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On the spectrum and fine spectrum of an upper triangular double-band matrix on sequence spaces

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Abstract

In this paper, we investigate the spectrum and fine spectrum of the upper triangular double-band matrix Δ^{uv} on cs sequence space. We also determine the approximate point spectrum, the defect spectrum and the compression spectrum of this matrix on cs.

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1. Introduction

We denote the space of all real or complex valued sequences by w. We represente bounded variation, convergent series and absolutely summable spaces by bv, cs and ℓ_1 respectively.

Let X and Y be Banach spaces and $T : X \to Y$ be a bounded linear operator. By $R(T)$, we denote the range of T. i.e., $R(T) = \{y \in Y : y = Tx,\ x \in X\}$.

By $B(X)$, we denote the set of all bounded linear operator on X into itself. If X is any Banach space and $T \in B(X)$ then the adjoint T^* of T is a bounded linear operator on the dual X^* of X defined by $(T^*f)(x) = f(Tx)$, for all $f \in X^*$ and $x \in X$. We need some basic concepts which are given in [11] as follows:

Let $X \neq \{\theta\}$ be a complex normed space, where θ is the zero element and $T : D(T) \to X$ is a linear operator with domain $D(T) \subseteq X$. With T, we associate the operator $T_\lambda = T - \lambda I$, where λ is a complex number and I is the identity operator on $D(T)$. If T_λ has an inverse which is linear, we denote it by T_λ^{-1}, that is, $T_\lambda^{-1} = (T - \lambda I)^{-1}$ and we call it the resolvent operator of T. A regular value λ of T is a complex number such that

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$(R1)$ T_λ^{-1} exists,
$(R2)$ T_λ^{-1} is bounded,
$(R3)$ T_λ^{-1} is defined on a set which is dense in X.

By $\rho(T,X)$ we denote the resolvent set of T. It is a set of all regular values λ of T. Its complement $\sigma(T,X) = \mathbb{C} - \rho(T,X)$ in the complex plane $\mathbb{C}$ is called the spectrum of T. Furthermore, the spectrum $\sigma(T,X)$ is partitioned into three disjoint sets as follows:

The point spectrum $\sigma_p(T;X)$ is the set of all $\lambda \in \mathbb{C}$ such that T_λ^{-1} does not exist. The element of $\sigma_p(T,X)$ is called eigenvalue of T.

The continuous spectrum $\sigma_c(T;X)$ is the set of all $\lambda \in \mathbb{C}$ such that T_λ^{-1} exists and satisfies (R3) but not (R2), that is, T_λ^{-1} is unbounded.

The residual spectrum $\sigma_r(T;X)$ is the set of all $\lambda \in \mathbb{C}$ such that T_λ^{-1} exists but does not satisfy (R3), that is, the domain of T_λ^{-1} is not dense in X.

To avoid trivial misunderstandings, we can say that some of the sets defined above, may be empty. This is an existence problem, which shall have to discuss. Indeed, it is well known that $\sigma_c(T,X) = \sigma_r(T,X) = \varnothing$ and the spectrum $\sigma(T,X)$ consists of only the set $\sigma_p(T,X)$ in the finite dimensional case.

From Goldberg [9], if X is a Banach space and $T \in B(X)$, then there are three possibilities for $R(T)$ and T^{-1}:
(A) $R(T) = X$,
(B) $R(T) \neq \overline{R(T)} = X$,
(C) $\overline{R(T)} \neq X$
and
(1) T^{-1} exists and is continuous,
(2) T^{-1} exists but is discontinuous,
(3) T^{-1} does not exist.

Applying Goldberg [9] classification to T_λ, we have the following possibilities;
(A) T_λ is surjective,
(B) $R(T_\lambda) \neq \overline{R(T_\lambda)} = X$,
(C) $\overline{R(T_\lambda)} \neq X$.
and
(1) T_λ is injective and T_λ^{-1} is continuous.
(2) T_λ is injective and T_λ^{-1} is discontinuous.
(3) T_λ is not injective.

If these possibilities are combined in all possible ways, nine different states are created. These are labelled by: $A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2$ and C_3. If λ is a complex number such that $T_\lambda \in A_1$ or $T_\lambda \in B_1$, then λ is in the resolvent set $\rho(T,X)$ of T on X. The other classifications give rise to the fine spectrum of T. We use $\lambda \in B_2\sigma(T,X)$ means the operator $T_\lambda \in B_2$, i.e., $R(T_\lambda) \neq \overline{R(T_\lambda)} = X$ and T_λ is injective but T_λ^{-1} is discontinuous, similarly others. Following Appell et. al. [4], we define the three more subdivisions of the spectrum as follows:

Let T be a bounded linear operator in a Bancah space X, we call a sequence (x_k) in X as a Weyl sequence for T if $\|x_k\| = 1$ and $\|Tx_k\| \to 0$ as $k \to \infty$. In what follows, the sets are called

$\sigma_{ap}(T,X) = \{\lambda \in \mathbb{C} : \text{there exists a Weyl sequence for } T - \lambda I\}$ the approximate point spectrum of T.

$\sigma_\delta(T,X) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not surjective}\}$, the defect spectrum of T.

We can write spectrum as a form of subdivision of two subspectra given (not necessarily disjoint) $\sigma(T,X) = \sigma_{ap}(T,X) \cup \sigma_\delta(T,X)$.

There is another spectrum $\sigma_{co}(T,X) = \left\{\lambda \in \mathbb{C} : \overline{R(T-\lambda I)} \neq X\right\}$ which is called the compression spectrum of T. Then we have another property such as $\sigma(T,X) = \sigma_{ap}(T,X) \cup \sigma_{co}(T,X)$.

From the definitions which are given above the subdivisions spectrum are illustrated in the Table 1.

Proposition 1.1 [4] Spectra and subspectra of an operator $T \in B(X)$ and its adjoint $T^* \in B(X^*)$ have some relationships given as follows:
(a) $\sigma_p(T^*, X^*) = \sigma_{co}(T,X)$.
(b) $\sigma_{ap}(T^*, X^*) = \sigma_\delta(T,X)$.
(c) $\sigma_\delta(T^*, X^*) = \sigma_{ap}(T,X)$.

Lemma 1.2 [9] T has a dense range if and only if T^* is one to one.

Lemma 1.3 [9] T has a bounded inverse if and only if T^* is onto.

We know that $cs = \left\{x = (x_n) \in w : \lim_n \sum\limits_i x_i \text{ exists}\right\}$ is a Banach space with the norm $\|x\|_{cs} = \sup_n \left|\sum\limits_{i=0}^{n} x_i\right|$. The main purpose of this paper determines the spectrum and fine spectrum of the upper triangular matrix Δ^{uv} on the sequence space cs. Also we examine the approximate point spectrum, the defect spectrum and the compression spectrum on cs. If we take $v_k = r$ and $u_k = s$ we obtain the matrix representation of the operator $U(r,s)$ which were given in [12]. Hence our results are a generalization of results which were given in [12]. The fine spectrum of the difference operator Δ over the sequence spaces ℓ_p and bv_p, $(1 \le p < \infty)$ is studied by Akhmedov and Başar in [1] and [2]. Also Başar and Altay have determined the fine spectrum of the difference operator Δ over the sequence spaces c_0, c and ℓ_p, $(0 < p < 1)$ in [5] and [3]. The fine spectrum of the operator Δ_{uv} over the sequence space c_0 has been examined by Fathi and Lashkaripour in [7]. They also studied the fine spectrum of generalized upper triangular double band matrices Δ^v and Δ^{uv} over the sequence ℓ_1 in [8]. Some other authors studied spectrum and fine spectrum of various matrix operators (see [6], [10], [13]).

		1	2	3
		T_λ^{-1} *exists and it is bounded*	T_λ^{-1} *exists and it is not bounded*	T_λ^{-1} *does not exist*
A	$R(T-\lambda I)$	$\lambda \in \rho(T,X)$	$\cdots$	$\lambda \in \sigma_p(T,X)$ $\lambda \in \sigma_{ap}(T,X)$
B	$\overline{R(T-\lambda I)} = X$	$\lambda \in \rho(T,X)$	$\lambda \in \sigma_c(T,X)$ $\lambda \in \sigma_{ap}(T,X)$ $\lambda \in \sigma_\delta(T,X)$	$\lambda \in \sigma_p(T,X)$ $\lambda \in \sigma_{ap}(T,X)$ $\lambda \in \sigma_\delta(T,X)$
C	$\overline{R(T-\lambda I)} \neq X$	$\lambda \in \sigma_c(T,X)$ $\lambda \in \sigma_\delta(T,X)$ $\lambda \in \sigma_{co}(T,X)$	$\lambda \in \sigma_r(T,X)$ $\lambda \in \sigma_{ap}(T,X)$ $\lambda \in \sigma_\delta(T,X)$ $\lambda \in \sigma_{co}(T,X)$	$\lambda \in \sigma_p(T,X)$ $\lambda \in \sigma_{ap}(T,X)$ $\lambda \in \sigma_\delta(T,X)$ $\lambda \in \sigma_{co}(T,X)$

Table 1. Subdivisions of spectrum of a linear operator

2. Main Results

The upper triangular double-band matrices Δ^{uv} is defined

$$\Delta^{uv}x = \Delta^{uv}(x_n) = (v_n x_n + u_{n+1}x_{n+1})_{n=0}^{\infty}.$$

Then, it is easy to verify that the double-band matrices Δ^{uv} can be represented by the matrix,

$$\Delta^{uv} = \begin{bmatrix} v_0 & u_1 & 0 & 0 & 0 & \cdots \\ 0 & v_1 & u_2 & 0 & 0 & \cdots \\ 0 & 0 & v_2 & u_3 & 0 & \cdots \\ 0 & 0 & 0 & v_3 & u_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

(u_k) is a sequence of positive real numbers such that $u_k \neq 0$ for each $k \in \mathbb{N}$ with $u = \lim_{k\to\infty} u_k \neq 0$ and (v_k) is either constant or strictly decreasing sequence of positive real numbers with $v = \lim_{k\to\infty} v_k \neq 0$, and $v_0 < u + v$.

Theorem 2.1 The operator $\Delta^{uv} : cs \to cs$ is a bounded linear operator and $\|\Delta^{uv}\|_{B(cs)} \leq \sup_k (|v_k| + |u_k|)$.

Proof.

$$\begin{aligned} |\Delta^{uv}(x)| &= \left|\sum_{k=0}^{\infty} v_k x_k + u_{k+1}x_{k+1}\right| \leq \left|\sum_{k=0}^{\infty} v_k x_k\right| + \left|\sum_{k=0}^{\infty} u_{k+1}x_{k+1}\right| \\ &\leq \sup_k |v_k| \|x\|_{cs} + \sup_k |u_k| \|x\|_{cs} = \sup_k (|v_k| + |u_k|) \|x\|_{cs}. \end{aligned}$$

Thus $\|\Delta^{uv}\|_{B(cs)} \leq \sup_k (|v_k| + |u_k|)$.

Theorem 2.2 Let $L_1 = \left\{\lambda \in \mathbb{C} : |\lambda - v| = u,\ \sum_k \prod_{i=1}^{k} \left(\frac{\lambda - v_{i-1}}{u_i}\right) < \infty\right\}$. Then the inclusion $\{\lambda \in \mathbb{C} : |\lambda - v| < u\} \cup L_1 \subseteq \sigma_p(\Delta^{uv}, cs)$ holds.

Proof. Firstly we suppose $v = (v_k)$ is a constant sequence, say, $v_k = v$ for all k. $\Delta^{uv}x = \lambda x$, for $x \neq 0 = (0, 0, 0, ...)$ in cs, which gives

$$\begin{aligned} v_0 x_0 + u_1 x_1 &= \lambda x_0 \\ v_1 x_1 + u_2 x_2 &= \lambda x_1 \\ v_2 x_2 + u_3 x_3 &= \lambda x_2 \\ &\vdots \\ v_k x_k + u_{k+1} x_{k+1} &= \lambda x_k \\ &\vdots \end{aligned}$$

If $x_0 = 0$, then $x_k = 0$ for all k. Hence $x_0 \neq 0$. Solving this equations, we get

$$x_n = \prod_{i=1}^{n} \left(\frac{\lambda - v_{i-1}}{u_i}\right) x_0$$

for all $n \in \mathbb{N}$. Now suppose $\lambda \in \mathbb{C}$ with $|\lambda - v| < u$. Then

$$\lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| = \lim_{n\to\infty}\left|\frac{v_n-\lambda}{u_{n+1}}\right| = \left|\frac{v-\lambda}{u}\right| < 1$$

therefore $(x_n) \in \ell_1 \subset cs$, and thus easily $L_1 \subseteq \sigma_p(\Delta^{uv}, cs)$ is seen, consequently

$$\{\lambda \in \mathbb{C} : |\lambda - v| < u\} \cup L_1 \subseteq \sigma_p(\Delta^{uv}, cs).$$

Now we give an example to show this inclusion is strict. Take $r_k = \left(\frac{k+1}{k+3}\right)^2$ and $s_k = \left(\frac{k+1}{k+2}\right)^2$, $k \in \mathbb{N}$. These sequences provide the properties which are given in their definitions. Clearly, $0 \notin \{\lambda \in \mathbb{C} : |\lambda - v| < u\}$. But $0 \in \sigma_p(\Delta^{uv}, cs)$ since there exists $x = (x_0, x_1, ...)$ such that $x_0 \neq 0$, $x_1 \neq 0$ and $x_{k+1} = -\frac{r_{k-1}}{s_{k-1}} x_k$, $k \geq 1$ and

$$\sum_k |x_k| = |x_0| + |x_1| + 4|x_1| \sum_{k=3}^{\infty} \frac{1}{k^2} < \infty.$$

Theorem 2.3 $\sigma_p((\Delta^{uv})^*, cs^* \cong bv) = \varnothing$.

Proof. Suppose (v_k) is a constant sequence, say, $v_k = v$ for all k. Then there exists $f \neq \theta = (0,0,0,...)$ in bv such that $\Delta^{uv} f = \lambda f$. We have

$$\begin{aligned}
v_0 f_0 &= \lambda f_0 \\
v_1 f_0 + v_1 f_1 &= \lambda f_1 \\
v_2 f_1 + v_2 f_2 &= \lambda f_2 \\
&\vdots \\
v_k f_{k-1} + v_k f_k &= \lambda f_k \\
&\vdots
\end{aligned}$$

Let f_m be the first non-zero entry of the sequence (f_n). So we get $u_m f_{m-1} + v f_m = \lambda f_m$ which implies $\lambda = v$ and from the equation $u_{m+1} f_m + v f_{m+1} = \lambda f_{m+1}$ we get $f_m = 0$, which is a contradiction to our assumption. Therefore,

$$\sigma_p((\Delta^{uv})^*, cs) = \varnothing.$$

Suppose (v_k) is a strictly decreasing sequence. Consider $(\Delta^{uv})^* f = \lambda f$, for $f \neq 0 = (0,0,0,...)$ in bv, which gives above system of equations. Hence, for all $\lambda \notin \{v_0, v_1, v_2, ...\}$, we have $v_k = 0$ for all k, which is a contradiciton. So $\lambda \notin \sigma_p((\Delta^{vu})^*, bv)$. This shows that

$$\sigma_p((\Delta^{uv})^*, cs) \subseteq \{v_0, v_1, v_2, ...\}.$$

Let $\lambda = v_m$ for some m. Then $f_0 = f_1 = ... = f_{m-1} = 0$. Now if $f_m = 0$, then $f_k = 0$ for all k, which is contradiciton. Also if $f_m \neq 0$, then

$$f_{k+1} = \frac{u_{k+1}}{v_m - v_{k+1}} f_k, \text{ for all } k \geq m$$

and

$$f_k = \frac{u_k u_{k-1} ... u_1}{(\lambda - v_k)(\lambda - v_{k-1})(\lambda - v_{k-2}) ... (\lambda - v_1)} f_0 = \prod_{i=1}^{k} \frac{u_i}{\lambda - v_i} f_0, \qquad k \geq 1.$$

We have

$$\lim_{k\to\infty}|f_k| = \left|\frac{u_k u_{k-1}...u_1}{(\lambda - v_k)(\lambda - v_{k-1})(\lambda - v_{k-2})...(\lambda - v_1)}\right| |f_0| \neq 0$$

because of $v_0 < v + u$.

$$\lim_{k\to\infty}|f_{k+1} - f_k| \neq 0$$

then we have $f = (f_k) \notin bv$. Thus $\sigma_p\left(\left(\Delta^{uv}\right)^*, cs^*\right) = \varnothing$.

Theorem 2.4 $\Delta_\lambda^{uv} : cs \to cs$ has a dense range for any $\lambda \in \mathbb{C}$.

Proof. $\sigma_p\left(\left(\Delta^{uv}\right)^*, cs^*\right) = \varnothing$ hence $\left(\Delta^{uv} - \lambda I\right)^*$ is one to one for all $\lambda \in \mathbb{C}$ and from Lemma 1.2 we have the required result.

Theorem 2.5 $\sigma_r\left(\Delta^{uv}, cs\right) = \varnothing$.

Proof. It is a result of Lemma 1.2 and Theorem 2.4.

Theorem 2.6 $\sigma\left(\Delta^{uv}, cs\right) = \{\lambda \in \mathbb{C} : |\lambda - v| \leq u\}$.

Proof. Let $y \in cs$ and consider $\left(\Delta^{uv} - \lambda I\right)^* x = y$. Then we have the linear system of equations

$$\begin{aligned}
(v_0 - \lambda) x_0 &= y_0 \\
u_1 x_0 + (v_1 - \lambda) x_1 &= y_1 \\
u_2 x_0 + (v_2 - \lambda) x_2 &= y_2 \\
&\vdots \\
u_k x_{k-1} + (v_k - \lambda) x_k &= y_k \\
&\vdots
\end{aligned}$$

By solving this equations, we get

$$x_k = \frac{(-1)^k u_0 u_1 ... u_{k-1} y_0}{(v_2 - \lambda)(v_1 - \lambda)(v_0 - \lambda)...(v_k - \lambda)} + ... - \frac{u_{k-1} y_{k-1}}{(v_k - \lambda)(v_{k-1} - \lambda)} + \frac{y_k}{v_k - \lambda}.$$

Then

$$\sum_k |x_k| < \sum_k R_k |y_k|$$

where

$$R_k = \left|\frac{1}{v_k - \lambda}\right| + \left|\frac{u_k}{(v_k - \lambda)(v_{k+1} - \lambda)}\right| + \left|\frac{u_k u_{k+1}}{(v_k - \lambda)(v_{k+1} - \lambda)(v_{k+2} - \lambda)}\right| + ... \,.$$

While $k \to \infty$, $\left|\frac{u_k}{v_{k+1} - \lambda}\right| \to \left|\frac{v}{v - \lambda}\right| < 1$. For $k_0 \in \mathbb{N}$ and $q_0 \in \mathbb{R}$ we have $\left|\frac{u_k}{v_{k+1} - \lambda}\right| < q_0$ for $k \geq k_0$. Then,

$$R_k \leq \frac{1}{|v_k - \lambda|}\left(1 + q_0 + q_0^2 + ...\right)$$

for $k \geq k_0 + 1$. Also there exist $k_1 \in \mathbb{N}$ and a real number q_1 which provides $\left|\frac{1}{v_k - \lambda}\right| < q_1$ for all $k \geq k_1$. Then, $R_k \leq \frac{q_1}{1 - q_0}$ for all $k > \max\{k_0, k_1\}$ and $\sup_{k\in\mathbb{N}} R_k < \infty$. Consequently, since

$$\sum_k |x_k| \leq \sum_k R_k |y_k| \leq \sup_k |R_k| \sum_k |y_k| < \infty,$$

$x \in \ell_1 \subset bv$. Hence, for $u < |\lambda - v|$, $(\Delta^{uv} - \lambda I)^*$ is onto and by Lemma 1.3 $\Delta^{uv} - \lambda I$ has a bounded inverse. This means that

$$\sigma_c (\Delta^{uv}, cs) \subseteq \{\lambda \in \mathbb{C} : |\lambda - v| \leq u\}.$$

By Theorem 2.2 and Theorem 2.5, we get

$$\{\lambda \in \mathbb{C} : |\lambda - v| < u\} \subseteq \sigma (\Delta^{uv}, cs) \subseteq \{\lambda \in \mathbb{C} : |\lambda - v| \leq u\}.$$

Since the spectrum of any bounded operator is closed, we get

$$\sigma (\Delta^{uv}, cs) = \{\lambda \in \mathbb{C} : |\lambda - v| \leq u\}.$$

Theorem 2.7 $\sigma_c (\Delta^{uv}, cs) = \{\lambda \in \mathbb{C} : |\lambda - v| = u\} \backslash L_1$.

Proof. $\sigma (\Delta^{uv}, cs)$ is a disjoint union of the parts $\sigma_p (\Delta^{uv}, cs)$, $\sigma_r (\Delta^{uv}, cs)$ and $\sigma_c (\Delta^{uv}, cs)$ and so we obtain the required result.

Theorem 2.8 If $|\lambda - v| < u$, then $\lambda \in A_3\sigma (\Delta^{uv}, cs)$.

Proof. Let $|\lambda - v| < u$. Then by Theorem 2.2, $\lambda \in \sigma_p (\Delta^{uv}, cs)$ and hence $\lambda \in (3)$. We need to prove that $\Delta^{uv} - \lambda I$ is surjective when $|\lambda - v| < u$. Let $z = (z_0, z_1, z_2, ...) \in cs$ and consider the equation $(\Delta^{uv} - \lambda I) x = z$. Then we have the linear system of equations

$$\begin{aligned}
(v_0 - \lambda) x_0 + u_1 x_1 &= z_0 \\
(v_1 - \lambda) x_1 + u_2 x_2 &= z_1 \\
(v_2 - \lambda) x_2 + u_3 x_3 &= z_2 \\
&\vdots \\
(v_k - \lambda) x_k + u_{k+1} x_{k+1} &= z_k \\
&\vdots
\end{aligned}$$

Let $x_0 = 0$. Therefore, we obtain

$$x_k = \frac{(\lambda - v_1)(\lambda - v_2)\dots(\lambda - v_{k-1}) z_0}{u_1 u_2 ... u_k} + ... + \frac{(v_{k-1} - \lambda) z_{k-2}}{u_k u_{k-1}} + \frac{z_{k-1}}{u_k}.$$

Then, $\sum_k |x_k| \leq \sup_{k\in\mathbb{N}} S_k \sum_k |z_k|$, where

$$S_k = \left|\frac{1}{u_{k+1}}\right| + \left|\frac{v_{k+1} - \lambda}{u_{k+1}u_{k+2}}\right| + \left|\frac{(v_{k+1} - \lambda)(v_{k+2} - \lambda)}{u_{k+1}u_{k+2}u_{k+3}}\right| + ...$$

for all $k \in \mathbb{N}$. Since $\left|\frac{v_{k+1}-\lambda}{u_{k+1}}\right| \to \left|\frac{v-\lambda}{u}\right| < 1$ as $k \to \infty$, there exist $k_0 \in \mathbb{N}$ and a real number p_0 such that $|\frac{v_{k+1}-\lambda}{u_{k+1}}| < p_0$ for all $k \geq k_0$. Then, for all $k \geq k_0 + 1$,

$$S_k \leq \frac{1}{|u_{k+1}|} \left(1 + p_0 + p_0^2 + ...\right).$$

Also there exist $k_1 \in \mathbb{N}$ and a real number p_1 such that $|\frac{1}{u_{k+1}}| < p_1$ for all $k \geq k_1$. Then, $S_k \leq \frac{p_1}{1-p_0}$, for all $k > max\{k_0, k_1\}$. Thus, $\sup_{k\in\mathbb{N}} S_k < \infty$. Therefore, $\sum_k |x_k| \leq \sup_{k\in\mathbb{N}} S_k \sum_k |z_k| < \infty$. Hence $x \in cs$.

Corollary 2.9 Let (v_k) and (u_k) be constant sequences, say, $v_k = v$ and $u_k = u$ for all k, and $|\lambda - v| = u$. Then $\lambda \in B_2\sigma(\Delta^{uv}, cs)$.

Proof. When $|\lambda - v| = u$ by Theorem 2.7 we see that $\lambda \in A_2 \cup B_2$. Also $\Delta^{uv} - \lambda I$ is not surjective and hence $\lambda \in B_2\sigma(\Delta^{uv}, cs)$.

Corollary 2.10
(i) $\sigma_{co}(\Delta^{uv}, cs) = \varnothing$
(ii) $\sigma_\delta(\Delta^{uv}, cs) = \{\lambda \in \mathbb{C} : |\lambda - v| = |u|\}$
(iii) $\sigma_{ap}(\Delta^{uv}, cs) = \{\lambda \in \mathbb{C} : |\lambda - v| \leq |u|\}$.

Proof.
(i) From Proposition 1.1 (a) we have $\sigma_p((\Delta^{uv})^*, cs^*) = \sigma_{co}(\Delta^{uv}, cs) = \varnothing$.
(ii) We have that $\sigma_\delta(\Delta^{uv}, cs) = \sigma(\Delta^{uv}, cs) \setminus A_3\sigma(\Delta^{uv}, cs)$ from Table 1. Hence by Theorem 2.2 and Theorem 2.8 we obtain the required result.
(iii) From Table 1 $\sigma_{ap}(\Delta^{uv}, cs) = \sigma(\Delta^{uv}, cs) \setminus A_1\sigma(\Delta^{uv}, cs)$. Also
$\sigma_r(\Delta^{uv}, cs) = A_1\sigma(\Delta^{uv}, cs) \cup A_2\sigma(\Delta^{uv}, cs)$. By Theorem 2.5 $A_1\sigma(\Delta^{uv}, cs) = \varnothing$. Hence by Theorem 2.2 $\sigma_{ap}(\Delta^{uv}, cs) = \{\lambda \in \mathbb{C} : |\lambda - v| \leq |u|\}$.

Corollary 2.11
(i) $\sigma_{ap}((\Delta^{uv})^*, cs^* \cong bv) = \{\lambda \in \mathbb{C} : |\lambda - v| = |u|\}$
(ii) $\sigma_\delta((\Delta^{uv})^*, cs^* \cong bv) = \{\lambda \in \mathbb{C} : |\lambda - v| \leq u\}$.

Proof.
(i) We have $\sigma_{ap}((\Delta^{uv})^*, cs^* \cong bv) = \sigma_\delta(\Delta^{uv}, cs) = \{\lambda \in \mathbb{C} : |\lambda - v| = |u|\}$ from Proposition 1.1 (b).
(ii) $\sigma_\delta((\Delta^{uv})^*, cs^* \cong bv) = \sigma_{ap}(\Delta^{uv}, cs)$ is seen from Proposition 1.1 (c).

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