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On existence and uniqueness of solutions of a nonlinear Volterra-Fredholm integral equation

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Abstract

In this paper we investigate the existence and uniqueness for Volterra-Fredholm type integral equations and extension of this type of integral equations. The result is obtained by using the coupled fixed point theorems in the framework of Banach space $X = C([a, b], \mathbb{R})$. Finally, we give an example to illustrate the applications of our results.

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1. Introduction and Preliminaries

In this paper we intend to prove existence and uniqueness of the solutions of the following nonhomogeneous nonlinear Volterra-Fredholm integral equations.

$$u(x) = \varphi_1 \left[\int_a^x F(x, t, u(t))dt \right] + \varphi_2 \left[\int_a^b G(x, t, u(t))dt \right] + g(x) \tag{1.1}$$

where $-\infty < a < b < \infty$ and F and G are two continuous mappings on the domain $D := \{(x, t, u) : x \in [a, b], t \in [a, x], u \in X\}$ and $g : [a, b] \to \mathbb{R}$ is a continuous mapping.

Many authors use fixed point theorems to prove existence and uniqueness the solutions of integral equations(see [3],[5],[8]). We extend Volterra-Fredholm integral equation and discuss about the solutions of this category of integral equations, but for this intention not used common fixed

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point theorems, instead we used coupled fixed point theorems for mappings having mixed monotone property. First we recall some basic results which we will need in this paper.

Through this article, we consider the complete metric space (X, d) which $X = C([a, b], \mathbb{R})$ and $d(f, g) = sup_{x\in[a,b]}|f(x) - g(x)|$ for all $f, g \in X$.

Definition 1.1. ([2]) A pair $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if

$$F(x, y) = x, F(y, x) = y.$$

The concept of a mixed monotone property has been introduced by Bhaskar and Lakshmikantham in [2].

Definition 1.2. Let (X,$\preceq$) be a partially ordered set. We say that a mapping $F : X \times X \to X$ has the mixed monotone property if $F(x, y)$ is monotone nondecreasing in x and is monotone nonincreasing in y, that is, for any $x, y \in X$, we have:

$$x_1, x_2 \in X, x_1 \preceq x_2 \Rightarrow F(x_1, y) \preceq F(x_2, y)$$
$$y_1, y_2 \in X, y_1 \preceq y_2 \Rightarrow F(x, y_2) \preceq F(x, y_1).$$

In 2011, Berinde [1] established some generalized coupled fixed point results for the mixed monotone mappings that introduce as follows.

Theorem 1.3. Let $(X, \preceq)$ be a partially ordered set and suppose there exists a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a mixed monotone mapping for which there exists a constant $k \in [0, 1)$ such that

$$d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \leq k\ [d(x, u) + d(y, v)], \tag{1.2}$$

for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$. If there exist $x_0, y_0 \in X$ such that

$$x_0 \preceq F(x_0, y_0) \ \ and \ \ y_0 \succeq F(y_0, x_0),$$

or

$$x_0 \succeq F(x_0, y_0) \ \ and \ \ y_0 \preceq F(y_0, x_0),$$

then there exist $\bar{x}, \bar{y} \in X$ such that $\bar{x} = F(\bar{x}, \bar{y})$ and $\bar{y} = F(\bar{y}, \bar{x})$.

If every pair of elements in $X \times X$ has either a lower bound or an upper bound, which is known, (see Theorem 2 [9]); that is; for all $(x, y), (u, v) \in X \times X$, there exists $(w, z) \in X \times X$ such that is comparable to (x, y) and (u, v), then we have the following theorems [1].

Theorem 1.4. Adding above condition to the hypotheses of Theorem 1.3, we obtain the uniqueness of the coupled fixed point of F.

Similarly to [2], by assuming the same condition as in Theorem 1.4 but with respect to the ordered set X; that is; by assuming that every pair of elements of X have either an upper bound or a lower bound in X , we can show that even the components of the coupled fixed points are equal.

Theorem 1.5. In addition to the hypothesis of Theorem 1.3, suppose that $x_0, y_0 \in X$ are comparable. Then for the coupled fixed point $(\bar{x}, \bar{y})$ we have $\bar{x} = \bar{y}$, that is, F has a fixed point.

$$F(\bar{x}, \bar{x}) = \bar{x}$$

2. Main Results

In this section we use the previous theorems to show that the equation (1.1) has a unique solution. Let Φ denoted the class of those functions $\varphi : \mathbb{R} \to \mathbb{R}$ which satisfies the following conditions:
(a) φ is nondecreasing,
(b) φ is Lipschitz; that is; there exists $L > 0$ such that for every $t, s \in \mathbb{R}$ we have

$$|\varphi(t) - \varphi(s)| \leq L|t - s|.$$

For example every linear function on $\mathbb{R}$ belong to Φ.

Now we consider the equation (1.1) under the following conditions:

(i) $\varphi_1, \varphi_2 \in \Phi$ with constants L_1 and L_2 respectively.

(ii) There exists two integrable functions $p_1, p_2 : [a, b] \times [a, b] \to \mathbb{R}$ such that for every $u, v \in X$ with $v \preceq u$

$$0 \preceq F(x, t, u) - F(x, t, v) \preceq p_1(x, t)|u - v| \tag{2.1}$$

and

$$-p_2(x, t)|u - v| \preceq G(x, t, u) - G(x, t, v) \preceq 0. \tag{2.2}$$

(iii) There exist $K_1, K_2 \in [0, 1)$ such that

$$sup_{x\in[a,b]} \int_a^b p_1(x, t)dt \leq \frac{K_1}{2L_1} \tag{2.3}$$

$$sup_{x\in[a,b]} \int_a^b p_2(x, t)dt \leq \frac{K_2}{2L_2}. \tag{2.4}$$

(iv) There exist $\alpha, \beta \in C([a, b], \mathbb{R})$ such that

$$\alpha(x) \leq \varphi_1\left[\int_a^b F(x, t, \alpha(t))dt\right] + \varphi_2\left[\int_a^b G(x, t, \beta(t))dt\right] + g(x),$$

and

$$\beta(x) \geq \varphi_1\left[\int_a^b F(x, t, \beta(t))dt\right] + \varphi_2\left[\int_a^b G(x, t, \alpha(t))dt\right] + g(x),$$

for all $x \in [a, b]$.

Theorem 2.1. *Under the assumptions $(i) - (iv)$ the integral equation (1.1) has a unique solution in $C([a, b], \mathbb{R})$.*

Proof . Let $X = C([a, b], \mathbb{R})$. We endow the set X with the following partial order $\preceq$ defined by

$$u \preceq v \Leftrightarrow u(x) \preceq v(x) \text{ for all } x \in [a, b].$$

With the respect to the definition of partially ordered above, for every $u, v \in X$, $max\{u, v\}$ and $min\{u, v\}$ are in X. Therefor, for every $(u, v), (w, z) \in X \times X$ there exist a $(max\{u, w\}, min\{v, z\}) \in X \times X$ that is comparable to (u, v) and (w, z).

Also we consider the complete metric space (X, d) ,which

$$d(u, v) = sup_{x\in[a,b]}\{|u(x) - v(x)|\}$$

for all $u, v \in X$.

Now we define the operator $T : X \times X \to X$ as follows:

$$T(u, v)(x) = \varphi_1 \left[\int_a^x F(x, t, u(t))dt\right] + \varphi_2 \left[\int_a^b G(x, t, v(t))dt\right] + g(x).$$

At first we prove that T has the mixed monotone property. For this goal suppose that $u_1, u_2, v \in X$ such that $u_1 \preceq u_2$. Then

$$\begin{aligned} T(u_1, v)(x) - T(u_2, v)(x) &= \varphi_1 \left[\int_a^x F(x, t, u_1(t))dt\right] + \varphi_2 \left[\int_a^b G(x, t, v(t))dt\right] \\ &\quad -\varphi_1 \left[\int_a^x F(x, t, u_2(t))dt\right] - \varphi_2 \left[\int_a^b G(x, t, v(t))dt\right] \\ &= \varphi_1 \left[\int_a^x F(x, t, u_1(t))dt\right] - \varphi_1 \left[\int_a^x F(x, t, u_2(t))dt\right]. \end{aligned} \tag{2.5}$$

Taking into account that $u_1 \preceq u_2$ and our assumption,

$$F(x, t, u_1(t)) - F(x, t, u_2(t)) \leq 0.$$

Since φ_1 is nondecreasing and from (2.5) we obtain

$$T(u_1, v)(x) - T(u_2, v)(x) \leq 0.$$

Hence $T(u_1, v)(x) \leq T(u_2, v)(x)$ for all $x \in [a, b]$ and by definition of partially ordered relation we have $T(u_1, v) \preceq T(u_2, v)$.

Similarly if $v_1, v_2, u \in X$ such that $v_1 \preceq v_2$ we get $T(u, v_2) \preceq T(u, v_1)$. Thus $T(u, v)$ is monotone nondecreasing in u and monotone nonincreasing in v.

Now we assume that $u, v, z, w \in X$ such that $u \succeq w$ and $v \preceq z$. Then for all $x \in [a, b]$

$$\begin{aligned} |T(u, v)(x) - T(w, z)(x)| &= \Big|\varphi_1\Big[\int_a^x F(x, t, u(t))dt\Big] + \varphi_2\Big[\int_a^b G(x, t, v(t))dt\Big] + g(x) \\ &\quad -\varphi_1\Big[\int_a^x F(x, t, w(t))dt\Big] - \varphi_2\Big[\int_a^b G(x, t, z(t))dt\Big] - g(x)\Big| \\ &= \Big|\varphi_1\Big[\int_a^x F(x, t, u(t))dt\Big] - \varphi_1\Big[\int_a^x F(x, t, w(t))dt\Big] \\ &\quad +\varphi_2\Big[\int_a^b G(x, t, v(t))dt\Big] - \varphi_2\Big[\int_a^b G(x, t, z(t))dt\Big]\Big| \end{aligned}$$

and so

$$\begin{aligned}
& |T(u,v)(x) - T(w,z)(x)| \qquad (2.6)\\
\leq\ & \Big|\varphi_1\Big[\int_a^x F(x,t,u(t))dt\Big] - \varphi_1\Big[\int_a^x F(x,t,w(t))dt\Big]\Big|\\
& + \Big|\varphi_2\Big[\int_a^b G(x,t,v(t))dt\Big] - \varphi_2\Big[\int_a^b G(x,t,z(t))dt\Big]\Big|\\
\leq\ & L_1\Big|\int_a^x (F(x,t,u(t)) - F(x,t,w(t)))dt\Big| + L_2\Big|\int_a^b (G(x,t,v(t)) - G(x,t,z(t)))dt\Big|\\
\leq\ & L_1\int_a^x \Big|F(x,t,u(t)) - F(x,t,w(t))\Big|dt + L_2\int_a^b |G(x,t,v(t)) - G(x,t,z(t))|dt\\
\leq\ & L_1\int_a^b \Big|F(x,t,u(t)) - F(x,t,w(t))\Big|dt + L_2\int_a^b |G(x,t,v(t)) - G(x,t,z(t))|dt\\
\leq\ & L_1\int_a^b p_1(x,t)|u(t) - w(t)|dt + L_2\int_a^b p_2(x,t)|v(t) - z(t)|dt\\
\leq\ & L_1 d(u,w)\int_a^b p_1(x,t)dt + L_2 d(v,z)\int_a^b p_2(x,t)dt\\
\leq\ & L_1 d(u,w)\frac{K_1}{2L_1} + L_2 d(v,z)\frac{K_2}{2L_2}\\
=\ & \frac{1}{2}(K_1 d(u,w) + K_2 d(v,z)). \qquad (2.7)
\end{aligned}$$

Therefore:

$$d(T(u,v),T(w,z)) = sup_{x\in[a,b]}|T(u,v)(x) - T(w,z)(x)| \leq \frac{1}{2}(K_1 d(u,w) + K_2 d(v,z)). \tag{2.8}$$

Similarly, we can show that

$$d(T(v,u),T(z,w)) = sup_{x\in[a,b]}|T(v,u)(x) - T(z,w)(x)| \leq \frac{1}{2}(K_1 d(u,w) + K_2 d(v,z)). \tag{2.9}$$

By summing up the two above inequalities for $u \succeq w$ and $v \preceq z$, we get

$$d(T(u,v),T(w,z)) + d(T(v,u),T(z,w)) \leq K(d(u,w) + d(v,z)), \tag{2.10}$$

where $K = max\{K_1, K_2\}$. So the inequality (1.2) and all of the conditions of Theorem 1.3 are satisfied. Thus there exist $(\bar{x},\bar{y})$ such that $T(\bar{x},\bar{y}) = \bar{x}$ and $T(\bar{y},\bar{x}) = \bar{y}$.

Finally let α, β be two functions that appearing in assumption (iv) in Theorem 2.1. Then Theorem 1.5 and Theorem 1.4 give us, T has a unique fixed point and this complete the proof. □

Remark 2.2. By taking $\varphi_2 = 0$ in Theorem 2.1, we can generalized the solution of Volterra integral equation [3].

Remark 2.3. By taking $\varphi_1(x) = \varphi_2(x) = x$ in Theorem 2.1 we can conclude the Tidke-Aage-Salunke theorem [8].

3. Example

In this section we give an example to illustrate the usefulness of our results.

Example 3.1. Consider the following integral equation:

$$u(x) = \frac{1}{2}\int_0^x txu(t)dt + \frac{1}{3}\int_0^1 t^2xu(t)dt + \frac{1}{5}. \tag{3.1}$$

Suppose that $\varphi_1(x) = \frac{1}{2}x$ and $\varphi_2(x) = \frac{1}{3}x$. Obviously $\varphi_1, \varphi_2 \in \Phi$ with constants $L_1 = \frac{1}{2}$ and $L_2 = \frac{1}{3}$ respectively.
If $v \preceq u$, then

$$txu(t) - txv(t) = tx(u(t) - v(t)) = tx|u(t) - v(t)|.$$

By putting $p_1(x,t) = tx$ we get

$$sup_{x\in[0,1]} \int_0^1 txdt = \frac{1}{2} \leq \frac{K_1}{2 \times \frac{1}{2}}$$

where $K_1 = \frac{1}{2}$.
Also if $v \preceq u$, then

$$-t^2xu(t) + t^2xv(t) = t^2x(v(t) - u(t)) = t^2x|u(t) - v(t)|$$

If we define $p_2(x,t) = t^2x$, then

$$sup_{x\in[0,1]} \int_0^1 t^2xdt = \frac{1}{3} \leq \frac{K_2}{2 \times \frac{1}{3}}$$

where $K_2 = \frac{2}{9}$.

By choosing $\alpha = 0$ and $\beta = 1$, we have $\alpha, \beta \in C([0,1], \mathbb{R})$ and

$$\alpha(x) = 0 \leq 0 + \frac{1}{3}\int_0^1 t^2xdt + \frac{1}{5} = \varphi_1\left[\int_a^x F(x,t,\alpha(t))dt\right] + \varphi_2\left[\int_a^b G(x,t,\beta(t))dt\right] + g(x)$$

where $F(x,t,u(t)) = txu(t)$ and $G(x,t,u(t)) = t^2xu(t)$, and

$$\beta(x) = 1 \geq \frac{1}{2}\int_0^1 txdt + 0 + \frac{1}{5} = \varphi_1\left[\int_a^x F(x,t,\beta(t))dt\right] + \varphi_2\left[\int_a^b G(x,t,\alpha(t))dt\right] + g(x).$$

Therefore all of the conditions of Theorem 2.1 are satisfied and hence the equation (3.1) has a unique solution in $C([0,1], \mathbb{R})$.

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