambda'}}) \in (0,1)(\lambda' > 0)$).

We only prove the case of integrals (for the case of series, see [51]).

Proof. (a) For $p > 1$, setting $\widetilde{k}_\lambda(s)$ as: $\widetilde{k}_\lambda(s) := \int_0^\infty k_\lambda(1,u)u^{\frac{\lambda}{s}-1}du$, then it follows $k_\lambda(r) = \widetilde{k}_\lambda(s)$. In fact, setting $v = \frac{1}{u}$, we obtain

$$\widetilde{k}_\lambda(s) = \int_0^\infty k_\lambda(1,\frac{1}{v})v^{\frac{-\lambda}{s}+1}\frac{dv}{v^2} = \int_0^\infty k_\lambda(v,1)v^{\frac{\lambda}{r}-1}dv = k_\lambda(r).$$

For $x, y \in (0,\infty)$, define the weight functions $\omega_\lambda(r,y)$ and $\varpi_\lambda(s,x)$ as:

$$\omega_\lambda(r,y) := \int_0^\infty k_\lambda(x,y)\frac{y^{\frac{\lambda}{s}}}{x^{1-\frac{\lambda}{r}}}dx, \varpi_\lambda(s,x) := \int_0^\infty k_\lambda(x,y)\frac{x^{\frac{\lambda}{r}}}{y^{1-\frac{\lambda}{s}}}dy. \tag{5.8}$$

Setting $x = yu$ in the integral $\omega_\lambda(r,y)$ and $y = xv$ in the integral $\varpi_\lambda(s,x)$, we find that $\omega_\lambda(r,y) = k_\lambda(r)$ and $\varpi_\lambda(s,x) = \widetilde{k}_\lambda(s)$. Then it follows that

$$\omega_\lambda(r,y) = \varpi_\lambda(s,x) = k_\lambda(r), \text{ for } x, y \in (0,\infty). \tag{5.9}$$

By Hölder's inequality[52], in view of (5.8) and (5.9), for $y > 0$, we have

$$\begin{aligned}
&(\int_0^\infty k_\lambda(x,y)f(x)dx)^p = \{\int_0^\infty k_\lambda(x,y)[\frac{x^{(1-\frac{\lambda}{r})/q}}{y^{(1-\frac{\lambda}{s})/p}}f(x)][\frac{y^{(1-\frac{\lambda}{s})/p}}{x^{(1-\frac{\lambda}{r})/q}}]dx\}^p \\
\leq &\int_0^\infty k_\lambda(x,y)\frac{x^{(1-\frac{\lambda}{r})(p-1)}}{y^{1-\frac{\lambda}{s}}}f^p(x)dx\{\int_0^\infty k_\lambda(x,y)\frac{y^{(1-\frac{\lambda}{s})(q-1)}}{x^{1-\frac{\lambda}{r}}}dx\}^{p-1} \\
= &[k_\lambda(r)]^{p-1}y^{1-\frac{p\lambda}{s}}\int_0^\infty k_\lambda(x,y)\frac{x^{(1-\frac{\lambda}{r})(p-1)}}{y^{1-\frac{\lambda}{s}}}f^p(x)dx.
\end{aligned} \tag{5.10}$$

Then in view of (5.8), (5.9) and (5.10), by Fubini's Theorem[53], it follows

$$\begin{aligned}
J_\lambda &\leq [k_\lambda(r)]^{p-1}\int_0^\infty\int_0^\infty k_\lambda(x,y)\frac{x^{(1-\frac{\lambda}{r})(p-1)}}{y^{1-\frac{\lambda}{s}}}f^p(x)dxdy \\
&= [k_\lambda(r)]^{p-1}\int_0^\infty[\int_0^\infty k_\lambda(x,y)\frac{x^{(1-\frac{\lambda}{r})(p-1)}}{y^{1-\frac{\lambda}{s}}}dy]f^p(x)dx \\
&= [k_\lambda(r)]^{p-1}\int_0^\infty \varpi_\lambda(s,x)x^{p(1-\frac{\lambda}{r})-1}f^p(x)dx = k_\lambda^p(r)||f||_{p,\omega}^p.
\end{aligned} \tag{5.11}$$

We need to show that (5.10) keeps the strict sign-inequality. If for a fixed $y > 0$, inequality (5.10) keeps the form of equality, then [52], there exist constants A and B, such that they are not all zero and

$$A\frac{x^{(1-\frac{\lambda}{r})(p-1)}}{y^{1-\frac{\lambda}{s}}}f^p(x) = B\frac{y^{(1-\frac{\lambda}{s})(q-1)}}{x^{1-\frac{\lambda}{r}}}a.e.\, in\,(0,\infty).$$

It follows $Ax^{p(1-\frac{\lambda}{r})}f^p(x) = By^{q(1-\frac{\lambda}{s})}$ $a.e.$ in $(0,\infty)$. We affirm that $A \neq 0$ (otherwise $B = A = 0$). Hence $x^{p(1-\frac{\lambda}{r})-1}f^p(x) = [By^{q(1-\frac{\lambda}{s})}]/(Ax)$ $a.e.$ in $(0,\infty)$. This contradicts the fact that $0 < ||f||_{p,\omega} < \infty$. Hence (5.11) still preserves the strict sign-inequality and thus (5.2) is satisfied. By Hölder's inequality, we obtain

$$I_\lambda = \int_0^\infty[y^{\frac{-1}{p}+\frac{\lambda}{s}}\int_0^\infty k_\lambda(x,y)f(x)dx][y^{\frac{1}{p}-\frac{\lambda}{s}}g(y)]dy \leq J_\lambda^{\frac{1}{p}}||g||_{q,\varpi}. \tag{5.12}$$

In view of inequality (5.2), one obtains (5.1). Suppose that (5.1) is valid. Setting $g(y) := y^{\frac{p\lambda}{s}-1}(\int_0^\infty k_\lambda(x,y)f(x)dx)^{p-1}$, then $||g||_{q,\varpi}^q = J_\lambda$. If $J_\lambda = \infty$, then by (5.11), it leads to a contradiction; if $J_\lambda = 0$, then (5.2) is naturally valid. Assuming that $0 < J_\lambda < \infty$, by (5.1), we find

$$0 < ||g||_{q,\varpi}^q = J_\lambda = I_\lambda < k_\lambda(r)||f||_{p,\omega}||g||_{q,\varpi} < \infty,$$

$||g||_{q,\varpi}^{q-1} = J_\lambda^{1/p} < k_\lambda(r)||f||_{p,\omega}$, and we obtain (5.2), which is equivalent to (5.1).

For $0 < \varepsilon < min\{\frac{p\lambda}{r}, \frac{q\lambda}{s}\}$, setting $f_\varepsilon(x) = g_\varepsilon(x) = 0$, for $x \in (0,1)$; $f_\varepsilon(x) = x^{\frac{\lambda}{r}-\frac{\varepsilon}{p}-1}$, $g_\varepsilon(x) = x^{\frac{\lambda}{s}-\frac{\varepsilon}{q}-1}$, for $x \in [1,\infty)$, if there exists a constant $0 < k \le k_\lambda(r)$, such that inequality (5.1) is still valid when we replace $k_\lambda(r)$ by k, then

$$H_\lambda := \varepsilon \int_0^\infty \int_0^\infty k_\lambda(x,y)f_\varepsilon(x)g_\varepsilon(y)dxdy < \varepsilon k||f_\varepsilon||_{p,\omega}||g_\varepsilon||_{q,\varpi} = k. \tag{5.13}$$

Setting $x = y/u$ in the following, by using Fubini's Theorem and (5.13), we find

$$\begin{aligned} k \;&>\; H_\lambda = \varepsilon \int_1^\infty y^{\frac{\lambda}{s}-\frac{\varepsilon}{q}-1}[\int_0^y k_\lambda(\frac{y}{u},y)(\frac{y}{u})^{\frac{\lambda}{r}-\frac{\varepsilon}{p}-1}\frac{y}{u^2}du]dy \\ &= \varepsilon \int_1^\infty y^{-1-\varepsilon}[\int_0^y k_\lambda(1,u)u^{\frac{\lambda}{s}+\frac{\varepsilon}{p}-1}du]dy \\ &= \int_0^1 k_\lambda(1,u)u^{\frac{\lambda}{s}+\frac{\varepsilon}{p}-1}du + \varepsilon\int_1^\infty y^{-1-\varepsilon}[\int_1^y k_\lambda(1,u)u^{\frac{\lambda}{s}+\frac{\varepsilon}{p}-1}du]dy \\ &= \int_0^1 k_\lambda(1,u)u^{\frac{\lambda}{s}+\frac{\varepsilon}{p}-1}du + \varepsilon\int_1^\infty [\int_u^\infty y^{-1-\varepsilon}dy]k_\lambda(1,u)u^{\frac{\lambda}{s}+\frac{\varepsilon}{p}-1}du \\ &= \int_0^1 k_\lambda(1,u)u^{\frac{\lambda}{s}+\frac{\varepsilon}{p}-1}du + \int_1^\infty k_\lambda(1,u)u^{\frac{\lambda}{s}-\frac{\varepsilon}{q}-1}du. \end{aligned} \tag{5.14}$$

Since $k_\lambda(1,u)u^{\frac{\lambda}{s}+\frac{\varepsilon}{p}-1} \le k_\lambda(1,u)u^{\frac{\lambda}{s}-1}$, $u \in (0,1)$; $k_\lambda(1,u)u^{\frac{\lambda}{s}-\frac{\varepsilon}{q}-1} \le k_\lambda(1,u)u^{\frac{\lambda}{s}-1}$, $u \in [1,\infty)$, and $\int_0^1 k_\lambda(1,u)u^{\frac{\lambda}{s}-1}du \le k_\lambda(r)$; $\int_1^\infty k_\lambda(1,u)u^{\frac{\lambda}{s}-1}du \le k_\lambda(r)$, then by Lebesgue's Control Convergence Theorem[53], it follows

$$\begin{aligned} \int_0^1 k_\lambda(1,u)u^{\frac{\lambda}{s}+\frac{\varepsilon}{p}-1}du &= \int_0^1 k_\lambda(1,u)u^{\frac{\lambda}{s}-1}du + o_1(1), \\ \int_1^\infty k_\lambda(1,u)u^{\frac{\lambda}{s}-\frac{\varepsilon}{q}-1}du &= \int_1^\infty k_\lambda(1,u)u^{\frac{\lambda}{s}-1}du + o_2(1)(\varepsilon \to 0^+). \end{aligned} \tag{5.15}$$

In view of (5.14) and (5.15), it follows $k \ge k_\lambda(r)(\varepsilon \to 0^+)$. Hence $k = k_\lambda(r)$ is the best value of (5.1). We conform that the constant factor in (5.2) is the best possible. Otherwise, by (5.12), we can get a contradiction that the constant factor in (5.1) is not the best possible.

(b) For $0 < p < 1$, by reverse Hölder's inequality applying the same way, we can obtain the equivalent reverse forms of (5.1) and (5.2). Suppose that there exists a constant $K \ge k_\lambda(r)$, such that the reverse of (5.1) is valid if we replace $k_\lambda(r)$ by K. In particular, for $0 < \varepsilon \le \varepsilon_0 < \frac{-q\lambda}{r}$, setting $f_\varepsilon, g_\varepsilon$ as (a), we can obtain the following inequality:

$$\begin{aligned} K &< H_\lambda = \int_0^1 k_\lambda(1,u)u^{\frac{\lambda}{s}+\frac{\varepsilon}{p}-1}du + \int_1^\infty k_\lambda(1,u)u^{\frac{\lambda}{s}-\frac{\varepsilon}{q}-1}du \\ &\le \int_0^1 k_\lambda(1,u)u^{\frac{\lambda}{s}-1}du + \int_1^\infty k_\lambda(1,u)u^{\frac{\lambda}{s}-\frac{\varepsilon}{q}-1}du. \end{aligned} \tag{5.16}$$

Setting (r_0, s_0): $\frac{1}{r_0} = \frac{1}{r} + \frac{\varepsilon_0}{q\lambda}, \frac{1}{s_0} = \frac{1}{s} - \frac{\varepsilon_0}{q\lambda}$, we find $r_0 > 1, \frac{1}{r_0} + \frac{1}{s_0} = 1$, and

$$\int_1^\infty k_\lambda(1,u)u^{\frac{\lambda}{s}-\frac{\varepsilon}{q}-1}du \le k_\lambda(r_0) < \infty.$$

By making use of Lebesgue's Control Convergence Theorem, we obtain the second expression of (5.15). Because of (5.16), it follows $K \le k_\lambda(r)(\varepsilon \to 0^+)$. Hence the constant factor $K = k_\lambda(r)$ in the reverse of (5.1) is the best possible. By the reverse inequality of (5.12) and the above result, we can show that the constant factor $k_\lambda^p(r)$ in the reverse of (5.2) is still the best possible. □

Remark 5.2. (a) For $\lambda = 1, s = p, r = q$, it follows that (5.1) and (5.2) reduce,respectively,to (2.5) and (2.6). It is obvious that Theorem 5.1 is partially an extension of Theorem 1.1; (b) we still can define an operator T to express Theorem 5.1 in the form (4.8); (c) in particular, for $k_\lambda(x,y) = \frac{1}{x^\lambda+y^\lambda}, \frac{1}{(x+y)^\lambda}$ in (5.1), we have (4.2) and (4.3); for $k_\lambda(x,y) = \frac{1}{\max\{x^\lambda,y^\lambda\}}, \frac{\ln(x/y)}{x^\lambda-y^\lambda}$ $(\lambda > 0)$ in (5.1), we obtain[54],[55]

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x^\lambda, y^\lambda\}}dxdy < \frac{rs}{\lambda}||f||_{p,\omega}||g||_{q,\varpi}, \tag{5.17}$$

$$\int_0^\infty \int_0^\infty \frac{\ln(x/y)}{x^\lambda - y^\lambda}f(x)g(y)dxdy < [\frac{\pi}{\lambda\sin(\frac{\pi}{r})}]^2||f||_{p,\omega}||g||_{q,\varpi}, \tag{5.18}$$

which are extensions of (2.9) and (2.13); for $k_\lambda(x,y) = \frac{\ln(x/y)}{\max\{x^\lambda,y^\lambda\}}(\lambda > 0)$, $(0 < \lambda < 1)$ in (5.1), we obtain[56],[57]

$$\int_0^\infty \int_0^\infty \frac{\ln(x/y)f(x)g(y)}{\max\{x^\lambda, y^\lambda\}}dxdy < \frac{r^2+s^2}{\lambda}||f||_{p,\omega}||g||_{q,\varpi}, \tag{5.19}$$

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{|x-y|^\lambda}dxdy < [B(1-\lambda, \frac{\lambda}{r}) + B(1-\lambda, \frac{\lambda}{s})]||f||_{p,\omega}||g||_{q,\varpi}, \tag{5.20}$$

which are new Hilbert-type integral inequalities with the particular kernels and best constant factors.

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[1] Department of Mathematics, Guangdong Education Institute, Guangzhou, Guangdong 510303, P. R. China

E-mail address: bcyang@pub.guangzhou.gd.cn

[2] Department of Mathematics, National Technical University of Athens, Zografou, Campus 15780 Athens, Greece.

E-mail address: trassias@math.ntua.gr