Int. J. Nonlinear Anal. Appl. 17 (2026) 2, 1–6
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2024.34597.5177

Generalized log orthogonal functions for solving a class of cordial Volterra integral equations

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(Communicated by Zakieh Avazzadeh)

Abstract

This paper deals with the numerical solution of a class cordial Volterra integral equation with the Mittag-Leffler solution. A numerical approach based on the generalized log orthogonal functions is proposed to solve this kind of Volterra integral equation. By using the generalized log orthogonal functions as a basis function, the presented numerical method can effectively approximate the solution of problems with singular behaviour. The error estimate with respect to L^2-norm is investigated. Finally, the accuracy of the method is illustrated through a numerical example.

Keywords: Cordial Volterra integral equation, Mittage−Leffler function, Generalized log orthogonal functions
2020 MSC: 45D05, 42C05

1 Introduction

This work is concerned with numerical solution for the second kind linear cordial Volterra integral equations (CVIEs) of the form

$$u(t) = f(t) + a\int_0^t t^{-1}\varphi(t^{-1}s)k(t,s)u(s)ds, \tag{1.1}$$

whose solution can be expressed in terms of the Mittag−Leffler function defined by $E_d(z) := \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(1+nd)}$, $z\in\mathbb{C}, d>0$, in which Γ denotes the gamma function. The function $f\in C^m(I)$, a stands for an arbitrary constant, $\varphi(t^{-1}s)=\frac{t^b(1-t^{-1}s)^{b-1}}{\Gamma(b)}$, $0<b<1$, and without loss of generality we assume $k(t,s)=1$. The cordial Volterra integral operator

$$(\mathcal{V}_\varphi u)(t) = \int_0^t t^{-1}\varphi(t^{-1}s)k(t,s)u(s)ds, \qquad t\in I := [0,T], \tag{1.2}$$

is inspired by Vainikko's studies [11, 12]. The function $\varphi\in L^1(0,1)$ is the core of the operator, and $k\in C^m(D)$ for some $m\geq 0$ where $D=\{(t,s): 0\leq s\leq t\leq T\}$. The cordial Volterra integral operators are a special class of Volterra integral operators with weak singular kernels that appear in the study of heat conduction problems with mixed boundary conditions and some Volterra integral operators with certain kernel singularities [1, 7]. Such operators and the associated Volterra integral equations have been studied by Vainikko [13, 14] and several other authors [5, 6, 16].

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Received: June 2024 *Accepted:* September 2024

It is of interest to know the CVIEs have singular behavior at the initial point $t = 0$. In general, facing singular problems, in order to develop accurate spectral methods, there are some strategies such as employing a local adaptive procedure in finite differences/finite elements [8], singular functions method [10], the enriched spectral methods [2, 3], and mapped spectral methods [9, 15]. In [4] authors suggested that mapped spectral methods on the non-uniformly Sobolev weighted spaces are more suitable for equations with singular behaviors. Actually, these methods lead to better convergence results than numerical methods for instance finite-element, finite-difference and spectral methods on usual Sobolev spaces. The choice of a log mapping to generalized Laguerre polynomials $\mathbf{L}_n^{(\alpha)}(x)$, $\alpha > -1$, seemed to be the best adapted to their theory. Thus, they introduced two new classes of orthogonal functions on the non-uniformly Sobolev weighted spaces, log orthogonal functions (LOFs) and generalized log orthogonal functions (GLOFs). Now, in order to solve numerically (1.1), we apply the spectral collocation method using the GLOFs as basis functions.

The layout of this paper is as follows: Section 2, presents definitions and some properties of the LOFs and their generalized type, approximation by the GLOFs along with operational matrices, and applies the well-known spectral collocation method for solving (1.1). The error estimation of the approximate solution will be studied in section 3. In Section 4, a numerical example is given to clarify the effectiveness of the proposed method. Finally, in the last section, we present our conclusion.

2 Generalized log orthogonal functions

In this section, the generalized log orthogonal functions will be introduced [4].

Definition 2.1. For $\alpha, \beta > -1$ the LOFs are defined by

$$\mathcal{S}_n^{(\alpha,\beta)}(t) := \mathbf{L}_n^{(\alpha)}(x(t)) = \mathbf{L}_n^{(\alpha)}(-(\beta+1)\log(t)), \qquad n = 0, 1, \dots,$$

with satisfying the following properties:

- Three-term recurrence relation

$$\begin{aligned}
&\mathcal{S}_0^{(\alpha,\beta)}(t) = 1,\\
&\mathcal{S}_1^{(\alpha,\beta)}(t) = (\beta+1)\log(t) + \alpha + 1,\\
&\mathcal{S}_{n+1}^{(\alpha,\beta)}(t) = \frac{2n+\alpha+1+(\beta+1)\log(t)}{n+1}\mathcal{S}_n^{(\alpha,\beta)}(t) - \frac{n+\alpha}{n+1}\mathcal{S}_{n-1}^{(\alpha,\beta)}(t), \qquad n = 1, 2, \dots,
\end{aligned}$$

- Orthogonality

$$\int_0^1 \mathcal{S}_n^{(\alpha,\beta)}(t)\mathcal{S}_m^{(\alpha,\beta)}(t)(-\log(t))^\alpha t^\beta dt = \gamma_n^{(\alpha,\beta)}\delta_{nm}, \qquad \gamma_n^{(\alpha,\beta)} = \frac{\Gamma(n+\alpha+1)}{(\beta+1)^{\alpha+1}\Gamma(n+1)}. \tag{2.1}$$

Definition 2.2. For $\alpha, \beta > -1$, $\lambda \in \mathbb{R}$, the GLOFs are defined by

$$\mathcal{S}_n^{(\alpha,\beta,\lambda)}(t) := t^{(\beta-\lambda)/2}\mathcal{S}_n^{(\alpha,\beta)}(t), \quad \lambda \in \mathbb{R}, \quad n \geq 0,$$

with satisfying orthogonality condition

$$\int_0^1 \mathcal{S}_n^{(\alpha,\beta,\lambda)}(t)\mathcal{S}_m^{(\alpha,\beta,\lambda)}(t)(-\log(t))^\alpha t^\lambda dt = \gamma_n^{(\alpha,\beta)}\delta_{mn}, \tag{2.2}$$

in which $\gamma_n^{(\alpha,\beta)}$ is already defined in (2.1). It is interesting to know that by choosing $\lambda = \beta$, the GLOFs are the same as the LOFs.

2.1 Approximation by the GLOFs

To obtain an approximation of any function $f \in L^2[0,1]$ in terms of the GLOFs, one can write

$$f(t) = \sum_{i=0}^{\infty} c_i \mathcal{S}_i^{(\alpha,\beta,\lambda)}(t), \qquad f(t) \simeq f_n(t) = \sum_{i=0}^{n} c_i \mathcal{S}_i^{(\alpha,\beta,\lambda)}(t) := C^T\Phi(t) = \Phi^T(t)C$$

where $c_i = \langle f(t), \mathcal{S}_i^{(\alpha,\beta,\lambda)}(t)\rangle$ in which $\langle .,. \rangle$ denotes the inner product with respect to the weight function $\chi^{\alpha,\lambda}(t) := (-\log(t))^\alpha t^\lambda$, and $(n+1)$-order vectors C, $\Phi(t)$ are given by

$$C = [c_0, c_1, \dots, c_n]^T, \qquad \Phi(t) = [\mathcal{S}_0^{(\alpha,\beta,\lambda)}(t), \mathcal{S}_1^{(\alpha,\beta,\lambda)}(t), \dots, \mathcal{S}_n^{(\alpha,\beta,\lambda)}(t)]^T. \tag{2.3}$$

Similarly, approximation of a two-variable function $\mathcal{K}(t,s) \in L^2([0,1]\times[0,1])$ is as follows

$$\mathcal{K}(t,s) \simeq \mathcal{K}_n(t,s) = \sum_{i=0}^{n}\sum_{j=0}^{n} \mathcal{K}_{ij}\mathcal{S}_i^{(\alpha,\beta,\lambda)}(t)\mathcal{S}_j^{(\alpha,\beta,\lambda)}(s) = \Phi^T(t)\mathcal{K}\Phi(s), \tag{2.4}$$

where $\mathcal{K}$ is an $(n+1)\times(n+1)$ matrix with coefficients $\mathcal{K}_{ij}$ are given by

$$\mathcal{K}_{ij} = \langle \mathcal{S}_i^{(\alpha,\beta,\lambda)}(t), \langle \mathcal{K}(t,s), \mathcal{S}_j^{(\alpha,\beta,\lambda)}(s)\rangle_{\chi^{\alpha,\lambda}(s)}\rangle_{\chi^{\alpha,\lambda}(t)}, \qquad i,j = 0,1,\dots,n.$$

2.2 Operational matrices

We can approximate the integration of the vector $\Phi(t)$ defined in (2.3) as follows

$$\int_0^t \Phi(\tau)d\tau \simeq \mathcal{P}\Phi(t), \tag{2.5}$$

where $\mathcal{P}$ is the GLOFs operational matrix of integration of order $(n+1)\times(n+1)$ with coefficients $\mathcal{P}_{ij}$ are given by

$$\mathcal{P}_{ij} = \frac{\left\langle \int_0^t \mathcal{S}_i^{(\alpha,\beta,\lambda)}(\tau)d\tau, \mathcal{S}_j^{(\alpha,\beta,\lambda)}(t)\right\rangle}{\left\langle \mathcal{S}_j^{(\alpha,\beta,\lambda)}(t), \mathcal{S}_j^{(\alpha,\beta,\lambda)}(t)\right\rangle}. \tag{2.6}$$

Furthermore, we have

$$\Phi(t)\Phi^T(t)C \simeq \tilde{C}^T\Phi(t), \qquad \Phi^T(t)C\Phi(t) = \hat{C}^T\Phi(t), \tag{2.7}$$

where $\tilde{C}$ is the product operation matrix of two GLOFs and using (2.2) the elements $\left\{\tilde{C}_{ij}\right\}_{i,j=0}^{n}$ can be calculated from

$$\tilde{C}_{ij} = \left(\gamma_j^{(\alpha,\beta)}\right)^{-1}\sum_{k=0}^{n} c_k g_{ijk},$$

where

$$g_{ijk} = \int_0^1 \mathcal{S}_i^{(\alpha,\beta,\lambda)}(t)\mathcal{S}_j^{(\alpha,\beta,\lambda)}(t)\mathcal{S}_k^{(\alpha,\beta,\lambda)}(t)(-\log(t))^\alpha dt.$$

Similarly, entries of matrix $\hat{C}$ can be obtained that are related to the vector C.

2.3 Methodology

Here, a numerical method based on the GLOFs and their operational matrices is introduced to solve (1.1). In order to solve (1.1) using the collocation method, we approximate the functions $u(t)$, $f(t)$, and $\mathcal{K}(t,s) := at^{-1}\varphi(t^{-1}s)k(t,s)$ by the GLOFs with coefficients determined by collocating (1.1) at the nodal points $\{t_i\}_{i=0}^n$, which are $(n+1)$ roots of Chebyshev polynomials $T_{n+1}(t)$ of degree $(n+1)$ on $[0,1]$. Assume

$$u(t) \simeq \sum_{i=0}^{n} c_i\mathcal{S}_i^{(\alpha,\beta,\lambda)}(t) = C^T\Phi(t) = \Phi^T(t)C, \quad f(t) \simeq \sum_{i=0}^{n} f_i\mathcal{S}_i^{(\alpha,\beta,\lambda)}(t) = F^T\Phi(t), \quad \mathcal{K}(t,s) \simeq \Phi^T(t)\mathcal{K}\Phi(s), \tag{2.8}$$

where C, $\mathcal{K}$ are defined in (2.3), (2.4), respectively, and $F = [f_0, f_1, \dots, f_n]^T$ is a known vector defined similarly to C. It is obtained by substituting (2.8) into (1.1), using (2.5) and (2.7), and setting $Y := \mathcal{K}\tilde{C}^T\mathcal{P}$

$$\begin{aligned} C^T\Phi(t) &\simeq F^T\Phi(t) + \int_0^t \Phi^T(t)\mathcal{K}\Phi(s)\Phi^T(s)Cds \\ &= F^T\Phi(t) + \Phi^T(t)\mathcal{K}\tilde{C}^T\int_0^t \Phi(s)ds \\ &= F^T\Phi(t) + \Phi^T(t)\mathcal{K}\tilde{C}^T\mathcal{P}\Phi(t) \\ &= F^T\Phi(t) + \hat{Y}^T\Phi(t). \end{aligned}$$

It follows that

$$\left(C^T - F^T - \hat{Y}^T\right)\Phi(t) \simeq 0. \tag{2.9}$$

Now, if we collocate (2.9) in $(n+1)$ points $\{t_i\}_{i=0}^n$ and replace $\simeq$ with $=$, we achieve

$$\left(C^T - F^T - \hat{Y}^T\right)\Phi(t_i) = 0, \qquad i = 0, 1, \ldots, n. \tag{2.10}$$

The equation (2.10) produces a linear system of $(n+1)$ equations and $(n+1)$ unknowns that can be solved for the unknown vector C. Thus, the approximate solution of (1.1) will be obtained by $u(t) \simeq C^T\Phi(t)$.

3 Convergence

In this section, we present the approximation error by the GLOFs. The construction is due to Chen and Shen [4]. First of all, suppose that $u_n(t)$ is the approximate solution of (1.1). The error function will be

$$e_n(t) = u(t) - u_n(t) = \sum_{i=n+1}^{\infty} c_i \mathcal{S}_i^{(\alpha,\beta,\lambda)}(t).$$

Consider a pseudo−derivative with respect to the LOFs as follows

$$\hat{\partial}_t u := t\partial_t u.$$

Assume

$$A^k_{\alpha,\beta}(I) := \left\{\nu \in L^2_{\chi^{\alpha,\beta}}(I) : \hat{\partial}^j_t \nu \in L^2_{\chi^{\alpha+j,\beta}}(I), j = 1, 2, ..., k\right\}, \qquad k \in \mathbb{N},$$

is a non−uniformly weighted Sobolev space equipped with the semi−norm and norm

$$|\, \nu \,|_{A^m_{\alpha,\beta}} :=\| \, \hat{\partial}^m_t \nu \, \|_{\chi^{\alpha+m,\beta}}, \qquad \| \, \nu \, \|_{A^m_{\alpha,\beta}} := \left(\sum_{k=0}^{m} |\, \nu \,|^2_{A^k_{\alpha,\beta}}\right)^{1/2}.$$

The pseudo−derivative with respect to the GLOFs can be defined as

$$\hat{\partial}_{\gamma,t} u := t^{1+\gamma}\partial_t\left\{t^{-\gamma}u\right\}.$$

Furthermore, to better describe the approximability of $u_n(t)$ by the GLOFs, we need to define a non−uniformly weighted Sobolev space as

$$A^k_{\alpha,\beta,\lambda}(I) := \left\{\nu \in L^2_{\chi^{\alpha,\lambda}}(I) : \hat{\partial}^j_{\frac{\beta-\lambda}{2},t} \nu \in L^2_{\chi^{\alpha+j,\lambda}}(I), j = 1, 2, ..., k\right\}, \qquad k \in \mathbb{N},$$

equipped with semi−norm and norm as

$$|\, \nu \,|_{A^m_{\alpha,\beta,\lambda}} :=\| \, \hat{\partial}^m_{\frac{\beta-\lambda}{2},t} \nu \, \|_{\chi^{\alpha+m,\lambda}}, \qquad \| \, \nu \, \|_{A^m_{\alpha,\beta,\lambda}} := \left(\sum_{k=0}^{m} |\, \nu \,|^2_{A^k_{\alpha,\beta,\lambda}}\right)^{1/2}.$$

Theorem 3.1. [4] Given $f(t) = t^r(-\log(t))^k, r \geq 0, k \in \mathbb{N}_0$. Let $\lambda > -1 - 2r, \alpha, \beta > -1$ and $\beta > \lambda$. Then, we have

$$f \in L^2_{\chi^{\alpha,\lambda}} \quad \text{and} \quad \mathbf{R}_{r,\beta,\lambda} = \left|\frac{2r + \lambda - \beta}{2r + 2 + \lambda + \beta}\right| < 1,$$

and

$$\| \, f - f_n \, \|_{\chi^{\alpha,\lambda}} \leq c(k+1)! n^{\frac{\alpha+1}{2}+k}(\mathbf{R}_{r,\beta,\lambda})^n \quad \text{when} \quad n > -\frac{2k + \alpha + 2}{2log(\mathbf{R}_{r,\beta,\lambda})},$$

where

$$c \approx \sqrt{\frac{2^{\alpha+1+k}(\beta+1)^{2\alpha+2-k}}{(\beta+\lambda+2r+2)^{\alpha+1+k}}}.$$

In particular, if $\alpha = \lambda = 0$, then an accurate estimate for the GLOFs to singular functions in L^2-norm is obtained as

$$\| f - f_n \| \leq \sqrt{2}^k (\beta+1)^{-k} k! n^k \sqrt{2(\beta+1)n} \left| \frac{2r-\beta}{2r+\beta+2} \right|^{n-k}.$$

Also, for $f(t) = t^r$, $r \geq 0$, we have

$$\| f - f_n \| \leq \sqrt{2(\beta+1)n} \left| \frac{2r-\beta}{2r+\beta+2} \right|^{n}.$$

Theorem 3.2. [4] Let $m, n, k \in \mathbb{N}, \lambda \in \mathbb{R}$ and $\alpha, \beta > -1$. For any $u \in A^m_{\alpha,\beta,\lambda}(I)$ and $0 \leq k \leq \tilde{m} = \min\{m, n+1\}$, we have

$$\| \hat{\partial}^k_{\frac{\beta-\lambda}{2},t} (u - u_n) \|_{\chi^{\alpha+k,\lambda}} \leq \sqrt{(\beta+1)^{k-\tilde{m}} \frac{(n-\tilde{m}+1)!}{(n-k+1)!}} \| \hat{\partial}^{\tilde{m}}_{\frac{\beta-\lambda}{2},t} u \|_{\chi^{\alpha+\tilde{m},\lambda}} ,$$

In particular, in the case of $\alpha = \beta = \lambda = k = 0$ and $m < n+1$, it holds that

$$\| u - u_n \| \leq c n^{-m/2} \| \hat{\partial}^m_t u \|_{\chi^m} ,$$

where $\chi^m = \chi^{m,0} = (-logt)^m$.

4 Numerical examples

In this section, we implement the collocation method given in Subsection 2.3 numerically for

$$u(t) = 1 - \sqrt{\pi} \int_0^t t^{-1} \varphi(t^{-1}s) u(s) ds, \qquad 0 \leq t \leq 1, \tag{4.1}$$

which has the exact solution $u(t) = E_{1/2}(\sqrt{\pi t})$. Here $\varphi(t^{-1}s) = \frac{t^{1/2}(1-t^{-1}s)^{-1/2}}{\Gamma(1/2)}$. Table 1 and Figure 1 illustrate the asymptotics of the proposed method numerically for this example. Table 1 exhibits the errors obtained by using the GLOFs with $\alpha = 0, \beta = 1, \lambda = -1$. It can be seen that as the number of the GLOFs increases the accuracy of the solution will reasonably improve. In Figure 1, we have shown the graphic representation of the exact and approximate solution of (4.1) for $n = 6$ with $\alpha = 0$, $\beta = 1$, $\lambda = -1$.

Table 1: The absolute errors with $\alpha = 0$, $\beta = 1$, $\lambda = -1$ in (4.1).

t	$n = 2$	$n = 4$	$n = 6$
0	0	0	0
0.125	$9.507E-2$	$1.921E-3$	$6.181E-6$
0.250	$9.409E-2$	$9.815E-4$	$3.570E-6$
0.375	$3.022E-2$	$5.985E-4$	$5.473E-6$
0.500	$3.158E-2$	$9.382E-5$	$2.173E-6$
0.625	$6.673E-2$	$2.710E-4$	$1.425E-6$
0.750	$6.411E-2$	$2.620E-5$	$1.797E-6$
0.875	$1.839E-2$	$1.970E-4$	$1.320E-6$
1.000	$7.298E-2$	$1.648E-4$	$1.292E-6$

5 Conclusion

The log orthogonal functions and their generalized type were introduced. The distinctive feature of these functions is that they are very useful in resolving singularities. These functions were used to numerically solve equation (1.1). An illustrative example is presented to assess the effectiveness of the method.

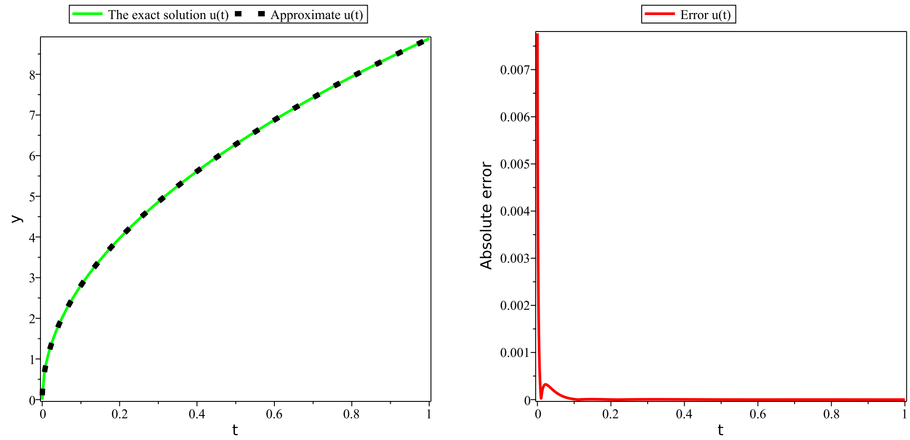

Figure 1: The approximate and the exact solution (left) and the absolute error (right) for $n = 6$ with $\alpha = 0$, $\beta = 1$, $\lambda = -1$ in (4.1).

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