Int. J. Nonlinear Anal. Appl. 7 (2016) No. 2, 351-362
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2017.1241.1296

Existence of three solutions for a class of fractional boundary value systems

Samad Mohseni Kolagar[a,*], Ghasem A. Afrouzi[a], Armin Hadjian[b]

[a] Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran
[b] Department of Mathematics, Faculty of Basic Sciences, University of Bojnord, P.O. Box 1339, Bojnord 94531, Iran

(Communicated by M.B. Ghaemi)

Abstract

In this paper, under appropriate oscillating behaviours of the nonlinear term, we prove some multiplicity results for a class of nonlinear fractional equations. These problems have a variational structure and we find three solutions for them by exploiting an abstract result for smooth functionals defined on a reflexive Banach space. We also give an example to illustrate the obtained result.

Keywords: fractional differential equations; Riemann-Liouville fractional derivatives; variational methods; three solutions.
2010 MSC: Primary 34A08; Secondary 35A15.

1. Introduction

Fractional differential equations have been proved to be valuable tools in the modeling of many phenomena in various fields of physics, chemistry, biology, engineering and economics. There has been significant development in fractional differential equations; one can see the monographs of Miller and Ross [20], Samko *et al.* [24], Podlubny [21], Hilfer [13], Kilbas *et al.* [16] and the papers [2, 3, 4, 5, 6, 7, 17, 18, 25, 26, 28] and references therein.

Critical point theory has been very useful in determining the existence of solutions for integer order differential equations with some boundary conditions; see for instance, in the vast literature on the subject, the classical books [19, 22, 27] and references therein. But until now, there are a few results for fractional boundary value problems (briefly BVP) which were established exploiting this

*Corresponding author
Email addresses: mohseni.samad@gmail.com (Samad Mohseni Kolagar), afrouzi@umz.ac.ir (Ghasem A. Afrouzi), a.hadjian@ub.ac.ir, hadjian83@gmail.com (Armin Hadjian)

Received: March 2016 *Revised:* November 2016

approach, since it is often very difficult to establish a suitable space and variational functional for fractional problems.

In this paper, we are interested in ensuring the existence of at least three solutions for the following system

$$\begin{cases} {}_tD_T^{\alpha_i}\Big(a_i(t){}_0D_t^{\alpha_i}u_i(t)\Big) & = \lambda F_{u_i}(t,u_1(t),\dots,u_n(t)) + \mu G_{u_i}(t,u_1(t),\dots,u_n(t)) \\ & \quad +h_i(u_i(t)) \qquad \text{a.e. } t\in[0,T], \\ u_i(0)=u_i(T)=0, & \end{cases} \tag{1.1}$$

for $1\le i\le n$, where $\alpha_i\in(0,1]$, ${}_0D_t^{\alpha_i}$ and ${}_tD_T^{\alpha_i}$ are the left and right Riemann-Liouville fractional derivatives of order α_i respectively, $a_i\in L^\infty([0,T])$ with $a_{i0}:=\operatorname{ess\,inf}_{[0,T]}a_i>0$ for $1\le i\le n$, λ and μ are positive parameters, $F,G:[0,T]\times\mathbb{R}^n\to\mathbb{R}$ are measurable functions with respect to $t\in[0,T]$ for every $(x_1,\dots,x_n)\in\mathbb{R}^n$ and are C^1 with respect to $(x_1,\dots,x_n)\in\mathbb{R}^n$ for a.e. $t\in[0,T]$, F_{u_i} and G_{u_i} denotes the partial derivative of F and G with respect to u_i, respectively, and $h_i:\mathbb{R}\to\mathbb{R}$ are Lipschitz continuous functions with the Lipschitz constants $L_i>0$ for $1\le i\le n$, i.e.,

$$|h_i(x_1)-h_i(x_2)|\le L_i|x_1-x_2|, \tag{1.2}$$

for every $x_1,x_2\in\mathbb{R}$, and $h_i(0)=0$ for $1\le i\le n$. In this paper, we need the following conditions:

(H) $\alpha_i\in(\frac{1}{2},1]$ for $1\le i\le n$.

(F$_1$) for every $M>0$ and every $1\le i\le n$,

$$\sup_{|(x_1,\dots,x_n)|\le M}|F_{u_i}(t,x_1,\dots,x_n)|\in L^1([0,T]).$$

(F$_2$) $F(t,0,\dots,0)=0$ for a.e. $t\in[0,T]$.

(G) for every $M>0$ and every $1\le i\le n$,

$$\sup_{|(x_1,\dots,x_n)|\le M}|G_{u_i}(t,x_1,\dots,x_n)|\in L^1([0,T]).$$

In the present paper, motivated by [29] and [30], using a three critical points theorem obtained in [23] which we recall in the next section (Theorem 2.6), we ensure the existence of at least three solutions for system (1.1). This theorem has been successfully employed to establish the existence of at least three solutions for perturbed boundary value problems in the papers [1, 8, 9, 10, 11, 12].

This paper is organized as follows. In Section 2, we present some necessary preliminary facts that will be needed in the paper. In Section 3, our main result (Theorem 3.1) and some significative consequences (Corollaries 3.3 and 3.4) and an example (Example 3.2) are presented.

2. Preliminaries

In this section, we first introduce some necessary definitions and properties of the fractional calculus which are used in this paper.

Definition 2.1. (Kilbas *et al.* [16]) Let u be a function defined on $[a,b]$. The left and right Riemann-Liouville fractional derivatives of order $\alpha>0$ for a function u are defined by

$${}_aD_t^\alpha u(t) := \frac{d^n}{dt^n}{}_aD_t^{\alpha-n}u(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-s)^{n-\alpha-1}u(s)ds,$$

and

$$
{}_tD_b^{\alpha}u(t) := (-1)^n \frac{d^n}{dt^n} {}_tD_b^{\alpha-n}u(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^b (t-s)^{n-\alpha-1} u(s)ds,
$$

for every $t \in [a, b]$, provided the right-hand sides are pointwise defined on $[a, b]$, where $n - 1 \leq \alpha < n$ and $n \in \mathbb{N}$.

Here, $\Gamma(\alpha)$ is the standard gamma function given by

$$
\Gamma(\alpha) := \int_0^{+\infty} z^{\alpha-1} e^{-z} dz.
$$

Set $AC^n([a, b], \mathbb{R})$ the space of functions $u : [a, b] \to \mathbb{R}$ such that $u \in C^{n-1}([a, b], \mathbb{R})$ and $u^{(n-1)} \in AC([a, b], \mathbb{R})$. Here, as usual, $C^{n-1}([a, b], \mathbb{R})$ denotes the set of mappings having $(n - 1)$ times continuously differentiable on $[a, b]$. In particular, we denote $AC([a, b], \mathbb{R}) := AC^1([a, b], \mathbb{R})$.

Proposition 2.2. (Samko *et al.* [24]) The following property of fractional integration hold

$$
\int_a^b [{}_aD_t^{-\alpha}u(t)]v(t)dt = \int_a^b [{}_tD_b^{-\alpha}v(t)]u(t)dt, \quad \alpha > 0,
$$

provided that $u \in L^p([a, b], \mathbb{R})$, $v \in L^q([a, b], \mathbb{R})$ and $p \geq 1$, $q \geq 1$, $1/p + 1/q \leq 1 + \alpha$ or $p \neq 1$, $q \neq 1$, $1/p + 1/q = 1 + \alpha$.

Proposition 2.3. (Jiao and Zhou [15]) If $u(a) = u(b) = 0$, $u \in L^\infty([a, b], \mathbb{R}^N)$, $v \in L^1([a, b], \mathbb{R}^N)$, or $v(a) = v(b) = 0$, $v \in L^\infty([a, b], \mathbb{R}^N)$, $u \in L^1([a, b], \mathbb{R}^N)$, then

$$
\int_a^b [{}_aD_t^{\alpha}u(t)]v(t)dt = \int_a^b [{}_tD_b^{\alpha}v(t)]u(t)dt, \quad 0 < \alpha \leq 1.
$$

To establish a variational structure for the main problem, it is necessary to construct appropriate function spaces. Following [14], we denote by $C_0^\infty([0, T], \mathbb{R})$ the set of all functions $g \in C^\infty([0, T], \mathbb{R})$ with $g(0) = g(T) = 0$.

Definition 2.4. (Jiao and Zhou [14]) Let $0 < \alpha_i \leq 1$ for $1 \leq i \leq n$. The fractional derivative space $E_0^{\alpha_i}$ is defined by the closure of $C_0^\infty([0, T], \mathbb{R})$ with respect to the weighted norm

$$
\|u_i\| := \left(\int_0^T a_i(t) |{}_0D_t^{\alpha_i} u_i(t)|^2 dt + \int_0^T |u_i(t)|^2 dt \right)^{1/2}, \quad \forall u_i \in E_0^{\alpha_i}. \tag{2.1}
$$

Clearly, the fractional derivative space $E_0^{\alpha_i}$ is the space of functions $u_i \in L^2([0, T], \mathbb{R})$ having an α_i-order Caputo fractional derivative ${}_0D_t^{\alpha_i} u_i \in L^2([0, T], \mathbb{R})$ and $u_i(0) = u_i(T) = 0$ for $1 \leq i \leq n$. From [14, Proposition 3.1], we know for $0 < \alpha_i \leq 1$, the space $E_0^{\alpha_i}$ is a reflexive and separable Banach space.

For every $u_i \in E_0^{\alpha_i}$, set

$$
\|u_i\|_{L^s} := \left(\int_0^T |u_i(t)|^s dt \right)^{1/s}, \qquad s \geq 1,
$$

and

$$
\|u_i\|_\infty := \max_{t \in [0,T]} |u_i(t)|. \tag{2.2}
$$

Lemma 2.5. (Zhao *et al.* [29]) Let $\alpha_i \in (1/2, 1]$ for $1 \leq i \leq n$. For all $u_i \in E_0^{\alpha_i}$, we have

$$\|u_i\|_{L^2} \leq \frac{T^{\alpha_i}}{\Gamma(\alpha_i+1)\sqrt{a}_{i0}}\Big(\int_0^T a_i(t)|{}_0D_t^{\alpha_i}u_i(t)|^2dt\Big)^{1/2}, \tag{2.3}$$

$$\|u_i\|_\infty \leq \frac{T^{\alpha_i-1/2}}{\Gamma(\alpha_i)\sqrt{a_{i0}(2\alpha_i-1)}}\Big(\int_0^T a_i(t)|{}_0D_t^{\alpha_i}u_i(t)|^2dt\Big)^{1/2}. \tag{2.4}$$

Hence, we can consider $E_0^{\alpha_i}$ with respect to the norm

$$\|u_i\|_{\alpha_i} := \left(\int_0^T a_i(t)|{}_0D_t^{\alpha_i}u_i(t)|^2dt\right)^{1/2}, \quad \forall\, u_i \in E_0^{\alpha_i}, \tag{2.5}$$

for $1 \leq i \leq n$, which is equivalent to (2.1).

Throughout this paper, we let X be the Cartesian product of the n spaces $E_0^{\alpha_i}$ for $1 \leq i \leq n$, i.e., $X = E_0^{\alpha_1} \times E_0^{\alpha_2} \times \cdots \times E_0^{\alpha_n}$ equipped with the norm

$$\|u\| := \sum_{i=1}^n \|u_i\|_{\alpha_i}, \quad u = (u_1, u_2, \ldots, u_n),$$

where $\|u_i\|_{\alpha_i}$ is defined in (2.5). Obviously, X is compactly embedded in $(C([0,T],\mathbb{R}))^n$.

We mean by a *(weak) solution* of system (1.1), any $u = (u_1, u_2, \ldots, u_n) \in X$ such that

$$\begin{aligned}&\int_0^T \sum_{i=1}^n a_i(t){}_0D_t^{\alpha_i}u_i(t){}_0D_t^{\alpha_i}v_i(t) - \lambda\int_0^T\sum_{i=1}^n F_{u_i}(t,u_1(t),\ldots,u_n(t))v_i(t)dt\\&-\mu\int_0^T\sum_{i=1}^n G_{u_i}(t,u_1(t),\ldots,u_n(t))v_i(t)dt - \int_0^T\sum_{i=1}^n h_i(u_i(t))v_i(t)dt = 0,\end{aligned}$$

for all $v = (v_1, v_2, \ldots, v_n) \in X$.

Here, we recall the following result of [23, Theorem 1], with easy manipulations that we are going to use in the sequel.

Theorem 2.6. (Ricceri [23]) Let X be a reflexive real Banach space; $\Phi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, bounded on bounded subsets of X; $\Psi : X \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that

$$\Phi(0) = \Psi(0) = 0.$$

Assume that there exists $r > 0$ and $\overline{x} \in X$, with $r < \Phi(\overline{x})$, such that

(a$_1$) $\frac{\sup_{\Phi(x)\leq r}\Psi(x)}{r} < \frac{\Psi(\overline{x})}{\Phi(\overline{x})}$;

(a$_2$) for each $\lambda \in \Lambda_r := \left]\frac{\Phi(\overline{x})}{\Psi(\overline{x})}, \frac{r}{\sup_{\Phi(x)\leq r}\Psi(x)}\right[$, the functional $\Phi - \lambda\Psi$ is coercive.

Then, for each compact interval $[a,b] \subseteq \Lambda_r$, there exists $\rho > 0$ with the following property: for every $\lambda \in [a,b]$ and every C^1 functional $\digamma : X \to \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0,\delta]$, the equation

$$\Phi'(x) - \lambda\Psi'(x) - \mu\digamma'(x) = 0$$

has at least three solutions in X whose norms are less than ρ.

3. The main results

In the present section we discuss the existence of multiple solutions for system (1.1). For any $\gamma > 0$, we denote by $K(\gamma)$ the set

$$\left\{(x_1, \ldots, x_n) \in \mathbb{R}^n : \ \frac{1}{2}\sum_{i=1}^{n} |x_i|^2 \leq \gamma\right\}.$$

This set will be used in some of our hypotheses with appropriate choices of γ. For $u = (u_1, \ldots, u_n) \in X$, we define

$$\Upsilon(u) := \sum_{i=1}^{n} \Upsilon_i(u_i),$$

where

$$\Upsilon_i(x) := \int_0^T H_i(x(s))ds \quad \text{and} \quad H_i(x) := \int_0^x h_i(z)dz, \quad 1 \leq i \leq n,$$

for every $t \in [0, T]$ and $x \in \mathbb{R}$.

Moreover, let

$$c := \max_{1\leq i\leq n}\left\{\frac{T^{2\alpha_i - 1}}{(\Gamma(\alpha_i))^2 a_{i0}(2\alpha_i - 1)}\right\},$$

$$k := \min_{1\leq i\leq n}\left\{1 - \frac{L_i T^{2\alpha_i}}{(\Gamma(\alpha_i + 1))^2 a_{i0}}\right\},$$

$$\tau := \max_{1\leq i\leq n}\left\{1 + \frac{L_i T^{2\alpha_i}}{(\Gamma(\alpha_i + 1))^2 a_{i0}}\right\}.$$

Theorem 3.1. *Suppose that $k > 0$ and the conditions* (F_1), (F_2), (G) *and* (H) *are satisfied. Furthermore, assume that there exist a positive constant r and a function $w = (w_1, \ldots, w_n) \in X$ such that*

(i) $\sum_{i=1}^{n} \frac{\|w_i\|_{\alpha_i}^2}{2} > \frac{r}{k}$;

(ii) $2r\frac{\int_0^T F(t, w_1, \ldots, w_n)dt}{\sum_{i=1}^{n} \|w_i\|_{\alpha_i}^2 - 2\Upsilon(w_1, \ldots, w_n)} - \int_0^T \max_{(x_1, \ldots, x_n)\in K(\frac{cr}{k})} F(t, x_1, \ldots, x_n)dt > 0$;

(iii) $\limsup_{(|x_1|, \ldots, |x_n|)\to(+\infty, \ldots, +\infty)} \frac{\sup_{t\in[0,T]} F(t, x_1, \ldots, x_n)}{\sum_{i=1}^{n} \frac{|x_i|^2}{2}} \leq 0.$

Then, setting

$$\Lambda := \left]\frac{\sum_{i=1}^{n} \frac{\|w_i\|_{\alpha_i}^2}{2} - \Upsilon(w_1, \ldots, w_n)}{\int_0^T F(t, w_1(t), \ldots, w_n(t))dt}, \frac{r}{\int_0^T \max_{(x_1, \ldots, x_n)\in K(\frac{cr}{k})} F(t, x_1, \ldots, x_n)dt}\right[,$$

for each compact interval $[a, b] \subseteq \Lambda$, there exists $\rho > 0$ with the following property: for every $\lambda \in [a, b]$, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, system (1.1) *admits at least three solutions in X whose norms are less than ρ.*

Proof . For each $u = (u_1, \ldots, u_n) \in X$, define $\Phi, \Psi : X \to \mathbb{R}$ as

$$\Phi(u) := \sum_{i=1}^{n} \frac{\|u_i\|_{\alpha_i}^2}{2} - \Upsilon(u),$$

and

$$\Psi(u) := \int_0^T F(t, u_1(t), \ldots, u_n(t))dt.$$

Clearly, Φ and Ψ are continuously Gâteaux differentiable functionals whose Gâteaux derivatives at the point $u \in X$ are given by

$$\Phi'(u)(v) = \int_0^T \sum_{i=1}^{n} a_i(t)_0D_t^{\alpha_i}u_i(t)_0D_t^{\alpha_i}v_i(t) - \int_0^T \sum_{i=1}^{n} h_i(u_i(t))v_i(t)dt,$$

$$\Psi'(u)(v) = \int_0^T \sum_{i=1}^{n} (F_{u_i}(x, u_1(x), \ldots, u_n(x))v_i(x)dx,$$

for every $v = (v_1, \ldots, v_n) \in X$. Hence, $\Phi - \lambda\Psi \in C^1(X, \mathbb{R})$. Moreover, $\Psi' : X \to X^*$ is a compact operator (see the proof of [29, Theorem 3.1]). Furthermore, similar to the proof of [30, Theorem 3.1], we can show that Φ is sequentially weakly lower semicontinuous. As concerns functional Φ, it is easy to show that Φ is bounded on each bounded subset of X and its derivative admits a continuous inverse on X^*. Moreover, we have $\Phi(0) = \Psi(0) = 0$.

We show that required hypothesis $\Phi(\overline{x}) > r$ follows from (i) and the definition of Φ by choosing $\overline{x} = w$. Indeed, since (1.2) holds for every $x_1, x_2 \in \mathbb{R}$ and $h_1(0) = \cdots = h_n(0) = 0$, one has $|h_i(x)| \leq L_i|x|$, $1 \leq i \leq n$, for all $x \in \mathbb{R}$. It follows from (2.3) that

$$\begin{aligned} \Phi(w) &\geq \frac{\sum_{i=1}^{n} \|w_i\|_{\alpha_i}^2}{2} - \Big| \int_0^T \sum_{i=1}^{n} H_i(w_i(t))dt \Big| \\ &\geq \frac{\sum_{i=1}^{n} \|w_i\|_{\alpha_i}^2}{2} - \sum_{i=1}^{n} \frac{L_i}{2} \int_0^T |w_i(t)|^2 dt \\ &\geq \sum_{i=1}^{n} \Big(\frac{1}{2} - \frac{L_i T^{2\alpha_i}}{2(\Gamma(\alpha_i+1))^2 a_{i0}} \Big) \|w_i\|_{\alpha_i}^2 \\ &\geq \frac{k}{2} \sum_{i=1}^{n} \|w_i\|_{\alpha_i}^2 > r. \end{aligned} \tag{3.1}$$

From (2.2) and (2.4), for every $u_i \in E_0^{\alpha_i}$ we have

$$\max_{t\in[0,T]} |u_i(t)|^2 \leq c\|u_i\|_{\alpha_i}^2,$$

for $1 \leq i \leq n$. Hence

$$\max_{t\in[0,T]} \sum_{i=1}^{n} |u_i(t)|^2 \leq c \sum_{i=1}^{n} \|u_i\|_{\alpha_i}^2, \tag{3.2}$$

for each $u = (u_1, \ldots, u_n) \in X$. From (2.4), (3.1) and (3.2), for each $r > 0$ we obtain

$$\begin{aligned}
\Phi^{-1}((-\infty, r]) &= \left\{ u = (u_1, \ldots, u_n) \in X : \Phi(u) \le r \right\} \\
&\subseteq \left\{ u = (u_1, \ldots, u_n) \in X : \sum_{i=1}^{n} \frac{\|u_i\|_{\alpha_i}^2}{2} \le \frac{r}{k} \right\} \\
&\subseteq \left\{ u = (u_1, \ldots, u_n) \in X : \sum_{i=1}^{n} \frac{(\Gamma(\alpha_i))^2 a_{i0}(2\alpha - 1)}{2T^{2\alpha_i - 1}} \|u_i\|_{\infty}^2 \le \frac{r}{k} \right\} \\
&\subseteq \left\{ u = (u_1, \ldots, u_n) \in X : \frac{1}{2} \sum_{i=1}^{n} |u_i(t)|^2 \le \frac{cr}{k}, \quad \text{for all } \ t \in [0, T] \right\}.
\end{aligned}$$

Then,

$$\begin{aligned}
\sup_{u \in \Phi^{-1}((-\infty, r])} \Psi(u) &= \sup_{u \in \Phi^{-1}((-\infty, r])} \int_0^T F(t, u_1, \ldots, u_n) dt \\
&\le \int_0^T \max_{(x_1, \ldots, x_n) \in K(\frac{cr}{k})} F(t, x_1, \ldots, x_n) dt.
\end{aligned}$$

Therefore, from the condition (ii), we have

$$\begin{aligned}
\sup_{u \in \Phi^{-1}((-\infty, r])} \Psi(u) &\le \int_0^T \max_{(x_1, \ldots, x_n) \in K(\frac{cr}{k})} F(t, x_1, \ldots, x_n) dt \\
&< 2r \frac{\int_0^T F(t, w_1, \ldots, w_n) dt}{\sum_{i=1}^{n} \|w_i\|_{\alpha_i}^2 - 2\Upsilon(w_1, \ldots, w_n)} \\
&= r \frac{\int_0^T F(t, w_1, \ldots, w_n) dt}{\sum_{i=1}^{n} \frac{\|w_i\|_{\alpha_i}^2}{2} - \Upsilon(w_1, \ldots, w_n)} \\
&= r \frac{\Psi(w)}{\Phi(w)},
\end{aligned}$$

from which assumption (a_1) of Theorem 2.6 follows. Fix $0 < \epsilon < \frac{1}{2Tc\lambda}$. From (iii) there is a constant τ_ϵ such that

$$F(t, x_1, \ldots, x_n) \le \epsilon \sum_{i=1}^{n} |x_i|^2 + \tau_\epsilon, \tag{3.3}$$

for every $t \in [0, T]$ and for every $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Taking (2.4) into account, from (3.3), it follows that, for each $u \in X$,

$$\begin{aligned}
\Phi(u) - \lambda\Psi(u) &= \frac{1}{2} \sum_{i=1}^{n} \|u_i\|_{\alpha_i}^2 - \lambda \int_0^T F(t, u_1, \ldots, u_n) dt \\
&\ge \frac{1}{2} \sum_{i=1}^{n} \|u_i\|_{\alpha_i}^2 - T\lambda c\epsilon \sum_{i=1}^{n} \|u_i\|_{\alpha_i}^2 - \lambda\tau_\epsilon \\
&\ge (\frac{1}{2} - T\lambda c\epsilon) \sum_{i=1}^{n} \|u_i\|_{\alpha_i}^2 - \lambda\tau_\epsilon,
\end{aligned}$$

and thus

$$\lim_{\|u\|\to+\infty}(\Phi(u)-\lambda\Psi(u))=+\infty,$$

which means the functional $\Phi-\lambda\Psi$ is coercive for every parameter λ, in particular, for every $\lambda\in\Lambda\subseteq\left]\frac{\Phi(w)}{\Psi(w)},\frac{r}{\sup_{\Phi(u)\le r}\Psi(u)}\right[$. Then, also condition (a$_2$) holds.

In addition, since $G:[0,T]\times\mathbb{R}^n\to\mathbb{R}$ is a measurable function with respect to $t\in[0,T]$ for every $(x_1,\dots,x_n)\in\mathbb{R}^n$ belong to C^1 with respect to $(x_1,\dots,x_n)\in\mathbb{R}^n$ for a.e. $t\in[0,T]$, satisfying condition (G), the functional

$$\digamma(u)=\int_0^T G(t,u_1(t),\dots,u_n(t))dt$$

is well defined and continuously Gâteaux differentiable on X with a compact derivative, and

$$\digamma'(u)(v)=\int_0^T\sum_{i=1}^n G_{u_i}(t,u_1(t),\dots,u_n(t))v_i(t)dt$$

for all $u=(u_1,\dots,u_n),v=(v_1,\dots,v_n)\in X$. Thus, all the hypotheses of Theorem 2.6 are satisfied. Also note that the solutions of the equation $\Phi'(u)-\lambda\Psi'(u)-\mu\digamma'(u)=0$ are exactly the solutions of (1.1) (see [29]). So, the conclusion follows from Theorem 2.6. $\square$

Example 3.2. Consider the system

$$\begin{cases} {}_tD_1^{0.7}\Big((1+t)\,{}_0D_t^{0.7}u_1(t)\Big)=\lambda F_{u_1}(t,u_1(t),u_2(t))+\mu G_{u_1}(t,u_1(t),u_2(t))\\ \qquad\qquad\qquad\qquad\qquad +h_1(u_1(t))\quad \text{a.e. } t\in[0,1],\\ {}_tD_1^{0.75}\Big((2+t^3)\,{}_0D_t^{0.75}u_2(t)\Big)=\lambda F_{u_2}(t,u_1(t),u_2(t))+\mu G_{u_2}(t,u_1(t),u_2(t))\\ \qquad\qquad\qquad\qquad\qquad +h_2(u_2(t))\quad \text{a.e. } t\in[0,1],\\ u_1(0)=u_2(0)=u_1(1)=u_2(1)=0, \end{cases}\tag{3.4}$$

where $h_1(u_1)=\frac{1}{4}\sin u_1$, $h_2(u_2)=\frac{u_2}{2}$ and $G:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ is an arbitrary function which is measurable with to respect to $t\in[0,1]$ for every $(x_1,x_2)\in\mathbb{R}^2$ and is C^1 with respect to $(x_1,x_2)\in\mathbb{R}^2$ for a.e. $t\in[0,1]$, satisfying

$$\sup_{|(x_1,x_2)|\le M}|G_{u_i}(t,x_1,x_2)|\in L^1([0,1])$$

for every $M>0$ and $i=1,2$. Moreover, for all $(t,x_1,x_2)\in[0,1]\times\mathbb{R}^2$, put

$$F(t,x_1,x_2)=t^3\begin{cases}-(x_1^2+x_2^2)^2+2(x_1^2+x_2^2) & x_1^2+x_2^2\le 1,\\ 1 & x_1^2+x_2^2>1.\end{cases}$$

Obviously, $h_1,h_2:\mathbb{R}\to\mathbb{R}$ are two Lipschitz continuous functions with Lipschitz constants $L_1=\frac{1}{4}$, $L_2=\frac{1}{2}$ and $h_1(0)=h_2(0)=0$. Clearly, $F(t,0,0)=0$ for all $t\in[0,1]$. With the aid of direct computation we have

$$a_{10}=1,\qquad a_{20}=2,\qquad c\approx 1.4837,\qquad k\approx 0.9495.$$

By choosing, for instance, $w_1(t)=\frac{1}{2}(t^5-t^{10})$, $w_2(t)=3(t^5-t^{10})$, $r=\frac{1}{10^4}$, and by a simple calculation, we obtain

$$\|w_1\|_{0.7}^2\approx 0.003,\qquad \|w_2\|_{0.75}^2\approx 0.1743.$$

The conditions (i) and (ii) of Theorem 3.1 are satisfied. In fact,

$$\|w_1\|_{0.7}^2 + \|w_2\|_{0.75}^2 \approx 0.003 + 0.1743 \approx 0.1773 > \frac{2r}{k} \approx 0.0002,$$

and

$$\begin{aligned}\frac{\int_0^1 \max_{(x_1,x_2)\in K(\frac{cr}{k})} F(t,x_1,x_2)dt}{r} = \frac{c}{k} &\approx 1.5624\\ &< 2\frac{\int_0^1 F(t,w_1,w_2)dt}{\sum_{i=1}^2 \|w_i\|_{\alpha_i}^2 - 2\Upsilon(w_1,w_2)}\\ &\approx 2.1975.\end{aligned}$$

Note that

$$\limsup_{(|x_1|,|x_2|)\to(+\infty,+\infty)} \frac{\sup_{t\in[0,1]} F(t,x_1,x_2)}{\sum_{i=1}^2 \frac{|x_i|^2}{2}} = 0.$$

Hence, Theorem 3.1 is applicable to system (3.4) for

$$\Lambda \subseteq [0.455062, 0.64004].$$

Next, we want to give a verifiable consequence of Theorem 3.1 for a fixed text function w. For a given constant $h \in (0, \frac{1}{2})$ and for all $1 \le i \le n$, set

$$\begin{aligned}P_i(\alpha_i,h) \;:=\; & \frac{1}{2h^2T^2}\Bigg\{\int_0^T a_i(t)t^{2(1-\alpha_i)}dt + \int_{hT}^T a_i(t)(t-hT)^{2(1-\alpha_i)}dt\\ &+\int_{(1-h)T}^T a_i(t)(t-(1-h)T)^{2(1-\alpha_i)}dt - 2\int_{(1-h)T}^T a_i(t)(t^2-(1-h)Tt)^{1-\alpha_i}dt\\ &-2\int_{hT}^T a_i(t)(t^2-hTt)^{1-\alpha_i}dt + 2\int_{(1-h)T}^T a_i(t)(t^2-hTt+h(1-h)T^2)^{1-\alpha_i}dt\Bigg\},\end{aligned}$$

$$\Delta := \min\Bigg\{P_i(\alpha_i,h):\ \text{for } 1\le i\le n\Bigg\},$$

$$\Delta' := \max\Bigg\{P_i(\alpha_i,h):\ \text{for } 1\le i\le n\Bigg\}.$$

Corollary 3.3. *Let assumption (iii) in Theorem* 3.1 *holds. Assume that there exist positive constants d and η such that $\frac{d}{\Delta ckn} < \eta^2$, and also*

(j) $F(t,x_1,\dots,x_n) \ge 0$, *for each* $(t,x_1,\dots,x_n) \in [0,T]\times[0,+\infty)\times\cdots\times[0,+\infty)$;

(jj) $\frac{\int_0^T \max_{(x_1,\dots,x_n)\in K(d)} F(t,x_1,\dots,x_n)dt}{dk} < \frac{\int_{hT}^{(1-h)T} F(t,\Gamma(2-\alpha_1)\eta,\dots,\Gamma(2-\alpha_n)\eta)dt}{nc\tau\Delta'\eta^2}$.

Then, setting

$$\Lambda_1 := \left]\frac{n\tau\Delta'\eta^2}{\int_{hT}^{(1-h)T} F(t,\Gamma(2-\alpha_1)\eta,\dots,\Gamma(2-\alpha_n)\eta)dt}, \frac{dk}{c\int_0^T \max_{(x_1,\dots,x_n)\in K(d)} F(t,x_1,\dots,x_n)dt}\right[,$$

for each compact interval $[a,b] \subseteq \Lambda_1$, there exists $\rho > 0$ with the following property: for every $\lambda \in [a,b]$, there exists $\delta > 0$ such that, for each $\mu \in [0,\delta]$, system (1.1) *admits at least three solutions in X whose norms are less than ρ.*

Proof . For $h \in (0, \frac{1}{2})$ choose $w(t) = (w_1(t), \dots, w_n(t))$ for every $t \in [0, T]$ with

$$w_i(t) := \begin{cases} \frac{\Gamma(2-\alpha_i)\eta}{hT} t, & t \in [0, hT), \\ \Gamma(2-\alpha_i)\eta, & t \in [hT, (1-h)T], \\ \frac{\Gamma(2-\alpha_i)\eta}{hT}(T-t), & t \in ((1-h)T, T], \end{cases}$$

for $1 \le i \le n$. Clearly $w_i(0) = w_i(T) = 0$ and $w_i \in L^2([0,T], \mathbb{R})$ for $1 \le i \le n$. A direct calculation shows that

$${}_0D_t^{\alpha_i} w_i(t) = \begin{cases} \frac{\eta}{hT} t^{1-\alpha_i}, & t \in [0, hT), \\ \frac{\eta}{hT}\left(t^{1-\alpha_i} - (t-hT)^{1-\alpha_i}\right), & t \in [hT, (1-h)T], \\ \frac{\eta}{hT}\left(t^{1-\alpha_i} - (t-hT)^{1-\alpha_i} - (t-(1-h)T)^{1-\alpha_i}\right), & t \in ((1-h)T, T], \end{cases}$$

for $1 \le i \le n$. Furthermore,

$$\begin{aligned} \int_0^T a_i(t)|{}_0D_t^{\alpha_i} w_i(t)|^2 dt &= \int_0^T (a_i(t)|{}_0D_t^{\alpha_i} w_i(t)|^2 dt \\ &= \frac{\eta^2}{h^2T^2}\left\{\int_0^T a_i(t) t^{2(1-\alpha_i)} dt + \int_{hT}^T a_i(t)(t-hT)^{2(1-\alpha_i)} dt \right. \\ &\quad + \int_{(1-h)T}^T a_i(t)(t-(1-h)T)^{2(1-\alpha_i)} dt - 2\int_{hT}^T a_i(t)(t^2 - hTt)^{1-\alpha_i} dt \\ &\quad -2\int_{(1-h)T}^T a_i(t)(t^2-(1-h)Tt)^{1-\alpha} dt \\ &\quad \left. +2\int_{(1-h)T}^T a_i(t)(t^2 - hTt + h(1-h)T^2)^{1-\alpha_i} dt \right\} \\ &= 2P_i(\alpha_i, h)\eta^2, \end{aligned}$$

for $1 \le i \le n$. Thus, $w \in X$, and

$$\|w_i\|_{\alpha_i}^2 = \int_0^T a_i(t)|{}_0D_t^{\alpha_i} w_i(t)|^2 dt = 2P(\alpha_i, h)\eta^2,$$

for $1 \le i \le n$. This and (3.1) imply that

$$\begin{aligned} \Phi(w) = \Phi(w_1, \dots, w_n) &= \sum_{i=1}^n \frac{\|w_i\|_{\alpha_i}^2}{2} - \Upsilon(w) \\ &\ge \frac{k}{2}\sum_{i=1}^n \|w_i\|_{\alpha_i}^2 \\ &= k\eta^2 \sum_{i=1}^n P(\alpha_i, h) \\ &\ge nk\Delta\eta^2. \end{aligned} \tag{3.5}$$

Similarly to (3.1) and (3.5) we have $\Phi(w) \le n\tau\Delta'\eta^2$.

Let $r = \frac{dk}{c}$. From $\frac{d}{\Delta ckn} < \eta^2$ we have

$$\sum_{i=1}^n \frac{\|w_i\|_{\alpha_i}^2}{2} \ge \Phi(w) \ge nk\Delta\eta^2 > nk\Delta \times \frac{d}{\Delta ckn} = \frac{r}{k},$$

which is assumption (i) of Theorem 3.1.

On the other hand, by using assumption (j), we infer

$$\begin{aligned}\Psi(w) &= \int_0^T F(t, w_1(t), \ldots, w_n(t))dt \\ &\geq \int_{hT}^{(1-h)T} F(t, \Gamma(2-\alpha_1)\eta, \ldots, \Gamma(2-\alpha_n)\eta)dt \\ &= \int_{hT}^{(1-h)T} F(t, \Gamma(2-\alpha_1)\eta, \ldots, \Gamma(2-\alpha_n)\eta)dt.\end{aligned}$$

Moreover, by condition (jj) we have

$$\begin{aligned}\frac{\int_0^T \max_{(x_1,\ldots,x_n)\in K(\frac{cr}{k})} F(t, x_1, \ldots, x_n)dt}{r} &= \frac{c\int_0^T \max_{(x_1,\ldots,x_n)\in K(d)} F(t, x_1, \ldots, x_n)dt}{dk} \\ &< \frac{\int_{hT}^{(1-h)T} F(t, \Gamma(2-\alpha_1)\eta, \ldots, \Gamma(2-\alpha_n)\eta)dt}{n\tau\Delta'\eta^2} \\ &\leq \frac{\int_{hT}^{(1-h)T} F(t, \Gamma(2-\alpha_1)\eta, \ldots, \Gamma(2-\alpha_n)\eta)dt}{\Phi(w)} \\ &\leq \frac{2\int_0^T F(t, w_1, \ldots, w_n)dt}{\sum_{i=1}^n \|w_i\|_{\alpha_i}^2 - 2\Upsilon(w_1, \ldots, w_n)},\end{aligned}$$

which implies that (ii) is satisfied. Thus, all the assumptions of Theorem 3.1 are satisfied and the proof is complete. □

Corollary 3.4. Let $F : \mathbb{R}^n \to \mathbb{R}$ be a C^1-function and $F(0, \ldots, 0) = 0$. Assume that there exist positive constants d and η such that $\frac{d}{\Delta ckn} < \eta^2$, and also

(k) $F(x_1, \ldots, x_n) \geq 0$, for each $(x_1, \ldots, x_n) \in [0, +\infty) \times \cdots \times [0, +\infty)$;

(kk) $\frac{\max_{(x_1,\ldots,x_n)\in K(d)} F(x_1,\ldots,x_n)}{dk} < \frac{(1-2h)F(\Gamma(2-\alpha_1)\eta,\ldots,\Gamma(2-\alpha_n)\eta)}{nc\tau\Delta'\eta^2}$;

(kkk) $\limsup_{(|x_1|,\ldots,|x_n|)\to(+\infty,\ldots,+\infty)} \frac{F(x_1,\ldots,x_n)}{\sum_{i=1}^n \frac{|x_i|^2}{2}} \leq 0.$

Then, setting

$$\Lambda_2 := \left]\frac{n\tau\Delta'\eta^2}{T(1-2h)F(\Gamma(2-\alpha_1)\eta, \ldots, \Gamma(2-\alpha_n)\eta)}, \frac{dk}{cT \max_{(x_1,\ldots,x_n)\in K(d)} F(x_1, \ldots, x_n)}\right[,$$

for each compact interval $[a, b] \subseteq \Lambda_2$, there exists $\rho > 0$ with the following property: for every $\lambda \in [a, b]$, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the system

$$\begin{cases} {}_tD_T^{\alpha_i}\Big(a_i(t)\,{}_0D_t^{\alpha_i}u_i(t)\Big) & = \lambda F_{u_i}(u_1(t), \ldots, u_n(t)) + \mu G_{u_i}(t, u_1(t), \ldots, u_n(t)) \\ & \quad + h_i(u_i(t)) \quad \text{a.e. } t \in [0, T], \\ u_i(0) = u_i(T) = 0, \end{cases}$$

admits at least three solutions in X whose norms are less than ρ.

References

[1] G.A. Afrouzi, N.T. Chung and A. Hadjian, *Three solutions for a class of Neumann doubly eigenvalue boundary value systems driven by a $(p_1,\dots,p_n)$-Laplacian operator*, Matematiche (Catania) 67 (2012) 43–55.

[2] R.P. Agarwal, M. Benchohra and S. Hamani, *A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions*, Acta Appl. Math. 109 (2010) 973–1033.

[3] B. Ahmad and J.J. Nieto, *Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions*, Comput. Math. Appl. 58 (2009) 1838–1843.

[4] C. Bai, *Impulsive periodic boundary value problems for fractional differential equation involving Riemann-Liouville sequential fractional derivative*, J. Math. Anal. Appl. 384 (2011) 211–231.

[5] C. Bai, *Solvability of multi-point boundary value problem of nonlinear impulsive fractional differential equation at resonance*, Electron. J. Qual. Theory Differ. Equ. 89 (2011) 1–19.

[6] Z. Bai and H. Lü, *Positive solutions for boundary value problem of nonlinear fractional differential equation*, J. Math. Anal. Appl. 311 (2005) 495–505.

[7] M. Benchohra, S. Hamani and S.K. Ntouyas, *Boundary value problems for differential equations with fractional order and nonlocal conditions*, Nonlinear Anal. 71 (2009) 2391–2396.

[8] G. Bonanno and A. Chinnì, *Existence of three solutions for a perturbed two-point boundary value problem*, Appl. Math. Lett. 23 (2010) 807–811.

[9] G. Bonanno and G. D'Aguì, *Multiplicity results for a perturbed elliptic Neumann problem*, Abstr. Appl. Anal. 2010 (2010): 564363.

[10] P. Candito and G. D'Aguì, *Three solutions to a perturbed nonlinear discrete Dirichlet problem*, J. Math. Anal. Appl. 375 (2011) 594–601.

[11] S. Heidarkhani and J. Henderson, *Multiple solutions for a nonlocal perturbed elliptic problem of p-Kirchhoff type*, Comm. Appl. Nonlinear Anal. 19 (2012) 25–39.

[12] S. Heidarkhani and J. Henderson, *Critical point approaches to quasilinear second order differential equations depending on a parameter*, Topol. Methods Nonlinear Anal. 44 (2014) 177–197.

[13] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[14] F. Jiao and Y. Zhou, *Existence of solutions for a class of fractional boundary value problems via critical point theory*, Comput. Math. Appl. 62 (2011) 1181–1199.

[15] F. Jiao and Y. Zhou, *Existence results for fractional boundary value problem via critical point theory*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 22 (2012): 1250086.

[16] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[17] N. Kosmatov, *Integral equations and initial value problems for nonlinear differential equations of fractional order*, Nonlinear Anal. 70 (2009) 2521–2529.

[18] V. Lakshmikantham and A.S. Vatsala, *Basic theory of fractional differential equations*, Nonlinear Anal. 6 (2008) 2677–2682.

[19] J. Mawhin and M. Willem, *Critical Point Theorey and Hamiltonian Systems*, Springer, New York, 1989.

[20] K.S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.

[21] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[22] P.H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, in: CBMS, vol. 65, Amer. Math. Soc. 1986.

[23] B. Ricceri, *A three critical points theorem revisited*, Nonlinear Anal. 70 (2009) 3084–3089.

[24] S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integral and Derivatives: Theory and Applications*, Gordon and Breach, Longhorne, PA, 1993.

[25] J. Wang and Y. Zhou, *A class of fractional evolution equations and optimal controls*, Nonlinear Anal. Real World Appl. 12 (2011) 262–272.

[26] Z. Wei, W. Dong and J. Che, *Periodic boundary value problems for fractional differential equations involving a Riemann-Liouville fractional derivative*, Nonlinear Anal. 73 (2010) 3232–3238.

[27] M. Willem, *Minimax Theorems*, Birkhäuser, 1996.

[28] S. Zhang, *Positive solutions to singular boundary value problem for nonlinear fractional differential equation*, Comput. Math. Appl. 59 (2010) 1300–1309.

[29] Y. Zhao, H. Chen and B. Qin, *Multiple solutions for a coupled system of nonlinear fractional differential equations via variational methods*, Appl. Math. Comput. 257 (2015) 417–427.

[30] Y. Zhao, H. Chen and Q. Zhang, *Infinitely many solutions for fractional differential system via variational method*, J. Appl. Math. Comput. 50 (2016) 589–609.