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# Symmetry reduction and one-dimensional optimal system of the Hunter-Saxton equation

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**Abstract**

This study introduces the Hunter-Saxton equation to describe one model for nematic liquid crystals. We investigate the symmetry group of the Hunter-Saxton equation by applying the classical Lie symmetry methods. Also, by utilizing the classification of one-dimensional subalgebras of the symmetry algebra for this equation, we compute the optimal system of one-parameter subalgebras. Then by using this optimal system and differential invariants, we reduce the equation and obtain the group-invariant solutions and conservation laws for the Hunter-Saxton equation.

Keywords: Hunter-Saxton equation, Optimal system, Group-invariant solutions, Conservation laws
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## 1 Introduction

Nonlinear Evolution Equations (NLEEs) have been a subject of significant interest among mathematicians and physicists due to their relevance in a range of sophisticated applications, particularly in addressing exact solution problems. The examination of wave patterns for NLEEs holds paramount importance in comprehending nonlinear phenomena. NLEEs find extensive application in diverse fields, including plasma physics, numerical physics, optical fiber, computer science, water wave mechanics, control theory, meteorology, electromagnetic theory, mechanics, biogenetics, and other disciplines. Given their pervasive presence in various applications in physics, chemistry, engineering, control theory, finance, and dynamics, the study of wave patterns for Nonlinear Partial Differential Equations (NLPDEs) has elicited significant scholarly interest. Analytical findings about NLPDEs play a pivotal role in elucidating nonlinear physical phenomena [10]. Consider the Hunter-Saxton (H-S) equation, a crucial mathematical tool widely employed in the fields of physics and mathematics. This differential equation (DE) is used to modelling of nematic liquid crystals and it is a second-order equation which is defined as follows.

$$\left(\mathcal{H}_\xi + \mathcal{H}\mathcal{H}_\tau\right)_\tau = \frac{1}{2}\mathcal{H}^2{}_\tau.$$

The H-S equation that is a well known nonlinear hyperbolic in mathematical model which is described by the partial differential equation (PDE) [11]. Which has two independent variables $\tau = \left(\tau^1, \tau^2\right) = (\xi, \tau)$ and one dependent variable $\mathcal{H} = \mathcal{H}(\xi, \tau)$. Hunter and Saxton originally proposed this equation for the theoretical modeling of nematic liquid crystals [4].

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In the literature, the method of Lie symmetry classification has been established as a prominent technique for the derivation of precise and explicit solutions to nonlinear partial differential equations. This method, originally introduced by Sophus Lie in the 19th century, has found widespread application in addressing numerous physical problems across diverse scientific and engineering disciplines [1, 9]. Presently, it stands as a well-acknowledged and influential tool for the comprehensive analysis of nonlinear differential equations. In the context of our research, Lie theory has been employed to scrutinize the proposed model. Through the assessment of density at a singular point and the application of symmetry analysis, the complete vector field has been ascertained. It is noteworthy that the algebra yielded during this process forms an Abelian algebra. Building upon the established framework of concepts, the existence of an optimal Abelian algebra system has been ascertained. For this superior system, the identification of similarity variables has facilitated the reexpression of our assumed model as a nonlinear ordinary differential equation in terms of these analogous variables. Notably, all subsequent solutions have been obtained utilizing NEDAM applied to the derived nonlinear first-order ODE, manifesting as compelling wave packets expressed in terms of trigonometric and hyperbolic trigonometric functions.

Yao et al. [15] addressed the periodic H-S equation by presenting a variable coefficient through the classical approach to uncover invariant solutions. Similarly, Johnpillai and Khaliquo [5] applied the Lie classical method to identify exact solutions for an alternative extended form of the H-S equation. Typically, nematic liquid crystals necessitate two linearly independent vector fields for a comprehensive depiction [2], with one field characterizing fluid flow and the other describing molecular orientation as the director field [4]. The conservation laws in non-linear partial differential equations (NLPDEs) play a crucial role in simplifying the problem and finding solutions in various nonlinear scientific fields. Furthermore, Noether's theorem, formulated by Emmy Noether in 1918, states that every symmetry of a Lagrangian system's action integral corresponds to a conservation law. The literature documents several methods for creating these conservation laws, such as the multiplier method, Noether's theorem, Noether's approach, and the new conservation theorem. In this paper, we obtain the conservation laws of the H-S equation using the multiplier method, which leads to finding new conservation laws for this equation. The paper's structure is as follows: Section 2 presents essential definitions and theorems utilized in subsequent sections. In Section 3, an analysis of the Lie algebra of infinitesimal symmetries, the one-dimensional groups generated, and the Ω-invariant solutions of the H-S equation are conducted. Section 4 outlines the derivation of the one-dimensional optimal system of sub-algebras of the H-S equation via the classification of the adjoint representation orbits of the symmetry group. Section 5 is dedicated to establishing the reduced equation and group invariant solutions and conservation laws for all components of the optimal system, while Section 6 offers the concluding remarks.

## 2 Preliminaries

We use definitions 1 and 2 in section 3 and definitions 3 and 4 and theorem in section 4.

**Definition 2.1.** A system denoted by $\mathcal{L}$, comprising $r$-th order DE in $p$ independent and $q$ dependent variables, is defined as a set of equations,

$$\mathcal{D}_v\left(\tau, \mathcal{H}^{(r)}\right) = 0, \qquad v = 1, \cdots, l$$

involving $\tau = \left(\tau^1, \ldots, \tau^p\right)$ and $\mathcal{H} = \left(\mathcal{H}^1, \ldots, \mathcal{H}^q\right)$, along with their derivatives with respect to $\tau$ up to order $r$. The functions $\mathcal{D}\left(\tau, \mathcal{H}^{(r)}\right) = \left(\mathcal{D}_1\left(\tau, \mathcal{H}^{(r)}\right), \cdots, \mathcal{D}_l\left(\tau, \mathcal{H}^{(r)}\right)\right)$ are supposed to be differentiable in their terms, so $\mathcal{D}$ can be seen as a differentiable map from the jet space $\mathrm{T} \times \mathrm{H}^{(r)}$ to some $l$-dimensional Euclidean space

$$\mathcal{D} : \mathrm{T} \times \mathrm{H}^{(r)} \rightarrow \mathbb{R}^l.$$

The solutions of this differential equation are where the given mapping $\mathcal{D}$ vanishes on $\mathrm{T} \times \mathrm{H}^{(r)}$ , and thus determine a subvariety [12]

$$\mathcal{L}_{\mathcal{D}} = \left\{\left(\tau, \mathcal{H}^{(r)}\right) : \mathcal{D}\left(\tau, \mathcal{H}^{(r)}\right) = 0\right\} \subset \mathrm{T} \times \mathrm{H}^{(r)}$$

**Definition 2.2.** Assume that $M \subset \mathrm{T} \times \mathrm{H}$ is open and let $\lambda$ is a vector field on $M$ with related (local) one-dimensional group $\exp(\varepsilon\lambda)$. The $r$-th prolongation of $\lambda$ indicated by $pr^{(r)}\lambda$ will be a vector field on the $r$-jet space $M^{(r)}$ and is specified to be the infinitesimal generator of the interconnected prolonged one-dimensional group $pr^{(r)}\left[\exp(\varepsilon\lambda)\right]$; in other words, we have

$$pr^{(r)}\lambda\Big|_{\left(\tau, \mathcal{H}^{(r)}\right)} = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} pr^{(r)}\left[\exp(\varepsilon\lambda)\right]\left(\tau, \mathcal{H}^{(r)}\right),$$

for any $\left(\tau, \mathcal{H}^{(r)}\right) \in M^{(r)}$ [4].

**Definition 2.3.** Let $\tilde{\Omega}$ be the Lie algebra of Lie group $\Omega$. For every $\omega \in \Omega$, group conjugation $K_\omega(h) \equiv gh\omega^{-1}, h \in \Omega$, determines a diffeomorphism on $\Omega$. Moreover, $K_\omega \circ K_{\omega'} = K_{gg'}, K_e = 1_\Omega$, so $K_\omega$ specifies an action that is global of $\Omega$ on itself, and all conjugacy maps $K_\omega$ are homomorphisms: $K_\omega(hh') = K_\omega(h) K_\omega(h')$, etc. The differential operator $dK_\omega : T\Omega|_h \to T\Omega\big|_{K_\omega(h)}$ exhibits a clear propensity for maintaining the right-invariance properties of vector fields, consequently giving rise to a linear transformation on the Lie algebra of $\Omega$. This transformation is commonly referred to as the adjoint representation [12].

$$Adg(\lambda) \equiv dK_\omega(\lambda), \qquad \lambda \in \tilde{\Omega}$$

**Definition 2.4.** Let $\Omega$ be a Lie group. An $s$-parameter optimal system refers to a collection of subgroups where is $s$-parameter and not conjugate to each other and possess the property that any other subgroup can be brought into conjugacy with exactly one subgroup in the declared collection. Likewise, the optimal system for subalgebras is also expressed in the same way [12]

**Theorem 2.5.** (Infinitesimal Criterion) Suppose that

$$\mathcal{D}_\lambda\left(\tau, \mathcal{H}^{(r)}\right) = 0, \qquad \lambda = 1, \ldots, l,$$

is a maximal rank system of differential equations where is determined over $M \subset \mathrm{T} \times \mathrm{H}$ and $\Omega$ is a local group of transformations acting on $M$ , and

$$pr^{(r)}\lambda\left[\mathcal{D}_\lambda\left(\tau, \mathcal{H}^{(r)}\right)\right] = 0, \qquad \lambda = 1, \ldots, l \qquad \text{whenever} \qquad \mathcal{D}\left(\tau, \mathcal{H}^{(r)}\right) = 0,$$

for all infinitesimal generator $\lambda$ of $\Omega$, then $\Omega$ is a symmetry group of the system [13].

**Definition 2.6.** The zeroth-Euler operator for $\mathcal{H}$ is specified as

$$\mathbf{E}_\mathcal{H} = \partial/\partial\mathcal{H} + \sum_{s\geq 1}(-1)^s D_{j_1}\cdots D_{j_s}\partial/\partial\mathcal{H}_{j_1\cdots j_s}, \tag{2.1}$$

where $D$ is the total derivative operator. We use the Euler operator to calculate the coefficients of the conservation laws in the multiplier method.

## 3 Lie symmetry group for the H-S equation

We consider the second order PDE,

$$\mathcal{D}\left(\tau, \xi, \mathcal{H}^{(2)}\right) = \mathcal{H}_{\xi\tau} + \frac{\mathcal{H}^2{}_\tau}{2} + \mathcal{H}\mathcal{H}_{\tau\tau},$$

and let,

$$\lambda = \mathcal{S}(\tau, \xi, \mathcal{H})\frac{\partial}{\partial_\tau} + \mathcal{R}(\tau, \xi, \mathcal{H})\frac{\partial}{\partial_\xi} + \Lambda(\tau, \xi, \mathcal{H})\frac{\partial}{\partial_\mathcal{H}},$$

be a vector field on $\mathrm{T} \times \mathrm{H}$. We strive to uncover the coefficient functions $\mathcal{S}, \tau, \Lambda$ that would render the one-dimensional group $\exp(\varepsilon\lambda)$ for the group of symmetry for the H-S equation [7, 8]. As per the theorem, it is imperative to ascertain the second prolongation of $\lambda$.

$$pr^{(2)}\lambda = \lambda + \Lambda^\tau\frac{\partial}{\partial\mathcal{H}_\tau} + \Lambda^\xi\frac{\partial}{\partial\mathcal{H}_\xi} + \Lambda^{\tau\tau}\frac{\partial}{\partial\mathcal{H}_{\tau\tau}} + \Lambda^{\xi\xi}\frac{\partial}{\partial\mathcal{H}_{\xi\xi}} + \Lambda^{\tau\xi}\frac{\partial}{\partial\mathcal{H}_{\tau\xi}}.$$

Applying $pr^{(2)}\lambda$, we discover the invariant criterion must be satisfied whenever.

$$\mathcal{H}_{\xi\tau} = -\frac{\mathcal{H}^2{}_\tau}{2} - \mathcal{H}\mathcal{H}_{\tau\tau}.$$

Replacing $\mathcal{H}_{\xi\tau}$ by $-\dfrac{\mathcal{H}^2{}_\tau}{2} - \mathcal{H}\mathcal{H}_{\tau\tau}$, and equating the coefficients of the various monomials in the first and second-order partial derivatives of $\mathcal{H}$, we can effectively derive the equations that determine the symmetry group of the H-S equation [3]. The solution of the determining equations follows by

$$\begin{aligned}
\mathcal{S}(\tau,\xi,\mathcal{H}) &= f_2(\xi)\,\tau + f_3(\xi),\\
\mathcal{R}(\tau,\xi,\mathcal{H}) &= f_1(\xi),\\
\Lambda(\tau,\xi,\mathcal{H}) &= f_2(\xi)\,\mathcal{H},
\end{aligned}$$

for linear function $f_1, f_2, f_3$ depend on $\xi$. Let

$$\begin{aligned}
f_1(\xi) &= c_1\xi + c_2,\\
f_2(\xi) &= c_3,\\
f_3(\xi) &= c_4\xi + c_5,
\end{aligned}$$

where $c_1, \ldots, c_5$ are real scalar [6]. After consideration, we can assert that the map of coefficients for the most total symmetry of the H-S equation are as follows.

$$\begin{aligned}
\mathcal{S}(\tau,\xi,\mathcal{H}) &= c_3\tau + c_4\xi + c_5,\\
\mathcal{R}(\tau,\xi,\mathcal{H}) &= c_1\xi + c_2,\\
\Lambda(\tau,\xi,\mathcal{H}) &= c_3\mathcal{H}.
\end{aligned}$$

Hence, the Lie algebra of infinitesimal symmetries of the H-S equation is comprised of five vector fields.

$$\begin{aligned}
&\lambda_1 = \xi\partial_\xi, && \text{Dilatation}\\
&\lambda_2 = \partial_\xi, && \text{Time translation}\\
&\lambda_3 = \tau\partial_\tau + \mathcal{H}\partial_{\mathcal{H}}, && \textit{Dilatation}\\
&\lambda_4 = \xi\partial_\tau, && \\
&\lambda_5 = \partial_\tau. && \text{Space translation}
\end{aligned}$$

The commutation relations between these generators are detailed in the table below, with each entry in row $i$ and column $j$ representing $[\lambda_i, \lambda_j]$

Table 1: Lie algebra commutator table of the H-S equation

| [ , ] | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ | $\lambda_5$ |
|---|---|---|---|---|---|
| $\lambda_1$ | 0 | $-\lambda_2$ | 0 | $\lambda_4$ | 0 |
| $\lambda_2$ | $\lambda_2$ | 0 | 0 | $\lambda_5$ | 0 |
| $\lambda_3$ | 0 | 0 | 0 | $-\lambda_4$ | $-\lambda_5$ |
| $\lambda_4$ | $-\lambda_4$ | $-\lambda_5$ | $\lambda_4$ | 0 | 0 |
| $\lambda_5$ | 0 | 0 | $\lambda_5$ | 0 | 0 |

Let $f_i$ be the one-dimensional groups developed by the $\lambda_i$.

**Theorem 3.1.** Considering the above assumptions, it can be said that the transformed points $\exp(\varepsilon\lambda - i)(\tau,\xi,\mathcal{H}) = \left(\tilde{\tau},\tilde{\xi},\tilde{\mathcal{H}}\right)$ are

$$\begin{aligned}
f_1(\xi) &= (\tau, \xi e^{\varepsilon}, \mathcal{H}),\\
f_2(\xi) &= (\tau, \varepsilon + \xi, \mathcal{H}),\\
f_3(\xi) &= (\tau e^{\varepsilon}, \xi, \mathcal{H}e^{\varepsilon}),\\
f_4(\xi) &= (\varepsilon\xi + \tau, \xi, \mathcal{H}),\\
f_5(\xi) &= (\varepsilon + \tau, \xi, \mathcal{H}).
\end{aligned}$$

**Corollary 3.2.** Since each group $f_i$ is a symmetry group, implies that if $\mathcal{H} = f(\tau, \xi)$ is a solution of H-S equation, the following functions are also solutions.

$$\begin{aligned}
\mathcal{H}_1 &= f\left(\xi, \frac{\tau}{e^{\varepsilon}}\right), \\
\mathcal{H}_2 &= f(\xi, \tau - \varepsilon), \\
\mathcal{H}_3 &= f\left(\frac{\xi}{e^{\varepsilon}}, \tau\right) e^{\varepsilon}, \\
\mathcal{H}_4 &= f(-\varepsilon\tau + \xi, \tau), \\
\mathcal{H}_5 &= f(\xi - \varepsilon, \tau),
\end{aligned}$$

where $\varepsilon$ is any real number. Therefore, we obtained the $\Omega$-invariant solutions of the H-S equation.

Using this theorem, we can get the infinite solution of the equation by knowing only one solution. The important thing is that a large number of these obtained solutions may be placed in one group and do not differ much from the others. For this reason, categorizing and classifying these solutions is an important issue that we have to deal with. This makes us obtain the optimal system of subalgebras in the next section and prepare the foundations for proper classification.

## 4 Optimal System of H-S equation

In this section, we will derive the one-dimensional optimal system for the H-S equation using the symmetry group. Because every linear combination of infinitesimal symmetries is itself an infinitesimal symmetry, there are countless one-dimensional subgroups for $\Omega$. It is crucial to identify which subgroups lead to different solution types. To do this, we need to find invariant solutions that cannot be transformed into each other through symmetry transformations within the full symmetry group [6, 8]. Adjoint representation for every $\lambda_i,\ i = 1, \ldots, 5$ is expressed as follow:

$$Ad\left(\exp\left(\varepsilon\lambda_i\right)\right)\lambda_j = \lambda_j - \varepsilon\left[\lambda_i, \lambda_j\right] + \frac{\varepsilon^2}{2!}\left[\lambda_i, \left[\lambda_i, \lambda_j\right]\right] - \cdots,$$

where $\varepsilon$ is a parameter and $[\lambda_i, \lambda_j]$ is the commutator of the lie algebra for $i, j = 1, \cdots, 5$. By utilizing the commutator table, we can derive all adjoint representations related to the Lie group of the H-S equation. Now, we are pleased to introduce the following compelling theorem

**Theorem 4.1.** A one-dimensional optimal system for Lie algebra of the H-S equation is given by.

$$\begin{aligned}
&(1) \qquad 2\lambda_1 + \lambda_3, \\
&(2) \qquad \lambda_3, \\
&(3) \qquad \lambda_1,
\end{aligned}$$

**Proof .** Let $T_i^{\varepsilon} : \tilde{\Omega} \to \tilde{\Omega}$ be the adjoint transformation defined by $\lambda \to Ad\left(\exp\left(\varepsilon\lambda_i\right)\right)\lambda$ for $i = 1, \cdots, 5$ The matrix of $T_i^{\varepsilon},\ i = 1, \cdots, 5$, with respect to basis $[T_1, \cdots, T_5]$ is.

$$T^{\varepsilon}{}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & e^{\varepsilon} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & e^{-\varepsilon} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \ T^{\varepsilon}{}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -\varepsilon & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\varepsilon & 1 \end{bmatrix}, \ T^{\varepsilon}{}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & e^{\varepsilon} & 0 \\ 0 & 0 & 0 & 0 & e^{\varepsilon} \end{bmatrix}$$

$$T^{\varepsilon}{}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \varepsilon & 0 & -\varepsilon & 1 & 0 \\ 0 & \varepsilon & 0 & 0 & 1 \end{bmatrix}, T^{\varepsilon}{}_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\varepsilon & 0 & 1 \end{bmatrix},$$

Now, we try to vanish the coefficients of $\lambda$ that given a nonzero vector:

$$\lambda = a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 + a_4\lambda_4 + a_5\lambda_5.$$

By applying the adjoint representations ${T^\varepsilon}_i$ to $\lambda$ with carefully chosen parameters $\varepsilon$ at each step, we can streamline $\lambda$ as follows.

Case (1): If $a_3 \neq 0$ we can assume that $a_3 = 1, \varepsilon = \frac{a_5}{a_3}$ with ${T^\varepsilon}_2$ then we can make the coefficient of $\lambda_2$ vanish. With ${T^\varepsilon}_5$, $\varepsilon = a_5$ then we can make the coefficient of $\lambda_5$ vanish.

Case (2): If $a_1 \neq 0$ we can assume that $a_1 = 2, \varepsilon = -\frac{a_4}{{a_1}^{-1}}$ with ${T^\varepsilon}_4$ then let's aim to eliminate the coefficient of $\lambda_4$. so $\lambda$ is reduced:

$$2\lambda_1 + \lambda_3.$$

If $a_1 = 0$ we can assume that $\varepsilon = a_4$ with ${T^\varepsilon}_4$ then let's aim to eliminate the coefficient of $\lambda_4$. So $\lambda$ is reduced $\lambda_3$. If $a_3 = 0$ we can assume that $a_1 = 1, \varepsilon = \frac{a_5}{a_1}$ with ${T^\varepsilon}_2$ then we can make the coefficient of $\lambda_2$ vanish.

Case (3): If $a_4 \neq 0$ we can assume that $\varepsilon = a_4$ with ${T^\varepsilon}_4$ then let's aim to eliminate the coefficient of $\lambda_4$. So $\lambda$ is reduced $\lambda_1$. If $a_4 = 0$ with ${T^\varepsilon}_5$ we can make the coefficient of $\lambda_5$ vanish. So $\lambda$ is reduced $\lambda_1$.

We have discovered that the optimal system of one-dimensional subalgebras is constituted by the ones spanned by:

$$\begin{aligned}
\lambda_1 &= \xi\partial_\xi, \\
\lambda_3 &= \tau\partial_\tau + \mathcal{H}\partial_\mathcal{H}, \\
2\lambda_1 + \lambda_3 &= 2\xi\partial_\xi + \tau\partial_\tau + \mathcal{H}\partial_\mathcal{H},
\end{aligned}$$

and the proof is complete. □

## 5 Similarity Reductions and Conservation Laws of the H-S Equation

In this section, we obtain new solutions and conservation laws from the H-S equation.

### 5.1 Invariant Solutions

In this subsection, we will streamline the one-dimensional flow equation by re-communicating it in renewed coordinates. The H-S equation is formulated in the coordinates $(\tau, \xi, \mathcal{H})$, so we need to identify the appropriate coordinates for simplifying it. These renewed coordinates will be derived by identifying independent invariants $(z, w, f)$ that correspond to the symmetry group generators. To solve this equation, we must tackle the associated characteristic ordinary differential equation (ODE):

$$\frac{d_\tau}{\tau} = \frac{d_\xi}{2\xi} = \frac{d_\mathcal{H}}{2u}.$$

Hence, three functionally independent invariant $z = \xi\tau^{-2}, w = \xi\mathcal{H}^{-1}$ and $f = \mathcal{H}\tau^{-2}$ are obtained. The reduced equations are obtained as follows:

$$\begin{aligned}
ff_{zz} + \frac{{f_z}^2}{z} &= 0, \\
f_w + \frac{f^2}{2} &= 0, \\
\frac{f^2}{2} &= 0.
\end{aligned}$$

The solutions to the above equations in terms of $(z, w, f)$ variables are as follows:

$$\begin{aligned}
&f(z) = \left(\frac{3}{2}(c_1 z + c_2)\right)^{\frac{2}{3}}, \\
&f(w) = \frac{2}{2c_1 + w}, \\
&f(z) = 0.
\end{aligned}$$

Finally, we found group-invariant solutions of H-S equation.

$$
\begin{aligned}
f\left(\xi\tau^{-2}\right) &= \left(\frac{3}{2}\left(c_1\xi\tau^{-2}+c_2\right)\right)^{\frac{2}{3}},\\
f\left(\xi\mathcal{H}^{-1}\right) &= \frac{2}{2c_1+\xi\mathcal{H}^{-1}},\\
f\left(\xi\tau^{-2}\right) &= 0.
\end{aligned}
$$

### 5.2 Conservation Laws

To derive the conservation laws of the H-S equation, we make the crucial assumption of a set of non-singular local multipliers of a specific rank. These multipliers are then multiplied in the H-S equation, and through the application of the Euler operator and solving the system, we can obtain local multipliers. Finally, utilizing the inverse divergence operator, we derive $\mathrm{L}_\xi$ and $\mathrm{L}_\tau$ of the H-S equation.

**Theorem 5.1.** [14] Consider the significance of each local non-singular multiplier $\Lambda(\tau,\mathcal{H}^{(r)})$ as it stands for a conservation law for system $\mathcal{D}(\tau,\mathcal{H}^{(n)})$ when

$$
\mathbf{E}_{\mathcal{H}}(\Lambda(\tau,\mathcal{H}^{(r)})\cdot\mathcal{D}(\tau,\mathcal{H}^{(n)}))\equiv 0, \tag{5.1}
$$

where $\mathbf{E}$ is the Euler operator.

The following theorem is obtained by doing calculations, finding the coefficients of $\Lambda$, collecting the coefficients of powers of (5.1) and solving the resulting system of equations using Maple, which results in finding 4 new conservation laws for the H-S equation.

**Theorem 5.2.** Conservation laws of the H-S equation obtained as follows.

**Case 1:** $\Lambda_1=\frac{1}{2}\xi^2\mathcal{H}_\xi+\tau\xi\mathcal{H}_\tau-\tau$. Therefore, the conserved vector results as:

$$
\begin{aligned}
\mathrm{L}_\xi =& \frac{1}{4}\xi^2\mathcal{H}\mathcal{H}_{\tau\xi}+\frac{1}{12}\xi^2\mathcal{H}_\tau^2\mathcal{H}+\frac{1}{6}\xi^2\mathcal{H}^2\mathcal{H}_{2\tau}+\frac{1}{8}\xi^2\mathcal{H}_\xi\mathcal{H}_\tau+\frac{1}{4}\tau\xi\mathcal{H}_\tau^2-\frac{1}{2}\tau\mathcal{H}_\tau-\frac{1}{8}\xi^2\mathcal{H}\mathcal{H}_{\xi\tau}\\
&-\frac{1}{4}\xi\mathcal{H}\mathcal{H}_\tau-\frac{1}{4}\tau\xi\mathcal{H}\mathcal{H}_{2\tau}+\frac{1}{2}\mathcal{H},\\
\mathrm{L}_\tau =& \frac{1}{3}\xi^2\mathcal{H}\mathcal{H}_\tau\mathcal{H}_\xi+\frac{1}{2}\tau\xi\mathcal{H}\mathcal{H}_{\tau\xi}+\frac{1}{2}\tau\xi\mathcal{H}\mathcal{H}_\tau^2-\frac{5}{4}\tau\mathcal{H}\mathcal{H}_\tau+\frac{1}{8}\xi^2\mathcal{H}_\xi^2+\frac{1}{4}\tau\xi\mathcal{H}_\tau\mathcal{H}_\xi-\frac{1}{2}\tau\mathcal{H}_\xi-\frac{1}{4}\xi\mathcal{H}\mathcal{H}_\xi\\
&-\frac{1}{8}\xi^2\mathcal{H}\mathcal{H}_{2\xi}-\frac{1}{3}\xi\mathcal{H}^2\mathcal{H}_\tau-\frac{1}{6}\xi^2\mathcal{H}\mathcal{H}_\xi\mathcal{H}_\tau+\frac{1}{2}\tau\mathcal{H}\mathcal{H}_\tau+\frac{1}{2}\mathcal{H}^2.
\end{aligned}
$$

**Case 2:** $\Lambda_2=\xi\mathcal{H}_\xi+\tau\mathcal{H}_\tau$. Therefore, the following conserved vector is obtained:

$$
\begin{aligned}
\mathrm{L}_\xi =& \frac{1}{2}\xi\mathcal{H}\mathcal{H}_{\tau\xi}+\frac{1}{6}\xi\mathcal{H}\mathcal{H}_\tau^2+\frac{1}{3}\xi\mathcal{H}^2\mathcal{H}_{2\tau}+\frac{1}{4}\xi\mathcal{H}_\xi\mathcal{H}_\tau+\frac{1}{4}\tau\mathcal{H}_\tau^2-\frac{1}{4}\xi\mathcal{H}\mathcal{H}_{\xi\tau}-\frac{1}{4}\mathcal{H}\mathcal{H}_\tau-\frac{1}{4}\tau\mathcal{H}\mathcal{H}_{2\tau},\\
\mathrm{L}_\tau =& \frac{1}{2}\tau\mathcal{H}\mathcal{H}_\tau^2+\frac{1}{2}\tau\mathcal{H}\mathcal{H}_{\tau\xi}+\frac{1}{3}\tau\mathcal{H}^2\mathcal{H}_{2\tau}+\frac{1}{4}\xi\mathcal{H}_\xi^2+\frac{1}{4}\tau\mathcal{H}_\xi\mathcal{H}_\tau-\frac{1}{4}\mathcal{H}\mathcal{H}_\xi-\frac{1}{4}\xi\mathcal{H}\mathcal{H}_{2\xi}-\frac{1}{3}\xi\mathcal{H}^2\mathcal{H}_{\xi\tau}-\frac{1}{3}\mathcal{H}^2\mathcal{H}_\tau.
\end{aligned}
$$

**Case 3:** $\Lambda_3=\xi\mathcal{H}_\xi+\mathcal{H}$. The conserved quantities are computed:

$$
\begin{aligned}
\mathrm{L}_\xi =& \frac{1}{4}\xi\mathcal{H}\mathcal{H}_{\tau\xi}+\frac{1}{6}\xi\mathcal{H}\mathcal{H}_\tau^2+\frac{1}{3}\xi\mathcal{H}^2\mathcal{H}_{2\tau}+\frac{1}{4}\xi\mathcal{H}_\xi\mathcal{H}_\tau+\frac{1}{4}\mathcal{H}\mathcal{H}_\tau-\frac{1}{4}\mathcal{H}\mathcal{H}_\tau,\\
\mathrm{L}_\tau =& \frac{1}{3}\xi\mathcal{H}\mathcal{H}_\tau\mathcal{H}_\xi+\frac{2}{3}\mathcal{H}^2\mathcal{H}_\tau-\frac{1}{2}\mathcal{H}\mathcal{H}_\xi-\frac{1}{4}\xi\mathcal{H}\mathcal{H}_{2\xi}+\frac{1}{4}\xi\mathcal{H}_\xi^2+\frac{1}{4}\mathcal{H}\mathcal{H}_\xi-\frac{1}{3}\xi\mathcal{H}^2\mathcal{H}_{\xi\tau}-\frac{2}{3}\mathcal{H}^2\mathcal{H}_\tau.
\end{aligned}
$$

**Case 4:** $\Lambda_4=\mathcal{H}_\xi$. Hence, we get,

$$
\begin{aligned}
\mathrm{L}_\xi =& \frac{1}{4}\mathcal{H}\mathcal{H}_{\tau\xi}+\frac{1}{6}\mathcal{H}\mathcal{H}_\tau^2+\frac{1}{3}\mathcal{H}^2\mathcal{H}_{2\tau}+\frac{1}{4}\mathcal{H}_\tau\mathcal{H}_\xi,\\
\mathrm{L}_\tau =& -\frac{1}{4}\mathcal{H}\mathcal{H}_{\xi\xi}+\frac{1}{4}\mathcal{H}_\xi^2-\frac{1}{3}\mathcal{H}^2\mathcal{H}_{\xi\tau}+\frac{1}{3}\mathcal{H}\mathcal{H}_\tau\mathcal{H}_\xi.
\end{aligned}
$$

## 6 Conclusion

In this paper, by using the theory of infinitesimal criterion and the influence of prolonged on the vector field $\lambda$ we compute the Lie symmetry group of the H-S equation. Also, by determining the equation of functions $\mathcal{S}, \tau, \Lambda$ one-parameter groups generator and optimal system of symmetry are obtained for H-S equation, five flows are acquired which two of them are dilatation, another one is time translation and the last one is space translation. Finally, we achieved the $\Omega$-invariant solutions of H-S equation under symmetry group $\Omega$. Through the utilization of the adjoint representation for the group of symmetry on its Lie algebra, we successfully have established an optimal system of one-parameter subalgebras. Furthermore, we have derived the similarity reductions for each component of the optimal system, along with their group invariant solutions and conservation laws for the H-S equation.

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