



# A New Class of Function Spaces on Domains of $\mathbb{R}^d$ and Its Relations to Classical Function Spaces

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## Abstract

A new class of function spaces on domains (i.e., open and connected subsets) of  $\mathbb{R}^d$ , by means of the asymptotic behavior of modulations of functions and distributions, is defined. This class contains the classes of Lebesgue spaces and modulation spaces. Main properties of this class are studied, its applications in the study of function spaces and its relations to classical function spaces are discussed.

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## 1. Introduction and preliminaries

The main aim of this paper is to introduce a new class of function spaces on domain  $\Omega \subseteq \mathbb{R}^d$  which contain the class of Lebesgue spaces and modulation spaces. Considering the fact that in general it is not possible to define modulation spaces on manifolds (see [7]), this new class of function spaces is interesting as being an appropriate substitution for modulation spaces in such cases.

Let  $\mathcal{D}(\Omega)$  denote the space of compactly supported infinitely differentiable functions with its usual topology by a family of seminorms and  $\mathcal{D}'(\Omega)$  denote its dual, the space of generalized functions. Also, Let  $\mathcal{S}(\mathbb{R}^d)$  and  $\mathcal{S}'(\mathbb{R}^d)$  denote the Schwartz space and its dual space, the space of tempered distributions, respectively. Let  $\phi \in \mathcal{D}(\mathbb{R}^d)$  be a smooth radial bump function such that

$$\phi : \mathbb{R}^d \longrightarrow [0, 1]$$

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$$\text{SUPP}\phi \subseteq B(0, \sqrt{2d}) \quad (1.1)$$

$$\phi(\xi) = 1, \text{ for } \xi \in B(0, \sqrt{d/2}).$$

Let  $\phi_k(\xi) = \phi(\xi - k)$ ,  $k \in \mathbb{Z}^d$  and  $\xi \in \mathbb{R}^d$ . Since  $\phi_k(\xi) = 1$  in the unit closed cube  $Q_k$  with center  $k$  and  $\{Q_k\}_{k \in \mathbb{Z}^d}$  is a covering of  $\mathbb{R}^d$ , clearly  $\sum_{k \in \mathbb{Z}^d} \phi_k(\xi) \geq 1$  for all  $\xi \in \mathbb{R}^d$ . We write

$$\varphi_k(\xi) = \frac{\phi_k(\xi)}{\sum_{l \in \mathbb{Z}^d} \phi_l(\xi)}, \quad k \in \mathbb{Z}^d.$$

Consequently, there exist  $c, c_m > 0$  such that

$$\left\{ \begin{array}{l} |\varphi_k(\xi)| \geq c, \quad \forall \xi \in Q_k \\ \text{SUPP}\varphi_k \subseteq B(k, \sqrt{2d}), \\ \sum_{l \in \mathbb{Z}^d} \phi_l(\xi) = 1, \\ |\varphi_k^{(m)}(\xi)| \leq c_m, \quad \forall \xi \in \mathbb{R}^d. \end{array} \right. \quad (1.2)$$

Let

$$\Phi = \{ \{\varphi_k\}_{k \in \mathbb{Z}^d} : \{\varphi_k\}_{k \in \mathbb{Z}^d} \text{ satisfies (1.2)} \}.$$

By the construction above  $\Phi \neq \emptyset$ . Let  $\{\varphi_k\}_{k \in \mathbb{Z}^d} \in \Phi$  be an arbitrary sequence. For  $r \in \mathbb{R}$ , we consider the weight function  $w_r(\xi) = (1 + |\xi|^2)^{r/2}$  on  $\mathbb{R}^d$  and define the modulation spaces  $M_{p,q}^r(\mathbb{R}^d)$ ,  $0 < p, q \leq \infty$ , by

$$M_{p,q}^r(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) : \|f : M_{p,q}^r\| := \left( \sum_{k \in \mathbb{Z}^d} (w_r(k) \|\mathcal{F}^{-1}\varphi_k \mathcal{F}f\|_p)^q \right)^{1/q} < \infty\}. \quad (1.3)$$

The class of modulation spaces is defined by Hans G. Feichtinger in [3]. For a recent treatment of modulation spaces we refer to [4] and [5]. For  $r \in \mathbb{R}$  and  $0 < p, q \leq \infty$ ,  $M_{p,q}^r(\mathbb{R}^d)$  is a quasi-Banach space (a Banach space if  $p, q \geq 1$ ). Now we give definition of Bessel potential spaces. Let  $s \in \mathbb{R}$  and  $1 \leq p \leq \infty$ . The Bessel potential space  $H_p^s(\mathbb{R}^d)$  is defined by

$$H_p^s(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) : \|f : H_p^s\| := \|\mathcal{F}^{-1}w_s \mathcal{F}f\|_p < \infty\}. \quad (1.4)$$

For  $s \in \mathbb{R}$  and  $1 \leq p \leq \infty$ ,  $H_p^s(\mathbb{R}^d)$  is a Banach space. If we use the notation  $\mathcal{J}^s(f) = \mathcal{F}^{-1}w_s \mathcal{F}f$  for  $s \in \mathbb{R}$ , then  $H_p^s(\mathbb{R}^d) = \mathcal{J}^{-s}L_p(\mathbb{R}^d)$ . When  $1 < p < \infty$  and  $n \in \mathbb{N}$ ,  $H_p^n(\mathbb{R}^d)$  is equal to the classical Sobolev space  $W_p^n(\mathbb{R}^d)$ , defined by

$$W_p^n(\mathbb{R}^d) = \{f \in L_p(\mathbb{R}^d) : \|f : W_p^n\| := \sum_{|\alpha| \leq n} \|D^\alpha f\|_p < \infty\}. \quad (1.5)$$

Note that  $H_p^0 = W_p^0 = L_p$ . We refer to [1], [2], [8] and [9] for the theory of Sobolev and Bessel potential spaces.

## 2. Definition and Properties of $N_{p,q}^{r,s}(\Omega)$

Let  $\Omega \subseteq \mathbb{R}^d$  be a domain, that is, an open bounded set. The Sobolev space on domain  $\Omega$ ,  $W_p^n(\Omega)$ , may be defined as

$$W_p^n(\Omega) = \{f \in L_p(\Omega) : \|f : W_p^n(\Omega)\| := \sum_{|\alpha| \leq n} \|D^\alpha f\|_{L^p(\Omega)} < \infty\}. \tag{2.1}$$

Then we have the following restriction property

$$W_p^n(\Omega) = \{f \in L_p(\Omega) : \exists g \in W_p^n(\mathbb{R}^d) \text{ such that } f = g|_\Omega\} \tag{2.2}$$

and  $\|f : W_p^n(\Omega)\|' := \inf\{\|g : W_p^n(\mathbb{R}^d)\| : g \in W_p^n(\mathbb{R}^d), f = g|_\Omega\}$  defines an equivalent norm on  $\|f : W_p^n(\Omega)\|$ . For a general theory of function spaces on  $\mathbb{R}^d$  as well as on domains we refer to the references [1], [8] and [9]. The described procedure is not applicable to define Bessel potential spaces and modulation spaces on  $\Omega$ . Therefore we define Bessel potential spaces and modulation spaces on  $\Omega$  as follows:

**Definition 2.1.** Let  $\Omega \subseteq \mathbb{R}^d$  be a domain,  $1 \leq p, q \leq \infty$  and  $r, s \in \mathbb{R}$ . Then we define

$$H_p^s(\Omega) = \{f \in \mathcal{D}'(\Omega) : \exists g \in H_p^s(\mathbb{R}^d) \text{ such that } g|_\Omega = f\}$$

with norm  $\|f : H_p^s(\Omega)\| = \inf\{\|g : H_p^s(\mathbb{R}^d)\| : g \in H_p^s(\mathbb{R}^d), f = g|_\Omega\}$ , and also

$$M_{p,q}^r(\Omega) = \{f \in \mathcal{D}'(\Omega) : \exists g \in M_{p,q}^r(\mathbb{R}^d) \text{ such that } g|_\Omega = f\}$$

with norm  $\|f : M_{p,q}^r(\Omega)\| = \inf\{\|g : M_{p,q}^r(\mathbb{R}^d)\| : g \in M_{p,q}^r(\mathbb{R}^d), f = g|_\Omega\}$ .

The proof of the following theorem is similar to the proof of proposition 3.2.3 in [8], so omitted.

**Theorem 2.2.** For  $1 \leq p, q \leq \infty$  and  $r, s \in \mathbb{R}$ ,  $H_p^s(\Omega)$  and  $M_{p,q}^r(\Omega)$  are Banach spaces of generalized functions.

The operators of modulation and translation are defined as follows

$$M_a f(x) = e^{2\pi i a \cdot x} f(x) \quad \text{and} \quad T_a f(x) = f(x - a).$$

**Definition 2.3 (Definition of  $N_{p,q}^{r,s}(\Omega)$ ).** Let  $r, s \in \mathbb{R}$ ,  $1 \leq p, q \leq \infty$  and  $\Omega \subseteq \mathbb{R}^d$  be a domain. For  $f \in H_p^s(\Omega)$  we say that  $f \in N_{p,q}^{r,s}(\Omega)$  if

$$\|f : N_{p,q}^{r,s}(\Omega)\| := \left( \sum_{k \in \mathbb{Z}^d} (\|M_k f : H_p^s(\Omega)\| w_r(k))^q \right)^{1/q} < \infty.$$

**Theorem 2.4.** *Let  $r, s \in \mathbb{R}$  and  $1 \leq p, q \leq \infty$ .*

1.  $N_{p,q}^{r,s}(\Omega)$  is a Banach space of distributions on  $\Omega$ ,
2.  $N_{p,q}^{r,s}(\Omega) \hookrightarrow H_p^s(\Omega)$
3.  $N_{p,q}^{r,s}(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$
4.  $\mathcal{S}(\mathbb{R}^d) \hookrightarrow N_{p,q}^{r,s}(\mathbb{R}^d)$  if  $r + s \leq 0$ .

**Proof .**

1. Clearly  $N_{p,q}^{r,s}(\Omega)$  is a normed space. Let  $\{f_n\}$  be a Cauchy sequence in  $N_{p,q}^{r,s}(\Omega)$ . Then  $\{f_n\}$  is also a Cauchy sequence in  $H_p^s(\Omega)$  and therefore there exists  $f \in H_p^s(\Omega)$  such that  $f_n \rightarrow f$  in  $H_p^s(\Omega)$ , which in turn implies that  $M_k f_n \rightarrow M_k f$  in  $H_p^s(\Omega)$ , for all  $k \in \mathbb{Z}$ . On the other hand, since  $\{\mathcal{J}^{-s} M_k f_n\}$  is Cauchy in  $\ell_q^r(L_p)$  and  $\ell_q^r(L_p)$  is complete, there is  $\{h_k\}_{k \in \mathbb{Z}^d} \in \ell_q^r(L_p)$  such that

$$\{\mathcal{J}^{-s} M_k f_n\} \rightarrow \{h_k\} \quad (\text{in } \ell_q^r(L_p)), \quad (2.3)$$

which implies that  $\mathcal{J}^{-s} M_k f_n \rightarrow h_k$  in  $L_p$  or  $M_k f_n \rightarrow \mathcal{J}^s h_k$  in  $H_p^s(\Omega)$ , for each  $k \in \mathbb{Z}^d$ . Comparing this with the fact that  $M_k f_n \rightarrow M_k f$  in  $H_p^s(\Omega)$  shows that  $h_k = \mathcal{J}^{-s} M_k f$ . Now (2.3) implies that  $f_n \rightarrow f$  in  $N_{p,q}^{r,s}(\Omega)$ .

2. This follows from  $N_{p,q}^{r,s}(\mathbb{R}^d) \hookrightarrow H_p^s(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$ .
3.  $f \in \mathcal{S}(\mathbb{R}^d)$ . Since  $\ell_\infty^r(\mathbb{Z}^d) \hookrightarrow \ell_1^{r-t}(\mathbb{Z}^d)$ , for  $t > d$ , we have the embedding  $N_{p,\infty}^{r,s} \hookrightarrow N_{p,1}^{r+t,s} \hookrightarrow N_{p,q}^{r,s}$ . Therefore we need only to show that  $\mathcal{S}(\mathbb{R}^d) \hookrightarrow N_{p,1}^{r,s}$ . To this end we have

$$\begin{aligned} \|f : N_{p,\infty}^{r,s}\| &= \sup_{k \in \mathbb{Z}^d} \|M_k f : H_p^s\| w_r(k) \\ &= \sup_{k \in \mathbb{Z}^d} \|\mathcal{F}^{-1} w_s \mathcal{F} M_k f\|_p w_r(k) \\ &= \sup_{k \in \mathbb{Z}^d} \|\mathcal{F}^{-1} w_s(\cdot + k) \mathcal{F} f\|_p w_r(k) \\ &\leq \sup_{k \in \mathbb{Z}^d} \|w_L \mathcal{F}^{-1} w_s(\cdot + k) \mathcal{F} f\|_\infty w_r(k) \\ &= \sup_{k \in \mathbb{Z}^d} \|\mathcal{F}^{-1} (1 - \Delta)^L w_s(\cdot + k) \mathcal{F} f\|_\infty w_r(k) \\ &\leq C \sup_{k \in \mathbb{Z}^d} \|(1 - \Delta)^L w_s(\cdot + k) \mathcal{F} f\|_1 w_r(k) \\ &\leq C' \sup_{k \in \mathbb{Z}^d} \|(1 - \Delta)^L \mathcal{F} f\|_1 w_{r+s}(k), \end{aligned}$$

for sufficiently large  $L$ . Now the result follows from  $r + s \leq 0$  and the fact that  $\mathcal{S}(\mathbb{R}^d) \hookrightarrow L_1(\mathbb{R}^d)$ .

□

The following theorem shows that this new class contains the classes of modulation spaces and Lebesgue spaces. Part (2) of this theorem was proved and used in [6]. In the proof of this theorem,

we denote the norm of the operator  $f \rightarrow \mathcal{F}^{-1}\varphi\mathcal{F}f$  from  $L_p$  to  $L_p$  by  $\|\varphi : M_{\mathcal{F}}(L_p)\|$ , whenever it is bounded.

**Theorem 2.5.** *We have*

1.  $N_{p,q}^{r,s}(\Omega) = L_p(\Omega)$  if  $s = 0$  and  $r < -d/q$
2.  $N_{p,q}^{r,s}(\mathbb{R}^d) = M_{p,q}^r(\mathbb{R}^d)$  if  $r < -|s| - d$
3.  $N_{p,q}^{r,s}(\Omega) = \{0\}$  if  $s \geq 0$  and  $r > -\frac{d}{q}$

**Proof .** The proofs of (1) and (3) are simple. We prove (2). We have

$$\begin{aligned}
 \|f : M_{p,q}^r\| &= \|\{\|\mathcal{F}^{-1}\varphi_k\mathcal{F}f\| : L_p\}_{k \in \mathbb{Z}^d} : \ell_q^r\| \\
 &= \|\{\|\mathcal{F}^{-1}T_k w_s T_k w_{-s} \varphi_k \mathcal{F}f : L_p\|\}_{k \in \mathbb{Z}^d} : \ell_q^r\| \\
 &= \|\{\|\mathcal{F}^{-1}T_k w_s \varphi_k \mathcal{F}\mathcal{F}^{-1}T_k w_{-s} \mathcal{F}f\| : L_p\}_{k \in \mathbb{Z}^d} : \ell_q^r\| \\
 &\leq \|\{\|T_k w_s \varphi_k : M_{\mathcal{F}}(L_p)\|\|\mathcal{F}^{-1}T_k w_{-s} \mathcal{F}f : L_p\|\}_{k \in \mathbb{Z}^d} : \ell_q^r\| \\
 &\leq \|w_s \varphi : M_{\mathcal{F}}(L_p)\|\|\{\|\mathcal{F}^{-1}w_{-s} \mathcal{F}M_{-k}f : L_p\|\}_{k \in \mathbb{Z}^d} : \ell_q^r\| \\
 &\leq C\|\{M_{-k}f : H_p^{-s}\}\| : \ell_q^r\|.
 \end{aligned} \tag{2.4}$$

For  $j \in \mathbb{Z}^d$  let

$$j^* = \{i \in \mathbb{Z}^d : \text{supp}(\varphi_i) \cap \text{supp}(\varphi_j) \neq \emptyset\}.$$

Since

$$\begin{aligned}
 \|M_{-k}f : H_p^{-r}\| &= \|\mathcal{F}^{-1}w_{-r}\mathcal{F}(M_{-k}f) : L_p\| \\
 &= \|\mathcal{F}^{-1}(T_k w_{-r})\mathcal{F}f : L_p\|
 \end{aligned}$$

and

$$\begin{aligned}
 \mathcal{F}^{-1}(T_k w_{-r})\mathcal{F}f &= \sum_{j \in \mathbb{Z}^d} \sum_{i \in j^*} \mathcal{F}^{-1}(T_k w_{-r})\varphi_j \varphi_i \mathcal{F}f \\
 &= \sum_{j \in \mathbb{Z}^d} \sum_{i \in j^*} \mathcal{F}^{-1}\varphi_i(T_k w_{-r})\mathcal{F}\mathcal{F}^{-1}\varphi_j \mathcal{F}f,
 \end{aligned}$$

with convergence in  $\mathcal{S}'$ , we have

$$\|M_{-k}f : H_p^{-r}\| \leq \sum_{j \in \mathbb{Z}^d} \sum_{i \in j^*} \|\varphi_i(T_k w_{-r}) : M_{\mathcal{F}}(L_p)\|\|\mathcal{F}^{-1}\varphi_j \mathcal{F}f : L_p\|.$$

Since

$$\|\varphi_i T_k w_{-r} : M_{\mathcal{F}}(L_p)\| \lesssim w_{-r}(k - i),$$

we have

$$\|M_{-k}f : H_p^{-r}\| \lesssim \sum_{j \in \mathbb{Z}^d} \sum_{i \in j^*} w_{-r}(k-i) \|\mathcal{F}^{-1} \varphi_j \mathcal{F} f : L_p\|. \quad (2.5)$$

Since  $i \in j^*$ , we have  $w_{-r}(k-i) \sim w_{-r}(k-j)$ , and therefore we may replace  $i$  by  $j$  in (2.5). After this substitution, by using the fact  $|j^*| \sim 1$ , we see that the right hand side of (2.5) is actually a convolution of two sequences  $\{w_{-r}(j)\}_{j \in \mathbb{Z}^d}$  and  $\{\|\mathcal{F}^{-1} \varphi_j \mathcal{F} f : L_p\|\}_{j \in \mathbb{Z}^d}$ . By invoking Young's inequality and using the  $w_{|s|}$ -moderateness of the weight function  $w_s$ , i.e.

$$w_s(x+y) \leq w_{|s|}(x)w_s(y),$$

we have

$$\|\{\|M_{-k}f : H_p^{-r}\|\}_{k \in \mathbb{Z}^d} : \ell_q^s\| \lesssim \|\{w_{-r}(j)\}_{j \in \mathbb{Z}^d} : \ell_1^{|s|}\| \|\{\|\mathcal{F}^{-1} \varphi_j \mathcal{F} f : L_p\|\}_{j \in \mathbb{Z}^d} : \ell_q^s\|.$$

Since  $r > |s| + d$ , we have  $\|\{w_{-r}(j)\}_{j \in \mathbb{Z}^d} : \ell_1^{|s|}\| < \infty$ , and finally we arrive at

$$\|\{\|M_{-k}f : H_p^{-r}\|\}_{k \in \mathbb{Z}^d} : \ell_q^s\| \lesssim \|f : M_{p,q}^s\|. \quad (2.6)$$

Now (2.4) and (2.6) completes the proof.  $\square$

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