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Semi linear elliptic system at resonance

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Abstract

In this work, we investigate the existence of weak solutions for the following semi-linear elliptic system

$$\begin{cases} -\Delta u + p(x)u = \alpha u + \phi(x, v) & \text{in } \Omega, \\ -\Delta v + q(x)v = \beta v + \psi(x, u) & \text{in } \Omega, \end{cases}$$

with Dirichlet boundary condition, where Ω is a bounded open set of $\mathbb{R}^N$ $(N \geq 2)$, α, β two real parameters, $(p(x), q(x)) \in (L^\infty(\Omega))^2$ and $p(x), q(x) \geq 0$. using the Leray-Schauder's topological degree and under some suitable conditions for the non linearities ϕ and ψ, we show the existence of nontrivial solutions.

Keywords: Homotopy, Topological degree, Fixed point theorems
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1 Introduction

The Laplacian has significant applications in various fields, including mathematics, physics, computer science, and image processing. The Laplacian operator, denoted by ∇^2 (nabla squared), is a second-order differential operator that measures the rate of change of a quantity (such as temperature, pressure, or potential) with respect to spatial coordinates. Several studies related to the Laplacian, p-Laplacian, or in general the $p(x)$-Laplacian, operator have been reported (see for instance [2, 3, 4, 5] and the references therein).

Here are some of its key applications: **1)Physics and Engineering: Heat Diffusion :** In the field of heat conduction, the Laplacian is used to describe how heat diffuses through materials over time. The heat equation, which involves the Laplacian, describes this phenomenon and is essential in understanding heat transfer processes.

2) Image Processing and Computer Vision: Image Enhancement: The Laplacian is used in image enhancement techniques like Laplacian sharpening, where the Laplacian of an image is used to highlight edges and fine details.

3) Geometry and Differential Geometry: Surface Curvature: The Laplace-Beltrami operator, a generalization of the Laplacian to curved surfaces, is used to compute the curvature of surfaces in differential geometry.

These applications illustrate the versatility and importance of the Laplacian operator across a wide range of disciplines. Its ability to capture spatial variations and rates of change makes it a fundamental tool in understanding and analyzing various phenomena in the natural and scientific world.

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Semilinear elliptic equations are the first nonlinear generalization of linear elliptic partial differential equations. It is well known that linear elliptic equations, such as the ubiquitous Laplace and Poisson equations, provide models for a variety of physics problems. For this reason, they have been studied for more than two hundred years and continue to attract researchers today. Solutions to these equations also represent or describe the potential of force fields in various physical contexts, such as electromagnetism, gravitation, fluid dynamics etc.

Systems of nonlinear elliptic equations present some new and interesting phenomena not found when studying a single equation. In general, the systems are coupled or even strongly coupled in the dependent variables.

In this work we investigate semilinear elliptic systems which in the scalar case are reduced to equations of the form

$$-\Delta u = \alpha u + f(u) \qquad \text{in } \Omega,$$

under certain conditions of non-linearity called **Landesman–Lazer** type conditions according to the work of **Landesman & Lazer** [1], where analogous results were proved for the first time and we speak of resonance problems because the corresponding ordinary differential version describes them the resonance in electric circuits when α is an eigenvalue.

More specifically, we investigate the existence of weak solutions for the following elliptic systems

$$\begin{cases} -\Delta u + p(x)u = \alpha u + \phi(x,v) & \text{in } \Omega, \\ -\Delta v + q(x)v = \beta v + \psi(x,u) & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

where Ω is a bounded open set of $\mathbb{R}^N$ $(N \geq 2)$, α, β two real parameters, $(p(x), q(x)) \in (L^\infty(\Omega))^2$ and $p(x), q(x) \geq 0.$, $\varphi, \psi : \Omega \times \mathbb{R} \to \mathbb{R}$ two continuous functions satisfying

$$\begin{cases} |\phi(x,s)| \leq C_1(1+|s|), \\ |\psi(x,t)| \leq C_2(1+|t|). \end{cases} \tag{1.2}$$

where C_1, C_2 are two positive constants. Verifying also

$$\begin{cases} \lim\limits_{s\to\infty} \frac{\phi(x,s)}{s} = \lim\limits_{t\to\infty} \frac{\psi(x,t)}{t} = 0 & \text{uniformely in } \Omega, \\ \lim\limits_{-s\to\infty} \frac{\phi(x,s)}{s} = \lim\limits_{-t\to\infty} \frac{\psi(x,t)}{t} = 0 & \text{uniformely in } \Omega. \end{cases} \tag{1.3}$$

2 Preliminaries

Assume the space

$$Z = H_0^1(\Omega) \times H_0^1(\Omega)$$

which is a Banach space equipped with the norm, which we shall denote by $\|.\|_Z$

$$\|(u,v)\|_Z^2 = \|u\|_{H_0^1(\Omega)}^2 + \|v\|_{H_0^1(\Omega)}^2$$

and let us take

$$Y = L^2(\Omega) \times L^2(\Omega).$$

In the following, $\|.\|_{L^2(\Omega)}$ and $\|.\|_{H_0^1(\Omega)}$ denote the usual norms for $L^2(\Omega)$ and $H_0^1(\Omega)$, respectively. Recalling that the operator $(-\triangle + p)$, given by

$$D(-\triangle + p) = \left\{u \in H_0^1(\Omega), \quad (\triangle + p)\, u \in L^2(\Omega)\right\}$$

define an inverse compact operator on $L^2(\Omega)$. It has a countable family of eigenvalues $(\lambda_k)_{k\in\mathbb{N}^*}$ that can be written as an increasing sequence of positive numbers tending to $+\infty$ when $n \to +\infty$ is defined by

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq ... \leq \lambda_k \leq ...$$

Each eigenvalue is repeated as many times as its multiplicity (which is finite) corresponds to. Let $\lambda_1 \in \mathbb{R}$ be defined as

$$\lambda_1 = \inf_{u \in H_0^1, u \neq 0} \frac{\int_\Omega \left(|\nabla u(x)|^2 + p(x) |u(x)|^2 \right) dx}{\int_\Omega |u(x)|^2 dx}$$

or equivalently as

$$\lambda_1 = \inf \left\{ \int_\Omega \left(|\nabla u(x)|^2 + p(x) |u(x)|^2 \right) dx : \int_\Omega |u(x)|^2 dx = 1, u \in H_0^1(\Omega), u \neq 0 \right\},$$

λ_1 is the first eigenvalue of the operator subject to the Dirichlet boundary conditions. There exists an orthonormal and complet Hilbertian basis $(\varphi_k)_{k \geq 1}$ be the sequence of all eigenfunctions such that

$$\langle \varphi_k, \varphi_j \rangle_2 = \begin{cases} 1, & \text{if } k = j, \\ 0, & \text{if } k \neq j \end{cases}$$

and

$$\int_\Omega \varphi_1(x)\, dx = 1.$$

Riesz representation theorem: Let H be a Hilbert space equipped with the product scalar denoted $\langle ., . \rangle$; $f \in H'$ a continuous linear form on H. Then there exist an unique y such that for every x of H we have $f(x) = \langle y, x \rangle$.

$$\exists! y \in H, \ \forall x \in H, \ f(x) = \langle y, x \rangle.$$

Defining the weak solution of the problem (1.1) as follows

Definition 2.1. We say that $(u, v) \in Z$ is a weak solution of the problem if we have

$$\begin{cases} \int_\Omega \nabla u(x) \nabla \overline{u}(x)\, dx + \int_\Omega p(x) u(x) \overline{u}(x)\, dx = \alpha \int_\Omega u(x) \overline{u}(x)\, dx + \int_\Omega \varphi(x, v) \overline{u}(x)\, dx, \\ \int_\Omega \nabla v(x) \nabla \overline{v}(x)\, dx + \int_\Omega q(x) v(x) \overline{v}(x)\, dx = \beta \int_\Omega v(x) \overline{v}(x)\, dx + \int_\Omega \psi(x, u) \overline{v}(x)\, dx, \\ (\bar{u}, \bar{v}) \in Z. \end{cases} \tag{2.1}$$

We recall the following propsition proved by **T. Gallouet** and **O. Kavian** (see [1]). The operator $A : Z \to Z$ defined by

$$\langle A(u, v), (\overline{u}, \overline{v}) \rangle_Z = \langle (A_1 u, A_2 v), (\overline{u}, \overline{v}) \rangle_Z; \qquad (u, v) \ , (\overline{u}, \overline{v}) \in Z \tag{2.2}$$

where

$$\langle A_1 u, \overline{u} \rangle_{H_0^1(\Omega)} = \int_\Omega u(x) \overline{u}(x)\, dx, \quad \overline{u} \in H_0^1(\Omega)$$

and

$$\langle A_2 v, \overline{v} \rangle_{H_0^1(\Omega)} = \int_\Omega v(x) \overline{v}(x)\, dx, \quad \overline{v} \in H_0^1(\Omega)$$

is positive, self-adjoint and compact.

For fixed $(u, v) \in Z$, we define the following linear forms on the space $H_0^1(\Omega)$

$$\begin{cases} \widetilde{T}_u(\overline{u}) = \int_\Omega \nabla u(x) \nabla \overline{u}(x)\, dx; \qquad \widetilde{T}_v(\overline{v}) = \int_\Omega \nabla v(x) \nabla \overline{v}(x)\, dx \\ \widetilde{S}_v(\overline{u}) = \int_\Omega \phi(x, v(x)) \overline{u}(x)\, dx; \qquad \widetilde{S}_u(\overline{v}) = \int_\Omega \psi(x, u(x)) \overline{v}(x)\, dx \end{cases} \tag{2.3}$$

The **Riesz** representation theorem shows that there exist uniquely determined elements $L_1(u)$, $L_2(v)$, $S_1(v)$, $S_2(u) \in H_0^1(\Omega)$ such that

$$\widetilde{T}_u(\overline{u}) = \langle L_1(u), \overline{u}\rangle_{H_0^1(\Omega)} \text{ for all } \overline{u} \in H_0^1(\Omega)$$

$$\widetilde{T}_v(\overline{v}) = \langle L_2(v), \overline{v}\rangle_{H_0^1(\Omega)} \text{ for all } \overline{v} \in H_0^1(\Omega)$$

and

$$\widetilde{S}_v(\overline{u}) = \langle S_1(v), \overline{u}\rangle_{H_0^1(\Omega)} \text{ for all } \overline{u} \in H_0^1(\Omega)$$

$$\widetilde{S}_u(\overline{v}) = \langle S_2(u), \overline{v}\rangle_{H_0^1(\Omega)} \text{ for all } \overline{v} \in H_0^1(\Omega).$$

We constat that the research of weak solution $(u, v) \in Z$ to the problem (1.1) is equivalent to the resolution of the operator equation

$$L(u, v) = BA(u, v) + S(u, v), \quad (u, v) \in Z, \tag{2.4}$$

with

$$B = \begin{pmatrix} \alpha - p(x) & 0 \\ 0 & \beta - q(x) \end{pmatrix}.$$

and

$$\begin{cases} S(u, v) = (S_1(v), S_2(u)) \\ \text{and} \\ L(u, v) = (L_1(u), L_2(v)). \end{cases}$$

Clearly

$$S : (u, v) \in Z \to (S_1(v), S_2(u)) \in Z$$

is compact and continuous operator. We will use the Leray-Schauder degree theory to prove the result.

3 The first case

In this case we study the existence of solutions when α, β are not respectively eigenvalues to the operator $-\Delta + k_i(x) I$; $k_1 = p, k_2 = q$. For $\tau \in [0, 1]$ and $(u, v) \in Z$ we define the following homotopy

$$H(\tau, u, v) = \begin{pmatrix} H_1(\tau, u, v) \\ \\ H_2(\tau, u, v) \end{pmatrix} = \begin{pmatrix} u + p(x) A_1(u) - \alpha A_1(u) - \tau S_1(v) \\ \\ v + q(x) A_2(v) - \beta A_2(v) - \tau S_2(u) \end{pmatrix},$$

in view of (2.4), we have

$$H(\tau, u, v) = \begin{pmatrix} u \\ v \end{pmatrix} - BA(u, v) - \tau\, S(u, v).$$

It is clear that

$$H : [0, 1] \times Z \to Y$$

is a compact homotopy and the existence of at least one solution of the system (1.1) would follow from

$$\deg(I - BA - S, B(0, R), 0) \neq 0.$$

Theorem 3.1. Under hypothesis $(1.2), (1.3)$ and if $\alpha, \beta \neq \lambda_k$, $k = 1, 2$, then the problem (1.1) have at least one solution.

The following lemma is necessarily for the proof of the theorem.

Lemma 3.2. There exists $R > 0$ such that

$$\begin{cases} \forall \tau \in [0, 1], \forall (u, v) \in Z, \|(u, v)\|_Z = R \\ \\ H(\tau, u, v) \neq 0 \end{cases}$$

4 Proof of the main results

Proof . By contradiction, Assume that no such $R > 0$ exists, i.e., we can find a sequence $\{(u_n, v_n)\}_{n=1}^{n=\infty} \in Z$ and $\{\tau_n\}_{n=1}^{n=\infty} \subset [0,1]$ such that $\|(u_n, v_n)\|_Z > n$ and

$$(u_n, v_n) - BA(u_n, v_n) - \tau_n \, S(u_n, v_n) = 0 \ , \tag{4.1}$$

where

$$B = \begin{pmatrix} \alpha - p(x) & 0 \\ 0 & \beta - q(x) \end{pmatrix}$$

Setting

$$w_n = (w_{1,n}, w_{2,n}) = \left(\frac{u_n}{\|(u_n, v_n)\|_Z}, \frac{v_n}{\|(u_n, v_n)\|_Z} \right)$$

then it follows with choice of w_n that

$$w_n = (w_{1,n}, w_{2,n}) \in D(-\triangle + p) \times D(-\triangle + q) \ \text{ and } \ \|w_n\|_Z = 1 \tag{4.2}$$

Indeed, it is easy to see that $\|w_n\|_Z = 1$. Let us show that $w_n \in D(-\triangle + p) \times D(-\triangle + q)$. We have

$$w_n - BA(w_n) - \tau_n \, \tfrac{S(u_n, v_n)}{\|(u_n, v_n)\|_Z} = 0 \ , \tag{4.3}$$

this is equivalent to

$$\begin{cases} \int\limits_\Omega \nabla w_{1,n} \nabla \overline{w}_1 dx + \int\limits_\Omega p(x) w_{1,n} \overline{w}_1 dx = \alpha \int\limits_\Omega w_{1,n} \overline{w}_1 dx - \tau_n \int\limits_\Omega \frac{\phi(x, v_n)}{\|(u_n, v_n)\|_Z} \overline{w}_1 dx \\ \int\limits_\Omega \nabla w_{2,n} \nabla \overline{w}_2 dx + \int\limits_\Omega q(x) w_{2,n} \overline{w}_2 dx = \beta \int\limits_\Omega w_{2,n} \overline{w}_2 dx - \tau_n \int\limits_\Omega \frac{\psi(x, u_n)}{\|(u_n, v_n)\|_Z} \overline{w}_2 dx. \end{cases} \tag{4.4}$$

From (1.2) , it is easy to obtain the following estimate

$$\begin{aligned} \int_\Omega |\phi(x, v_n)|^2 dx &\leq \int_\Omega c_1^2 (1 + |v_n|)^2 dx \\ &\leq c' \left(1 + \|v_n\|_{H_0^1}^2\right) \end{aligned}$$

where c' is positive constant. Therefore

$$\begin{aligned} \int_\Omega \frac{|\phi(x, v_n)|^2}{\|(u_n, v_n)\|_Z^2} dx &\leq c' \left(\frac{1}{\|(u_n, v_n)\|_Z^2} + \frac{\|v_n\|_{H_0^1}^2}{\|(u_n, v_n)\|_Z^2} \right) \\ &\leq c' \left(\frac{1}{n^2} + 1 \right) \\ &\leq 2c' \end{aligned}$$

that is, $\frac{|\phi(x,, v_n)|}{\|(u_n, v_n)\|_Z}$ is bounded in $L^2(\Omega)$. Similarly, the function $\frac{|\psi(x, u_n)|}{\|(u_n, v_n)\|_Z}$ is bounded in $L^2(\Omega)$. Then $w_n = (w_{1,n}, w_{2,n}) \in D(-\triangle + p) \times D(-\triangle + q)$. Since the embedding $(Z \hookrightarrow Y)$ is compact, we can extract a subsequence $(\tau_n, w_{1,n}, w_{2,n})$, still denoted by $(\tau_n, w_{1,n}, w_{2,n})$, which converges in $[0,1] \times Y$.

Let (τ, w_1, w_2) be the limit of $(\tau_n, w_{1,n}, w_{2,n})$ in $[0,1] \times Y$. From the hypothesis (1.3) , it follows that

$$\begin{cases} \frac{\phi(x,, v_n)}{\|(u_n, v_n)\|_Z} = \frac{v_n}{\|(u_n, v_n)\|_Z} \frac{\phi(x, v_n)}{v_n} = w_{2,n} . \frac{\phi(x, v_n)}{v_n} \underset{n \to \infty}{\to 0} \text{ a.e. in } \Omega \\ \frac{\psi(x, u_n)}{\|(u_n, v_n)\|_Z} = \frac{u_n}{\|(u_n, v_n)\|_Z} \frac{\psi(x, u_n)}{u_n} = w_{1,n} . \frac{\psi(x, u_n)}{u_n} \underset{n \to \infty}{\to 0} \text{ a.e. in } \Omega \end{cases}$$

and since the sequences $w_{1,n}, w_{2,n}$ are bounded in $L^2(\Omega)$, we get

$$\begin{cases} \frac{\phi(x, v_n)}{\|(u_n, v_n)\|_Z} \leq c_1 (1 + |w_{2,n}|) \leq c' \quad \text{a.e.} \quad \text{in } \Omega \\ \frac{\psi(x, u_n)}{\|(u_n, v_n)\|_Z} \leq c_2 (1 + |w_{1,n}|) \leq c'' \quad \text{a.e.} \quad \text{in } \Omega \end{cases}$$

where $c^{'}, c^{''}$ are real positive constants. Then, thanks to the Lebesgue's convergence theorem, we deduce that

$$\begin{cases} \frac{\phi(x,v_n)}{\|(u_n,v_n)\|_Z} \underset{n\to\infty}{\to} 0 & \text{in } L^2(\Omega) , \\ \frac{\psi(x,u_n)}{\|(u_n,v_n)\|_Z} \underset{n\to\infty}{\to} 0 & \text{in } L^2(\Omega) , \end{cases}$$

To summarize, we have

$$\tau_n \frac{S(u_n,v_n)}{\|(u_n,v_n)\|_Z} \to 0 \text{ in } Y$$

$$A(w_n) \to A(w) \text{ in } Z,$$

which gives

$$w - BA(w) = 0 , \qquad \|w\|_Z = 1.$$

Equivalently

$$\begin{cases} -\triangle w_1 + p(x) w_1 = \alpha w_1 \\ -\triangle w_2 + q(x) w_2 = \beta w_2 \end{cases}$$

which is a contradiction. □

Proof . Let

$$B(0,R) = \{(u,v) \in Z, \|(u,v)\|_Z < R\}$$

where R is as in the lemma 1. By invariance of the topological degree we have

$$\deg(H(\tau,.,.), B(0,R), 0), t \in [0,1],$$

is constant. Then

$$\deg(H(0,.,.), B(0,R), 0) = \deg(H(1,.,.), B(0,R), 0) = \pm 1$$

and theorem 1 was validated. □

5 The second case

Now we will examinate the situation where α, β are respectively eigenvalues of the operator $-\Delta + k_i(x) I$, $k_1 = p, k_2 = q$, under Dirichlet boundary conditions, i.e

$$\begin{cases} -\Delta u + p(x)u = \alpha u & \text{in } \Omega, \\ -\Delta v + q(x)v = \beta v & \text{in } \Omega, \\ u = v = 0 \text{ on } \Gamma, \end{cases}$$

and $\phi(x,v), \psi(x,u)$ are respectively of the form $\phi(v) - h_1(x), \psi(u) - h_2(x)$. Then it is clear that the problem (1.1) is equivalent to the following

$$\begin{cases} -\Delta u + p(x)u = \alpha u + \phi(v) - h_1(x) & \text{in } \Omega, \\ -\Delta v + q(x)v = \beta v + \psi(u) - h_2(x) & \text{in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega, \end{cases} \tag{5.1}$$

where $(h_1, h_2) \in \left(L^2(\Omega)\right)^2$, $\phi, \psi : \mathbb{R} \to \mathbb{R}$ two continuous functions with finit limit

$$\begin{cases} \lim\limits_{s\to\pm\infty} \phi(s) = \phi(\pm\infty), \\ \lim\limits_{t\to\pm\infty} \psi(t) = \psi(\pm\infty), \end{cases} \tag{5.2}$$

and such that for all $s, t \in \mathbb{R}$, we have

$$\begin{cases} \phi(-\infty) < \phi(s) < \phi(+\infty) \\ \psi(-\infty) < \psi(t) < \psi(+\infty). \end{cases} \tag{5.3}$$

Theorem 5.1. Under hypothesis $(1.2), (1.3), (5.2), (5.3)$ and if $\alpha, \beta = \lambda_k$, $k = 1, 2$, then the problem (5.1) have at least one solution if and only if

$$\begin{cases} \phi(-\infty) < \int\limits_{\Omega} h_1(x)\varphi_1(x)\,dx < \phi(+\infty) \\ \psi(-\infty) < \int\limits_{\Omega} h_2(x)\varphi_2(x)\,dx < \psi(+\infty). \end{cases} \tag{5.4}$$

Lemma 5.2. There exists $R_1 > 0$ such that

$$\begin{cases} \forall t \in [0,1], \forall (u,v) \in Z, \|(u,v)\|_Z = R_1 \\ H(\tau, u, v) \neq 0 \end{cases}$$

Proof . Let $\varepsilon > 0$ such that $]\lambda_i, \lambda_i + \varepsilon] \cap sp(-\triangle + q) = \emptyset$, $i = 1, 2\,(\lambda_1 = \alpha, \lambda_2 = \beta)$. For $\tau \in [0,1]$ and $(u,v) \in Z$ we define the following homotopy

$$H(\tau,u,v) = \begin{pmatrix} H_1(\tau,u,v) \\ H_2(\tau,u,v) \end{pmatrix} = \begin{pmatrix} u + p(x)A_1(u) - \alpha A_1(u) - \tau S_1(v) - (1-\tau)\varepsilon A_1(u) \\ v + q(x)A_2(u) - \beta A_2(v) - \tau S_2(u) - (1-\tau)\varepsilon A_2(v) \end{pmatrix}, \forall \varepsilon > 0$$

then

$$H(\tau,u,v) = \begin{pmatrix} u \\ v \end{pmatrix} - BA(u,v) - \tau\, S(u,v) - (1-\tau)\Lambda A(u,v) = 0\ , \forall \varepsilon > 0.$$

where

$$B = \begin{pmatrix} \alpha - p(x) & 0 \\ 0 & \beta - q(x) \end{pmatrix} \qquad \text{and} \qquad \Lambda = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}$$

here also

$$H : [0,1] \times Z \to Y$$

is a compact homotopy. We follow the same arguments to the proof of precedent lemma. Assume that no such $R_1 > 0$ exists, i.e. we can find a sequence $\{(u_n, v_n)\}_{n=1}^{n=\infty} \in Z$ and $\{\tau_n\}_{n=1}^{n=\infty} \subset [0,1]$ such that $\|(u_n, v_n)\|_Z > n$ and

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} - BA(u_n,v_n) - \tau_n\, S(u_n,v_n) - (1-\tau_n)\Lambda A(u_n,v_n) = 0\ , \forall \varepsilon > 0 \tag{5.5}$$

Setting

$$w_n = (w_{1,n}, w_{2,n}) = \left(\frac{u_n}{\|(u_n,v_n)\|_Z}, \frac{v_n}{\|(u_n,v_n)\|_Z}\right)$$

then it follows with choice of w_n that

$$w_n \in D(-\triangle + p) \times D(-\triangle + q) \text{ and } \|w_n\|_Z = 1. \tag{5.6}$$

Finally we arrive at

$$w - [B + (1-\tau)\Lambda]A(w) = 0\ , \|w\|_Z = 1, \forall \varepsilon > 0.$$

This is a contradiction if $\tau \neq 1$. Since

$$\begin{cases} -\triangle w_1 + p(x)w_1 = [\alpha + (1-\tau)]w_1 \\ -\triangle w_2 + q(x)w_2 = [\beta + (1-\tau)]w_2. \end{cases}$$

Let us assume $\tau = 1$ i.e $\tau_n \to 1$. Now, however, we have no contradiction since α, β are not eigenvalues and

$$w - BA(w) = 0,$$

has a solution with $\|w\|_Z = 1$. We have to revise the last step when passing to the limit in

$$w_n - BA(w_n) - (1-\tau_n)\Lambda A(w_n) - \tau_n \frac{S(u_n,v_n)}{\|(u_n,v_n)\|_Z} = 0 \ , \forall\varepsilon > 0,$$

and employ special properties of S. Namely,

$$\begin{pmatrix} u_{n_k} \\ v_{n_k} \end{pmatrix} - BA(u_{n_k}, v_{n_k}) - \tau_{n_k} S(u_{n_k}, v_{n_k}) - (1-\tau_{n_k})\Lambda A(u_{n_k}, v_{n_k}) = 0 \ , \forall\varepsilon > 0$$

is equivalent to the integral identity

$$\begin{cases} \int\limits_\Omega \nabla u_{n_k} \nabla \overline{w}_1 dx + \int\limits_\Omega p(x) u_{n_k} \overline{w}_1 dx = [\alpha + (1-\tau_{n_k})\varepsilon] \int\limits_\Omega u_{n_k} \overline{w}_1 dx + \tau_{n_k} \int\limits_\Omega \phi(v_{n_k}) \overline{w}_1 dx - \tau_{n_k} \int\limits_\Omega h_1(x) \overline{w}_1 dx \\ \int\limits_\Omega \nabla v_{n_k} \nabla \overline{w}_2 dx + \int\limits_\Omega q(x) v_{n_k} \overline{w}_2 dx = [\beta + (1-\tau_{n_k})\varepsilon] \int\limits_\Omega v_{n_k} \overline{w}_2 dx + \tau_{n_k} \int\limits_\Omega \psi(u_{n_k}) \overline{w}_2 dx - \tau_{n_k} \int\limits_\Omega h_2(x) \overline{w}_2 dx \\ (\overline{w}_1, \overline{w}_2) \in Z. \end{cases} \tag{5.7}$$

Taking $(\overline{w}_1, \overline{w}_2) = (\varphi_1, \varphi_2)$ and using the fact that

$$\begin{cases} \int\limits_\Omega \nabla u_{n_k} \nabla \varphi_1(x) dx + \int\limits_\Omega p(x) u_{n_k} \varphi_1(x) dx = \alpha \int\limits_\Omega u_{n_k} \varphi_1(x) dx \\ \int\limits_\Omega \nabla v_{n_k} \nabla \varphi_2(x) dx + \int\limits_\Omega q(x) v_{n_k} \varphi_2(x) dx = \beta \int\limits_\Omega v_{n_k} \varphi_2(x) dx. \end{cases} \tag{5.8}$$

With (5.8), the expression (5.7) became

$$\begin{cases} (1-\tau_{n_k})\varepsilon \int\limits_\Omega u_{n_k} \varphi_1(x) dx + \tau_{n_k} \int\limits_\Omega \phi(v_{n_k}) \varphi_1(x) dx = \tau_{n_k} \int\limits_\Omega h_1(x) \varphi_1(x) dx \\ (1-\tau_{n_k})\varepsilon \int\limits_\Omega v_{n_k} \varphi_2(x) dx + \tau_{n_k} \int\limits_\Omega \psi(u_{n_k}) \varphi_2(x) dx = \tau_{n_k} \int\limits_\Omega h_2(x) \varphi_2(x) dx. \end{cases} \tag{5.9}$$

Similarly to the first step we have $w_{n_k} \to w$ in $L^2(\Omega)$ and we can write w_1, w_2 as follows

$$\begin{cases} w_1 = k\varphi_1 \\ w_2 = k'\varphi_2. \end{cases} , \qquad k, k' \neq 0.$$

Assume that $k, k' > 0$, then

$$\begin{cases} v_{n_k} \underset{k\to\infty}{\to} \infty & \text{a.e in } \Omega \\ u_{n_{k'}} \underset{k'\to\infty}{\to} \infty & \text{a.e in } \Omega. \end{cases}$$

With the passage to the limit in (5.9), the using of $\tau_{n_k} \to 1$ and thanks to the Lebesgue's convergence theorem we get

$$\begin{cases} \int\limits_\Omega h_1(x) \varphi_1(x) dx \geq \lim\limits_{k\to\infty} \int\limits_\Omega \phi(v_{n_k}) \varphi_1(x) dx = \int\limits_\Omega \phi(+\infty) \varphi_1(x) dx = \phi(+\infty) \\ \int\limits_\Omega h_2(x) \varphi_2(x) dx \geq \lim\limits_{k'\to\infty} \int\limits_\Omega \psi(u_{n_{k'}}) \varphi_2(x) dx = \int\limits_\Omega \psi(+\infty) \varphi_2(x) dx = \psi(+\infty). \end{cases}$$

A Contradiction. If $k, k' < 0$ we get the contradiction with the first inequality. □

Proof . Choosing R_1 as in lemma 2. Let

$$B(0, R_1) = \{(u, v) \in Z, \|(u, v)\|_Z < R_1\}$$

By invariance of the topological degree we have

$$\deg(H(\tau,.,.), B(0,R_1), 0), \qquad t \in [0,1],$$

is constant. Then

$$\deg(H(0,.,.), B(0,R_1), 0) = \deg(H(1,.,.), B(0,R_1), 0) = \pm 1$$

and theorem 2 was validated. □

References

[1] T. Gallouet and O. Kavian, *Résultats d'existence et de non-existence pour certains problèmes demi-linéaires a l'infini*, Ann. Fac. Sci. Toulouse **3** (1981), no. 3-4, 201–246.

[2] S. Heidari, A. Razani, *Infinitely many solutions for (p(x),q(x))-Laplacian-like systems*, Commun. Korean Math. Soc. **36** (2021), no. 1, 51–62

[3] A. Khaleghi and A. Razani, *Solutions to a (p(x),q(x))-biharmonic elliptic problem on a bounded domain*, Bound. Value Prob. **2023** (2023), Article number: 53.

[4] M.A. Ragusa, A. Razani, and F. Safari, *Existence of radial solutions for a p(x)-Laplacian Dirichlet problem*, Adv. Differ. Equ. **2021** (2021), Article number: 215.

[5] A. Razani and G.M. Figueiredo, *Weak Solution by the Sub-Super solution method for a nonlocal system involving Lebrsgue generalized spaces*, Electronic J. Differ. Equ. **2022** (2022), no. 36, 1–18.