Int. J. Nonlinear Anal. Appl. 7 (2016) No. 2, 229-242
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2017.10573.1510

A contribution to approximate analytical evaluation of Fourier series via an Applied Analysis standpoint; an application in turbulence spectrum of eddies

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(Communicated by M. Eshaghi)

Abstract

In the present paper, we shall attempt to make a contribution to approximate analytical evaluation of the harmonic decomposition of an arbitrary continuous function. The basic assumption is that the class of functions that we investigate here, except the verification of Dirichlet's principles, is concurrently able to be expanded in Taylor's representation, over a particular interval of their domain of definition. Thus, we shall take into account the simultaneous validity of these two properties over this interval, in order to obtain an alternative equivalent representation of the corresponding harmonic decomposition for this category of functions. In the sequel, we shall also implement this resultant formula in the investigation of turbulence spectrum of eddies according to known from literature Von Karman's formulation, making the additional assumption that during the evolution of such stochastic dynamic effects with respect to time, the occasional time-returning period can be actually supposed to tend to infinity.

Keywords: Orthogonal functions; Abel's summability; Poisson's kernel; Von Karman's spectrum.
2010 MSC: Primary 42B35.

1. Introduction and preliminaries

It is known from Stone-Weierstrass Theorem, that every continuous function is actually able to be represented, as the pointwise or uniform convergence to a limit function of a functional sequence [4, 14].

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Received: February 2014 *Revised:* November 2015

On the other hand, Fejer Theorem asserts us that every 2π-periodic continuous function is able to be approximated by linear combinations of the fundamental functions: $\sin \kappa t$, $\cos \kappa t$, $\kappa \in \mathbb{N}$ [8, 12].

Moreover, the closed-form solutions of several boundary value problems of Dirichlet's or Neumann's forms, are mostly represented in the general pattern of the limit function, (pointwise or uniform), of an iterated power series, which has mainly trigonometric form in the usual practical problems [6, 7, 9].

However, we have to denote here that the pointwise or uniform convergence of such series is indeed very slow, from computational point of view and generally these forms are obviously not suitable in current use, as approximate solutions of boundary value problems of Mathematical Physics, especially for the necessary calculations in conceptual design process, by engineering standpoint.

Specifically, the algebraic rates of the three components of instantaneous velocity vector: $\vec{V}(x, y, z, t)$ by means of Eulerian representation for turbulent flow fields, are actually able to be written out, via the following cubic form [15]:

$$
\begin{aligned}
V_x(x,y,z,t) &= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\left(h_j(y)EXP\left(\frac{2\pi ik}{L_x}+\frac{2\pi il}{L_z}\right)\cdot V_{x_{jkl}}(t)\right)\\
V_y(x,y,z,t) &= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\left(h_k(y)EXP\left(\frac{2\pi ik}{L_y}+\frac{2\pi il}{L_x}\right)\cdot V_{y_{jkl}}(t)\right)\\
V_z(x,y,z,t) &= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\left(h_l(y)EXP\left(\frac{2\pi ik}{L_z}+\frac{2\pi il}{L_y}\right)\cdot V_{z_{jkl}}(t)\right),
\end{aligned}
\tag{1.1}
$$

where

$$
\mathrm{EXP}\left(\frac{2\pi ik}{L_x}+\frac{2\pi il}{L_z}\right)=\cos\left(\frac{2\pi k}{L_x}+\frac{2\pi l}{L_z}\right)+i\sin\left(\frac{2\pi k}{L_x}+\frac{2\pi l}{L_z}\right). \tag{1.2}
$$

The above statement, actually represents a sequence of instantaneous images of velocity vector field for turbulent flow patterns and is motivated indeed via a random partition of domain in spectral elementary segments and concurrent implementation of periodic boundary conditions [3].

Since these functions are orthogonal, the three-dimensional original problem can actually be decomposed in a superposition of a finite number of one-dimensional ones for each circumstantial pair of the indicators (k, l) in an orthogonal functional space.

2. Alternative representation of Fourier series

The quantitative estimation of a trigonometric power series, could generally be actualized in Orthogonal Spaces, ensuing the following procedure:
Let us consider, the harmonic decomposition of a periodic continuous function: $f : \mathbb{R} \to \mathbb{R}$ being defined on an interval: $[0, 2L]$

$$
f(x)=\frac{a_0}{2}+\sum_{\kappa=1}^{\infty}\left(a_\kappa\cdot\cos\Big(\kappa\cdot\frac{\pi x}{L}\Big)+b_\kappa\cdot\sin\Big(\kappa\cdot\frac{\pi x}{L}\Big)\right), \tag{2.1}
$$

where:

$$
a_0 = \frac{1}{2L}\int_0^{2L} f(x)\,dx
$$

$$
\begin{aligned}
a_\kappa &= \frac{1}{L}\int_0^{2L} f(x)\cos\Big(\frac{\kappa\pi x}{L}\Big)\,dx \\
b_\kappa &= \frac{1}{L}\int_0^{2L} f(x)\sin\Big(\frac{\kappa\pi x}{L}\Big)dx
\end{aligned} \tag{2.2}
$$

are the so-called Fourier coefficients. If we keep in mind some basic results of Orthogonal Functions Theory, we can exhibit as follows, the general form of Fourier coefficients in an arbitrary Orthogonal Space: $\{\phi_1, \phi_2, \dots, \phi_\kappa\}$, whose cardinality is obviously: "aleph zero" [9].

$$
c_\kappa = \frac{1}{\|\phi_\kappa\|}\int_0^{2L} f(x)\cdot\phi_\kappa(x)\,dx \tag{2.3}
$$

where the norm $\|\phi_\kappa\|$ is defined as follows:

$$
\|\phi_\kappa\| = \left(\int_0^{2L}\phi_\kappa^2(x)dx\right)^{\frac{1}{2}}. \tag{2.4}
$$

From Real Analysis point of view, the definition of the norm: $\|\phi_\kappa\|$ can be actually evident as follows: It is known that every functional space: $C^n[a,b]$, $n \in \mathbb{N}$ which in fact consists in a linear space, can be converted to a metric space, via the addition of a specific metric, such that to be induced by the following norm:

$$
\|f, g\|_\infty = \sup\{|f(x) - g(y)|^\rho\}, \qquad \rho \in \mathbb{Z}^*. \tag{2.5}
$$

Then, at the aforementioned space: $C^n[a,b]$, we are able indeed to define afterwards the following norms in the general pattern:

$$
\|f\|_p = \left(\int_a^b |f^p(x)|\,dx\right)^{\frac{1}{p}}, \quad p \in \mathbb{Z}^*. \tag{2.6}
$$

Next, let us consider, an arbitrary vector: $\vec{a} = (a_1, a_2, \dots, a_n) \in (\Re^n, \|\cdot\|_\rho)$ where $\Re^n$ denotes an arbitrary compact and connected ordered linear space. Then, the particular metric d onto space $\Re^n$, such that: $d(\vec{x}, \vec{y}) \equiv \left(\sum_{\kappa=1}^n |x-y|^\rho\right)^{\frac{1}{\rho}}$, (which is induced by the aforementioned norm $\|\phi_\kappa\|$), defines indeed univocally any vector of the space $\Re^n$.

Thus, in the specific case we study, we can deduce proportionally that the norm: $\|\phi_\kappa\| = \left(\int_0^{2L}\phi_\kappa^2(x)dx\right)^{\frac{1}{2}}$ induces in fact a metric onto linear space R^2 such that to be able to define univocally any element of the afore-mentioned orthogonal space: $\{\phi_1, \phi_2, \dots, \phi_\kappa\}$.

Returning now to the initial problem of approximate estimation of Fourier series, we can actually denote that the coefficients: c_κ for all $\kappa \in \mathbb{N}$ can arise, if we integrate over the interval $[0, 2L]$ selecting firstly an appropriate weight function $w(x)$ and also taking into account the following property for the inner product of two orthogonal functions:
Considering two functions: ϕ_m, ϕ_n such that: $\{\phi_m, \phi_n\} \subset \{\phi_1, \phi_2, \dots, \phi_\kappa\}$, it is valid that:

$$
\langle\phi_m \cdot \phi_n\rangle = \int_a^b \phi_m(x)\cdot\phi_n(x) = \begin{cases} 0, & \text{for } m \neq n \\ k_n, & \text{for } m = n, \end{cases} \tag{2.7}
$$

where the constant: k_n depends on the circumstantial selection of the weight function $w(x)$ and the function ϕ_n, e.g. for trigonometric functions we can conclude:

$$
\int_0^{2L} w(x)\cos\left(\frac{n\pi x}{L}\right)\cdot\cos\left(\frac{m\pi x}{L}\right)dx = 0 \text{ for } w(x) = 1 \wedge (m,n) \in \mathbb{N}^2 : \ m \neq n
$$

and:

$$\int_0^{2L} w(x)\sin\left(\frac{n\pi x}{L}\right)\cdot\sin\left(\frac{m\pi x}{L}\right)dx = 0 \text{ for } w(x)=1 \wedge (m,n)\in\mathbb{N}^2:\ m\neq n,$$

where we have expanded here the following countable functional space, since the amplitude of the period is: $2L$

$$\{\phi_1,\phi_2,\ldots,\phi_\kappa\} = \Big\{1, \cos\left(\frac{\pi x}{L}\right), \sin\left(\frac{\pi x}{L}\right), \cos\left(\frac{2\pi x}{L}\right), \sin\left(\frac{2\pi x}{L}\right), \ldots$$
$$\ldots, \cos\left(\frac{\kappa\pi x}{L}\right), \sin\left(\frac{\kappa\pi x}{L}\right)\Big\}.$$

Obviously, each pair of two orthogonal functions ought to be linearly independent to one another, since it is valid that:

$$\sum_{\kappa=0}^{\kappa} c_\kappa\phi_\kappa = 0. \tag{2.8}$$

Thus, every function $\sigma_\kappa \in \{\sigma_\kappa\}, \kappa\in N$ can emerge with respect to the above general basis from the following recursion formula [10]:

$$\sigma_\kappa(x) = q_{\kappa\kappa}\left(\phi_\kappa(x) - \sum_{j=1}^{\kappa-1}\frac{\langle\phi_\kappa(x),\sigma_j(x)\rangle}{\langle\sigma_j(x),\sigma_j(x)\rangle}\sigma_j(x)\right). \tag{2.9}$$

The arbitrary coefficients $q_{\kappa\kappa}$ can be also selected such that $\langle\sigma_j(x),\sigma_j(x)\rangle = 1$, which means that the orthogonal space $\{\sigma_\kappa\}$ consists in an orthonormal one.

On the other hand, we can also relate these functions to singular Sturm-Liouville's systems, (they can constitute the eigen-functions of them), from which we are able to derive their qualitative properties. Nevertheless, in order one to follow such methods, using qualitative properties of orthogonal functions, an advanced level of Real and Functional Analysis should be provided.

Besides, these evident expressions have rather qualitative character and are not directly useful in conceptual design calculations. Thus, it is preferable firstly one to apply some classic techniques from advanced Calculus.

From elementary Complex Analysis it is valid:

$$\begin{aligned}\cos\left(\kappa\cdot\tfrac{\pi x}{L}\right) &= \frac{1}{2}\left(\exp\left(i\kappa\cdot\frac{\pi x}{L}\right)+\exp\left(-i\kappa\cdot\frac{\pi x}{L}\right)\right)\\ \sin\left(\kappa\cdot\tfrac{\pi x}{L}\right) &= \frac{1}{2i}\left(\exp\left(i\kappa\cdot\frac{\pi x}{L}\right)-\exp\left(-i\kappa\cdot\frac{\pi x}{L}\right)\right).\end{aligned} \tag{2.10}$$

Hence, eq. (2.1) can be performed equivalently:

$$f(x) = \frac{a_0}{2} + \sum_{\kappa=1}^{\infty}\left(\frac{a_\kappa - ib_\kappa}{2}\ e^{i\kappa\cdot\frac{\pi x}{L}} + \frac{a_\kappa+ib_\kappa}{2}\ e^{-i\kappa\cdot\frac{\pi x}{L}}\right). \tag{2.11}$$

Based on the latter equation, we can also obtain the following equivalent form by making the following substitutions:

$$\begin{aligned}\frac{a_0}{2} &\leftrightarrow \alpha_0\\ \frac{a_\kappa - ib_\kappa}{2} &\leftrightarrow \alpha_\kappa\\ \frac{a_\kappa + ib_\kappa}{2} &\leftrightarrow \alpha_{-\kappa}.\end{aligned} \tag{2.12}$$

Hence, it implies:

$$f(x) = \frac{1}{2}\sum_{-\infty}^{+\infty}\left(\alpha_\kappa e^{i\kappa\cdot\frac{\pi x}{L}}\right), \tag{2.13}$$

where the above infinite summation extends over all integers $\kappa \in \mathbb{Z} \cup \{-\infty, +\infty\}$.

According to the statements about the inner product in orthogonal functional spaces mentioned above, we can also deduce:

$$\int_0^{2L} e^{i\kappa\cdot\frac{\pi x}{L}}\,dx = \begin{cases} 0 & \text{if } \kappa \neq 0 \\ 2L & \text{if } \kappa = 0. \end{cases} \tag{2.14}$$

Besides, we can also pinpoint that α_κ, $\alpha_{-\kappa}$ are conjugate complex values.

Concurrently, if we suppose that the above series verifies Abel's summability Criterion, then it follows that:

$$f_r(x) = \frac{1}{2}\sum_{-\infty}^{+\infty}\left(r^\kappa \alpha_\kappa e^{i\kappa\cdot\frac{\pi x}{L}}\right) \quad \text{is convergent} \quad \forall r \in (0,1) \tag{2.15}$$

where: $f_r(x)$ is a continuous function $\forall x \in [0, 2L]$.

Additionally, Poisson's kernel is defined as follows [11, 12]:

$$\ker P(x) = \frac{1}{2}\cdot\frac{1-r^2}{1-2r\cos x + r^2} \quad \forall r \in [0,1). \tag{2.16}$$

Obviously this quantity, is strictly positive $\forall r \in [0,1)$.

Besides, the following statement asserts us for the uniform convergence of $f(x)$. Letting $r \to 1^-$, there exists $r_0(\varepsilon)$, such that

$$|f(x) - f_r(x)| < \varepsilon \text{ for all } r \in (r_0, 1) \tag{2.17}$$

where: $\varepsilon \geq \ker P(x)$ for all $r \in [0,1)$.

In order now to calculate the two inceptive Fourier coefficients a_κ, b_κ we can put into effect the initial aforementioned assumption, that the circumstantial continuous function: f is also able to be represented in Taylor's formula over the interval $[0, 2L]$, past the rate: $x_0 = 0$.

Thus we can write out:

$$f(x) = f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \cdots. \tag{2.18}$$

In the sequel, let us assume that there exists a unique polynomial: $\Pi(x) = \sum_{n=0}^{n} g_n x^n$ which belongs to the functional space:

$$C^n[0, 2L] = \{f : [0, 2L] \to \mathbb{R} \,:\, f \text{ is } n\text{-times differentiable function}\}$$

and having concurrently the following property:

$$\Pi(x) \equiv f(x), \ \forall x \in [0, 2L]. \tag{2.19}$$

In accessory, it is also known from elementary Analysis that every arbitrary polynomial in the generic representation: $\Pi(x) = \sum_{n=0}^{n} g_n x^n$ can be written out equivalently in the next alter-ego form:

$$\Pi(x) = \Pi(0) + \frac{\Pi^{(1)}(0)}{1!}x + \frac{\Pi^{(2)}(0)}{2!}x^2 + \cdots. \tag{2.20}$$

Hence, it infers that:

$$f(0) = \Pi(0) \wedge f^{(n)}(0) = \Pi^{(n)}(0) \quad \forall n \in \mathbb{N}^*. \tag{2.21}$$

Yet, it is also known from single-valued Calculus, that for every real-valued polynomial in the generic form: $g_0 + \sum_{i=1}^{n} g_n x^n$, the corresponding polynomial function of its *μ-order derivative*, is given by the following formula [13]:

$$P^{(\mu)}(x) = \mu! \sum_{\lambda=\mu}^{n} g_\lambda \frac{\lambda!}{\mu!(\lambda-\mu)!} x^{\lambda-\mu}. \tag{2.22}$$

Afterwards, as for the linear space derived by the antiderivatives: $\int \cos(Ax)\Pi(x)dx$, $\forall A \in \mathbb{R}^*$ the following relationship comes along, via step by step integration:

$$\begin{aligned}
&\int \cos(Ax)\Pi(x)dx = \\
&\quad = \frac{1}{A} \cdot \left(\sin(A \cdot x)\Pi(x) + \frac{1}{A^1} \cdot \Pi^{(1)}(x) \cdot \sin\left(A \cdot x + 1 \cdot \frac{\pi}{2}\right) + \right. \\
&\qquad + \frac{1}{A^2} \cdot \Pi^{(2)}(x) \cdot \sin\left(A \cdot x + 2 \cdot \frac{\pi}{2}\right) + \cdots \\
&\qquad \left. \cdots + \frac{1}{A^n} \cdot \Pi^{(n)}(x) \sin\left(A \cdot x + n \cdot \frac{\pi}{2}\right)\right) + C_1,
\end{aligned}$$

since: $\Pi^{(n+1)}(x) \equiv 0$. Hence for the case we study, the following expression can be inferred equivalently:

$$\begin{aligned}
&\int \cos(Ax)\Pi(x)dx = \\
&\quad = \frac{1}{A} \cdot \sin(Ax)\Pi(x) + \frac{1}{A^2} \cdot \Pi^{(1)}(x) \cdot \sin\left(Ax + 1 \cdot \frac{\pi}{2}\right) + \\
&\qquad + \frac{1}{A^3} \cdot \Pi^{(2)}(x) \cdot \sin\left(Ax + 2 \cdot \frac{\pi}{2}\right) + \cdots \\
&\qquad \cdots + \frac{1}{A^{n+1}} \cdot \Pi^{(n)}(x) \sin\left(A \cdot x + n \cdot \frac{\pi}{2}\right) + C_1.
\end{aligned}$$

Thus, the inceptive Fourier coefficient: $a_\kappa = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{\kappa\pi}{L} \cdot x\right)dx$ can be evaluated after all for $\kappa \in \mathbb{N}^*$:

$$\begin{aligned}
a_\kappa \;=\; & \frac{1}{L}\Big[\frac{1}{(\kappa\pi/L)} \cdot \sin\Big(\frac{\kappa\pi}{L}x\Big)\Pi(x) + \frac{1}{(\kappa\pi/L)^2} \cdot \Pi^{(1)}(x) \cdot \sin\Big(\frac{\kappa\pi}{L}x + 1 \cdot \frac{\pi}{2}\Big) \\
& + \frac{1}{(\kappa\pi/L)^3} \cdot \Pi^{(2)}(x) \cdot \sin\Big(\frac{\kappa\pi}{L}x + 2 \cdot \frac{\pi}{2}\Big) + \cdots \\
& \cdots + \frac{1}{(\kappa\pi/L)^{n+1}} \cdot \Pi^{(n)}(x) \sin\Big(\frac{\kappa\pi}{L} \cdot x + n \cdot \frac{\pi}{2}\Big)\Big]_0^{2L}.
\end{aligned}$$

$\Leftrightarrow$

$$\begin{aligned} a_\kappa &= \Big[\frac{1}{\kappa\pi}\cdot\sin(2\kappa\pi)\Pi(2L) + \frac{L}{(\kappa\pi)^2}\cdot\Pi^{(1)}(2L)\cdot\sin\Big(2\kappa\pi + 1\cdot\frac{\pi}{2}\Big) \\ &\quad + \frac{L^2}{(\kappa\pi)^3}\cdot\Pi^{(2)}(2L)\cdot\sin\Big(2\kappa\pi + 2\cdot\frac{\pi}{2}\Big) + \cdots \\ &\quad \cdots + \frac{L^n}{(\kappa\pi)^{n+1}}\cdot\Pi^{(n)}(2L)\sin\Big(2\kappa\pi + n\cdot\frac{\pi}{2}\Big)\Big] \\ &\quad -\Big[\frac{1}{\kappa\pi}\cdot\sin(0)\Pi(0) + \frac{L}{(\kappa\pi)^2}\cdot\Pi^{(1)}(0)\cdot\sin\Big(1\cdot\frac{\pi}{2}\Big) \\ &\quad + \frac{L^2}{(\kappa\pi)^3}\cdot\Pi^{(2)}(0)\cdot\sin\Big(2\cdot\frac{\pi}{2}\Big) + \cdots \\ &\quad \cdots + \frac{L^n}{(\kappa\pi)^{n+1}}\cdot\Pi^{(n)}(0)\sin\Big(n\cdot\frac{\pi}{2}\Big)\Big] \end{aligned}$$

$\Leftrightarrow$

$$\begin{aligned} a_\kappa &= \Big[\frac{L}{(\kappa\pi)^2}\cdot\Pi^{(1)}(2L)\cdot\sin\Big(1\cdot\frac{\pi}{2}\Big) + \frac{L^2}{(\kappa\pi)^3}\cdot\Pi^{(2)}(2L)\cdot\sin\Big(2\cdot\frac{\pi}{2}\Big) + \cdots \\ &\quad \cdots + \frac{L^n}{(\kappa\pi)^{n+1}}\cdot\Pi^{(n)}(2L)\sin\Big(n\cdot\frac{\pi}{2}\Big)\Big] \\ &\quad -\Big[\frac{L}{(\kappa\pi)^2}\cdot\Pi^{(1)}(0)\cdot\sin\Big(1\cdot\frac{\pi}{2}\Big) + \frac{L^2}{(\kappa\pi)^3}\cdot\Pi^{(2)}(0)\cdot\sin\Big(2\cdot\frac{\pi}{2}\Big) + \cdots \\ &\quad \cdots + \frac{L^n}{(\kappa\pi)^{n+1}}\cdot\Pi^{(n)}(0)\sin\Big(n\cdot\frac{\pi}{2}\Big)\Big] \end{aligned}$$

$\Leftrightarrow$

$$\begin{aligned} a_\kappa &= \Big(\Pi^{(1)}(2L) - \Pi^{(1)}(0)\Big)\,\frac{L^1}{(\kappa\pi)^2}\cdot\sin\Big(1\cdot\frac{\pi}{2}\Big) + \cdots \\ &\quad \cdots + \Big(\Pi^{(n)}(2L) - \Pi^{(n)}(0)\Big)\,\frac{L^n}{(\kappa\pi)^{n+1}}\cdot\sin\Big(n\cdot\frac{\pi}{2}\Big) \end{aligned}$$

$\Leftrightarrow$

$$a_\kappa = \sum_{n=1}^{n}\Big(\Pi^{(n)}(2L) - \Pi^{(n)}(0)\Big)\,\frac{L^n}{(\kappa\pi)^{n+1}}\cdot\sin\Big(n\cdot\frac{\pi}{2}\Big). \tag{2.23}$$

Here we have indispensably to require, that the integer variables k and n must be functionally independent.

Next, we can obtain a proportional representation for the coefficient:

$$b_\kappa = \frac{1}{L}\int_0^{2L} f(x)\sin\Big(\frac{\kappa\pi x}{L}\Big)\,dx.$$

As concerns the antiderivatives: $\int \sin(Ax)\Pi(x)dx$, we can also infer:

$$\int \sin(Ax)\Pi(x)\,dx = \frac{1}{A}\int \sin(Ax)\Pi(x)\,d(Ax) = -\frac{1}{A}\int \Pi(x)\,d\left(\cos(Ax)\right).$$

$\Leftrightarrow$

$$\int \sin(Ax)\Pi(x)\,dx = -\frac{1}{A}\cos(Ax)\Pi(x) + \frac{1}{A}\int \cos(Ax)\Pi^{(1)}(x)\,dx. \tag{2.24}$$

The quantity: $\frac{1}{A}\int \cos(Ax)\Pi^{(1)}(x)\,dx$ can be estimated, via the same formula that we have explicated above for the integral: $\int \cos(Ax)\Pi(x)\,dx$, if we substitute instead of the polynomial: $\Pi(x)$, its first derivative: $\Pi'(x) = \Pi^{(1)}(x)$, whose order is obviously: $(n-1)$

Hence we can obtain:

$$\begin{aligned}
&\int \cos(Ax)\Pi'(x)\,dx = \\
&\quad = \frac{1}{A}\cdot\Big[\sin(A\cdot x)\Pi'(x) + \frac{1}{A^1}\cdot\Pi'^{(1)}(x)\cdot\sin\Big(A\cdot x + 1\cdot\frac{\pi}{2}\Big) \\
&\qquad + \frac{1}{A^2}\cdot\Pi'^{(2)}(x)\cdot\sin(A\cdot x + 2\cdot\frac{\pi}{2}) + \cdots \\
&\qquad \cdots + \frac{1}{A^{n-1}}\cdot\Pi'^{(n-1)}(x)\sin\Big(A\cdot x + (n-1)\cdot\frac{\pi}{2}\Big)\Big] + C_2,
\end{aligned} \tag{2.25}$$

since: $\Pi'(n)(x) \equiv 0$. Thus, after the necessary algebraic manipulation we can eventually infer:

$$\begin{aligned}
&\int \cos\Big(\frac{\kappa\pi}{L}x\Big)\Pi'(x)\,dx = \\
&\quad = \frac{1}{\left(\frac{\kappa\pi}{L}\right)}\cdot\sin\Big(\frac{\kappa\pi}{L}x\Big)\Pi^{(1)}(x) + \frac{1}{\left(\frac{\kappa\pi}{L}\right)^2}\cdot\Pi^{(2)}(x)\cdot\sin\Big(\frac{\kappa\pi}{L}x + 1\cdot\frac{\pi}{2}\Big) \\
&\qquad + \frac{1}{\left(\frac{\kappa\pi}{L}\right)^3}\cdot\Pi^{(3)}(x)\cdot\sin\Big(\frac{\kappa\pi}{L}x + 2\cdot\frac{\pi}{2}\Big) + \cdots \\
&\qquad \cdots + \frac{1}{\left(\frac{\kappa\pi}{L}\right)^n}\cdot\Pi^{(n)}(x)\sin\Big(\frac{\kappa\pi}{L}\cdot x + (n-1)\cdot\frac{\pi}{2}\Big) + C_2
\end{aligned}$$

$\Leftrightarrow$

$$\begin{aligned}
&\frac{1}{\left(\frac{\kappa\pi}{L}\right)}\cdot\int \cos\Big(\frac{\kappa\pi}{L}x\Big)\Pi'(x)\,dx = \\
&\quad = \frac{1}{\left(\frac{\kappa\pi}{L}\right)^2}\cdot\sin\Big(\frac{\kappa\pi}{L}x\Big)\Pi^{(1)}(x) + \frac{1}{\left(\frac{\kappa\pi}{L}\right)^3}\cdot\Pi^{(2)}(x)\cdot\sin\Big(\frac{\kappa\pi}{L}x + 1\cdot\frac{\pi}{2}\Big) \\
&\qquad + \frac{1}{\left(\frac{\kappa\pi}{L}\right)^4}\cdot\Pi^{(3)}(x)\cdot\sin\Big(\frac{\kappa\pi}{L}x + 2\cdot\frac{\pi}{2}\Big) + \cdots \\
&\qquad \cdots + \frac{1}{\left(\frac{\kappa\pi}{L}\right)^{n+1}}\cdot\Pi^{(n)}(x)\sin\Big(\frac{\kappa\pi}{L}\cdot x + (n-1)\cdot\frac{\pi}{2}\Big) + C_2.
\end{aligned}$$

Evidently, we can deduce that:

$$\begin{aligned}
b_\kappa &= \frac{1}{L}\left[-\frac{1}{\left(\frac{\kappa\pi}{L}\right)}\cos\Big(\frac{\kappa\pi}{L}x\Big)\Pi(x)\right]_0^{2L} \\
&\quad + \frac{1}{L}\Bigg[\frac{1}{\left(\frac{\kappa\pi}{L}\right)^2}\cdot\sin\Big(\frac{\kappa\pi}{L}x\Big)\Pi^{(1)}(x) + \frac{1}{\left(\frac{\kappa\pi}{L}\right)^3}\cdot\Pi^{(2)}(x)\cdot\sin\Big(\frac{\kappa\pi}{L}x + 1\cdot\frac{\pi}{2}\Big) \\
&\qquad + \frac{1}{\left(\frac{\kappa\pi}{L}\right)^4}\cdot\Pi^{(3)}(x)\cdot\sin\Big(\frac{\kappa\pi}{L}x + 2\cdot\frac{\pi}{2}\Big) + \cdots
\end{aligned}$$

$$\cdots + \frac{1}{\left(\frac{\kappa\pi}{L}\right)^{n+1}} \cdot \Pi^{(n)}(x) \sin\left(\frac{\kappa\pi}{L} \cdot x + (n-1) \cdot \frac{\pi}{2}\right)\Bigg]_0^{2L}$$

$$\begin{aligned} = \; & \frac{1}{\kappa\pi}\left(\Pi(0) - \Pi(2L)\right) + \frac{1}{L}\Bigg[\frac{1}{\left(\frac{\kappa\pi}{L}\right)^2} \cdot \sin\left(\frac{\kappa\pi}{L}x\right)\Pi^{(1)}(x) \\ & + \frac{1}{\left(\frac{\kappa\pi}{L}\right)^3} \cdot \Pi^{(2)}(x) \cdot \sin\left(\frac{\kappa\pi}{L}x + 1 \cdot \frac{\pi}{2}\right) \\ & + \frac{1}{\left(\frac{\kappa\pi}{L}\right)^4} \cdot \Pi^{(3)}(x) \cdot \sin\left(\frac{\kappa\pi}{L}x + 2 \cdot \frac{\pi}{2}\right) + \cdots \\ & \cdots + \frac{1}{\left(\frac{\kappa\pi}{L}\right)^{n+1}} \cdot \Pi^{(n)}(x) \sin\left(\frac{\kappa\pi}{L} \cdot x + (n-1) \cdot \frac{\pi}{2}\right)\Bigg]_0^{2L}. \end{aligned}$$

Hence,

$$\begin{aligned} b_\kappa = \; & \frac{1}{\kappa\pi}\left(\Pi(0) - \Pi(2L)\right) + \Bigg[\frac{L}{(\kappa\pi)^2} \cdot \sin(2\kappa\pi)\Pi^{(1)}(2L) \\ & + \frac{L^2}{(\kappa\pi)^3} \cdot \Pi^{(2)}(2L) \cdot \sin\left(2\kappa\pi + 1 \cdot \frac{\pi}{2}\right) \\ & + \frac{L^3}{(\kappa\pi)^4} \cdot \Pi^{(3)}(2L) \cdot \sin\left(2\kappa\pi + 2 \cdot \frac{\pi}{2}\right) + \cdots \\ & \cdots + \frac{L^n}{(\kappa\pi)^{n+1}} \cdot \Pi^{(n)}(2L) \sin\left(2\kappa\pi + (n-1) \cdot \frac{\pi}{2}\right)\Bigg] \\ & - \Bigg[\frac{L}{(\kappa\pi)^2} \cdot \sin(0)\Pi^{(1)}(0) + \frac{L^2}{(\kappa\pi)^2} \cdot \Pi^{(2)}(0) \cdot \sin\left(1 \cdot \frac{\pi}{2}\right) \\ & + \frac{L^3}{(\kappa\pi)^3} \cdot \Pi^{(3)}(0) \cdot \sin\left(2 \cdot \frac{\pi}{2}\right) + \cdots \\ & \cdots + \frac{L^n}{(\kappa\pi)^{n+1}} \cdot \Pi^{(n)}(0) \sin\left((n-1) \cdot \frac{\pi}{2}\right)\Bigg]. \end{aligned}$$

Thus, we are led eventually to the following power series which concerns the extraction of b_κ:

$$\begin{aligned} b_\kappa = \; & \frac{L^0}{(\kappa\pi)^1}\left(\Pi^{(0)}(2L) - \Pi^{(0)}(0)\right)\sin\left((-1) \cdot \frac{\pi}{2}\right) \\ & + \frac{L^1}{(\kappa\pi)^2}\left(\Pi^{(1)}(2L) - \Pi^{(1)}(0)\right)\sin\left(0 \cdot \frac{\pi}{2}\right) \\ & + \frac{L^2}{(\kappa\pi)^3}\left(\Pi^{(2)}(2L) - \Pi^{(2)}(0)\right)\sin\left(1 \cdot \frac{\pi}{2}\right) + \cdots \\ & \cdots + \frac{L^n}{(\kappa\pi)^{n+1}}\left(\Pi^{(n)}(2L) - \Pi^{(n)}(0)\right)\sin\left((n-1) \cdot \frac{\pi}{2}\right) \end{aligned}$$

$\Leftrightarrow$

$$b_\kappa = \sum_{n=0}^{n} \frac{L^n}{(\kappa\pi)^{n+1}}\left(\Pi^{(n)}(2L) - \Pi^{(n)}(0)\right)\sin\left((n-1) \cdot \frac{\pi}{2}\right). \tag{2.26}$$

Concurrently, regarding the two latter summations which concern the initial Fourier coefficients: a_κ, b_κ, we can also put into effect the known from Calculus Euler's formula, which holds in general for every complex-valued arc [14, 16]:

$$\sin\left(\frac{n\pi}{2}\right) = \left(\frac{n\pi}{2}\right) \cdot \prod_{\xi=1}^{+\infty} \left(1 - \frac{\left(\frac{n\pi}{2}\right)^2}{\xi^2 \cdot \pi^2}\right), \tag{2.27}$$

where we have also to require here, that the integer variables ξ and n must be functionally independent.

Apparently, the above form warrants more expeditious convergence. We can return now, to (2.13) formula of harmonic decomposition, making the relevant substitutions according to the relations from eq. (2.23) and (2.26). Hence, it implies that:

$$f(x) = \frac{1}{2} \sum_{-\infty}^{+\infty} \left((a_\kappa - i b_\kappa)\, e^{i\kappa \cdot \frac{\pi x}{L}} \right)$$

$\Leftrightarrow$

$$\begin{aligned} f(x) \;&=\; \frac{1}{2}\left(\sum_{-\infty}^{+\infty}\left[e^{i\kappa\cdot\frac{\pi x}{L}} \sum_{n=1}^{n} \left(\Pi^{(n)}(2L) - \Pi^{(n)}(0)\right) \frac{L^n}{(\kappa\pi)^{n+1}} \cdot \sin\left(n \cdot \frac{\pi}{2}\right)\right]\right) \\ &-\frac{i}{2}\sum_{-\infty}^{+\infty}\left[e^{i\kappa\cdot\frac{\pi x}{L}} \sum_{n=0}^{n} \frac{L^n}{(\kappa\pi)^{n+1}} \left(\Pi^{(n)}(2L) - \Pi^{(n)}(0)\right) \sin\left((n-1)\cdot\frac{\pi}{2}\right)\right]. \end{aligned} \tag{2.28}$$

We remind that according to eq. (2.22) it is valid:

$$\Pi^{(\mu)}(x) = \mu! \sum_{\lambda=\mu}^{n} g_\lambda \frac{\lambda!}{\mu!(\lambda-\mu)!} x^{\lambda-\mu}.$$

3. Implementation in turbulence spectrum of eddies

However, for the evolution of some dynamical effects like Brownian motion or turbulence the time returning period can be actually assumed to tend to infinity.

Now we have to examine the asymptotic behaviour of every term of the expression above, if the time returning period is letting tend to infinity. This actually means, that two same instantaneous images of a turbulent flow field appear with infinite time-length between them.

We have to take into account here, that the integer variables κ, L, n are in fact functionally independent to each other, which means that if we let them simultaneously tend to infinity*, then the rate of the quotient: $\frac{L}{\kappa}$ must indispensably tend to unity.

This is actually very useful for the calculation or Fourier integral. However, a turbulent flow field can be faced as a superposition of the time-averaging velocity and the three time-fluctuating components at Ox, Oy , Oz axes respectively, according to the general pattern of Von Karman's

We have assumed here that the function f still remains analytical over the extended interval $[0, 2L]$ letting L tend to infinity, thus eq. (2.28) still holds spectrum of eddies, which can be decomposed in terms of Fourier series:

$$\begin{aligned} V_x &= \left\{\frac{a_0}{2} + \sum_{n=1}^{+\infty}\left[[(V_{\max})_x]_n \cdot \cos\left(\frac{2\pi}{\mathrm{T}_n}\cdot t + \theta_n\right)\right]\right\} \\ V_y &= \left\{\frac{b_0}{2} + \sum_{n=1}^{+\infty}\left[[(V_{\max})_y]_n \cdot \cos\left(\frac{2\pi}{\mathrm{T}_n}\cdot t + \theta_n\right)\right]\right\} \\ V_z &= \left\{\frac{c_0}{2} + \sum_{n=1}^{+\infty}\left[[(V_{\max})_z]_n \cdot \cos\left(\frac{2\pi}{\mathrm{T}_n}\cdot t + \theta_n\right)\right]\right\}, \end{aligned} \tag{3.1}$$

where: $(\mathrm{V}_{\max})\mathrm{x} = \frac{2\pi}{\mathrm{T}}\cdot \mathrm{X}_{\max}$, $(\mathrm{V}_{\max})\mathrm{y} = \frac{2\pi}{\mathrm{T}}\cdot \mathrm{Y}_{\max}$, $(\mathrm{V}_{\max})\mathrm{z} = \frac{2\pi}{\mathrm{T}}\cdot \mathrm{Z}_{\max}$ and: $\mathrm{X}_{\max}$, $\mathrm{Y}_{\max}$, $\mathrm{Z}_{\max}$ are the coordinates which define the domain, inside which the anisotropic turbulence is dominated.

Obviously, the time-averaging terms are defined as follows:

$$\begin{aligned} \overline{V}_x &= \frac{1}{T}\cdot\int_0^T V_x \cdot dt \\ \overline{V}_y &= \frac{1}{T}\cdot\int_0^T V_y \cdot dt \\ \overline{V}_z &= \frac{1}{T}\cdot\int_0^T V_z \cdot dt. \end{aligned}$$

For $t = 0$ we can write out the following initial condition for the components of $\vec{V}$:

$$\begin{aligned} V_x &= V_e(x) \\ V_y &= 0 \\ V_z &= 0. \end{aligned} \tag{3.2}$$

We can remark here that regarding wake flows, the quantity $V_e(x)$ is referred to as external local velocity. This initial condition above, can arise, motivated by the known from literature Taylor's Vorticity Transfer theory according to which, the vorticity of each fluid particle remains constant for a given time-period [5, 17]. Hence, one obtains a mixing length for vorticity in a manner similar to the momentum mixing length in Prandtl's formalism [5].

We can remind here, that a power series with terms of cosine form, is a decomposition of an even function, if and only if it holds true that: $\theta_n = 0$ for all $n \in \mathbb{N}^*$. So the formulation above, can describe indeed every periodic function, even or odd. Hence, according to this initial condition, we can result:

$$\begin{aligned} V_x\,|_{t=0} = V_e(x) &\Leftrightarrow \frac{a_0}{2} + \sum_{n=1}^{+\infty}\left[[(V_{\max})_x]_n \cdot \cos(\theta_n)\right] = V_e(x) \\ V_y\,|_{t=0} = 0 &\Leftrightarrow \frac{b_0}{2} + \sum_{n=1}^{+\infty}\left[[(V_{\max})_y]_n \cdot \cos(\theta_n)\right] = 0 \\ V_z\,|_{t=0} = 0 &\Leftrightarrow \frac{c_0}{2} + \sum_{n=1}^{+\infty}\left[[(V_{\max})_z]_n \cdot \cos(\theta_n)\right] = 0, \end{aligned} \tag{3.3}$$

where it also holds true that:

$$a_0 = \frac{1}{T}\int_{-T}^{T} f_1(t)dx \;\Rightarrow\; \frac{a_0}{2} = \frac{1}{T}\int_0^T f_1(t)dx$$

$$b_0 = \frac{1}{T}\int_{-T}^{T} f_2(t)dy \quad \Rightarrow \quad \frac{b_0}{2} = \frac{1}{T}\int_{0}^{T} f_2(t)dy \tag{3.4}$$
$$c_0 = \frac{1}{T}\int_{-T}^{T} f_3(t)dz \quad \Rightarrow \quad \frac{c_0}{2} = \frac{1}{T}\int_{0}^{T} f_3(t)dz.$$

In case now that the apparent returning period tends to infinity, something very useful indeed for the turbulent flow field for which the exact rate of this period is ruther inexplicit, we can use the Eulerian exponential form of a complex variable*, in proportion with eq. (1.1) for the turbulent velocity, during the representation the generalized Fourier Integral (Fourier-Transform) for a non-periodic function:

$$f(x) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=+\infty}\left(\int_{n=-\infty}^{n=+\infty} F(n)e^{i\cdot\omega\cdot n\cdot(x-n)}\, dn\right) d\omega \tag{3.5}$$

or equivalently in infinite summation notation:

$$f(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} d_n e^{i\cdot n\cdot x}, \tag{3.6}$$

where apparently it is valid that: $e^{i\cdot n\cdot x} = \cos(nx) + i\sin(nx)$. Concurrently, we can consider the range of every oscillation as a single-valued function of time in the form: $x_0 = \frac{2\pi}{T}t$, something compatible to Lagrangian formulation for fluid motion [1]. So, we can obtain:

$$\begin{aligned} V_x &= \frac{1}{\sqrt{2\pi}}\cdot\int_{\omega=-\infty}^{\omega=+\infty} C(\omega)e^{i\cdot n\cdot\frac{2\pi}{T}\cdot t}\, d\omega \\ V_y &= \frac{1}{\sqrt{2\pi}}\cdot\int_{\omega=-\infty}^{\omega=+\infty} C'(\omega)e^{i\cdot n\cdot\frac{2\pi}{T}\cdot t}\, d\omega \\ V_z &= \frac{1}{\sqrt{2\pi}}\cdot\int_{\omega=-\infty}^{\omega=+\infty} C''(\omega)e^{i\cdot n\cdot\frac{2\pi}{T}\cdot t}\, d\omega, \end{aligned} \tag{3.7}$$

where:

$$\begin{aligned} C(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(n)\cdot e^{i\cdot n\cdot\frac{2\pi}{T}\cdot t}\, dn \\ C'(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f'(n)\cdot e^{i\cdot n\cdot\frac{2\pi}{T}\cdot t}\, dn \\ C''(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f''(n)\cdot e^{i\cdot n\cdot\frac{2\pi}{T}\cdot t}\, dn. \end{aligned} \tag{3.8}$$

As concerns the rate of change of the circular frequency ω, it is also valid that:

$$\Delta\omega = \omega_{n+1} - \omega_n = \frac{2\,(n+1)\,\pi}{T} - \frac{2n\pi}{T} = \frac{2\pi}{T}$$

$\Leftrightarrow$

$$\frac{\Delta\omega}{\pi} = \frac{2}{T}. \tag{3.9}$$

The Fourier-Transform has its greatest range of validity at the complex plane, where it is equivalent with Laplace-Transform [2, 12].

Obviously, when the apparent returning period tends to infinity, we can conjecture that the term $\Delta\omega$ must get the infinitesimal form $d\omega$.

We also denote here, that the quantities $C(\omega)$, $C'(\omega)$, $C''(\omega)$ are derived by the so-called inverse Fourier-Transform F^{-1}. Hence, we infer:

$$\begin{aligned} F^{-1}\{C(\omega)\} &= V_x \\ F^{-1}\{C'(\omega)\} &= V_y \\ F^{-1}\{C''(\omega)\} &= V_z. \end{aligned} \tag{3.10}$$

Moreover, the known Parseval identity can be written out in the following form, provided that the generalized integral on the left member of the following equation converges:

$$\int_{-\infty}^{+\infty} |f(t)|^2 dt \equiv \frac{1}{2\pi} \int_{-\infty}^{+\infty} |F(\omega)|^2 d\omega. \tag{3.11}$$

Note here, that from Physical aspect the left member of the above, denotes the total energy of any circumstantial turbulent signal, hence the term in right member denotes the distribution of it, in the motivated scales of eddies.

Besides, taking into account eq. (3.7) we have:

$$\begin{aligned} V_x &= \overline{V_x} + \left\{ \sum_{n=-\infty}^{\infty} d_n e^{i \cdot n \cdot \frac{2\pi}{\mathrm{T}} \cdot t} \right\} \\ V_y &= \overline{V_y} + \left\{ \sum_{n=-\infty}^{\infty} d_n e^{i \cdot n \cdot \frac{2\pi}{\mathrm{T}} \cdot t} \right\} \\ V_z &= \overline{V_z} + \left\{ \sum_{n=-\infty}^{\infty} d_n e^{i \cdot n \cdot \frac{2\pi}{\mathrm{T}} \cdot t} \right\}, \end{aligned} \tag{3.12}$$

where:

$$\begin{aligned} d_n &= \frac{1}{\mathrm{T}} \int_0^{\mathrm{T}} f_1(t) \cdot e^{i \cdot n \cdot \frac{2\pi}{\mathrm{T}} \cdot t} \\ d_n &= \frac{1}{\mathrm{T}} \int_0^{\mathrm{T}} f_2(t) \cdot e^{i \cdot n \cdot \frac{2\pi}{\mathrm{T}} \cdot t} \\ d_n &= \frac{1}{\mathrm{T}} \int_0^{\mathrm{T}} f_3(t)(t) \cdot e^{i \cdot n \cdot \frac{2\pi}{\mathrm{T}} \cdot t}. \end{aligned} \tag{3.13}$$

We can pinpoint here, that these integrals, appearing in the latter group of equations, are in the generic form: $\int e^{Ax} h(x)$.

It is also known from single-valued Calculus that:

$$\int e^{Ax} h(x) = \frac{1}{A} e^{Ax} \sum_{n=0}^{n} \left[(-1)^n \frac{1}{A^n} h^{(n)}(x) \right], \tag{3.14}$$

(provided that the function h is analytic on its domain of definition).

If we put into effect now this latter formula, we are able to express eventually the coefficients in eq. (3.13), as single-valued functions of the coefficients: a_0, b_0, c_0 which occurred on eq. (3.4) successively.

Besides, if we also equate the right members of eq. (3.6) and eq. (2.28) supposing that the ratio $\frac{L}{\kappa}$ tends to 1, we will be able indeed to estimate the corresponding coefficients: d_n for the three velocity components in a sinusoidal summation form.

4. Discussion

In this paper we attempted to make a contribution to the approximate evaluation of the corresponding harmonic decomposition for an arbitrary continuous function, via an Applied Analysis approach.

It is known that the convergence of such power series in any orthogonal space of trigonometric functions, is indeed very slow from computational standpoint. Actually we focused our investigation here, on a class of continuous functions such that except the verification of Dirichlet's principles to be also analytic over a particular interval of their domain of definition.

An application in turbulence spectrum of eddies of the evident alternative representation has also taken place, assuming additionally that the time returning period for the evolution of such dynamical effects tends to infinity. Then the expansion of Karman's spectrum of eddies, is possibly able to be predicted via this calculating process even qualitatively.

However, one cannot be very sure indeed if this resultant closed-form representation that we have derived here, because of its qualitative character which has not been avoided yet, is unedited capable to participate directly in the contemporary computational procedures, (i.e. FFT or Discrete Fourier Tranform), which actually consist in the status quo in the mathematical approach for the evolution of such dynamical effects, without the concurrent encumbrance of computational operating cost.

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