



On Ψ -Conditional Asymptotic Stability of First Order Non-Linear Matrix Lyapunov Systems

G. Suresh Kumar^a, B. V. Appa Rao^a, M. S. N. Murthy^{b,*}

^aDepartment of Mathematics, Konenu Lakshmaiah University, Green Fields, Vaddeswaram-522 502, Guntur Dt., Andhra Pradesh, India.

^bDepartment of Mathematics, Acharya Nagarjuna University, Nagarjuna Nagar – 522510, Guntur, Andhrapradesh, ,India.

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Abstract

We provide necessary and sufficient conditions for Ψ -conditional asymptotic stability of the solution of a linear matrix Lyapunov system and sufficient conditions for Ψ -conditional asymptotic stability of the solution of a first order non-linear matrix Lyapunov system $X' = A(t)X + XB(t) + F(t, X)$.

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1. Introduction

The importance of matrix Lyapunov systems, which arise in a number of areas of control engineering problems, dynamical systems, and feedback systems are well known. In this paper, we focus our attention to study of Ψ -conditional asymptotic stability of solutions of the first order non-linear matrix Lyapunov system

$$X' = A(t)X + XB(t) + F(t, X) \quad (1.1)$$

as a perturbed system of

$$X' = A(t)X + XB(t), \quad (1.2)$$

where $A(t)$, $B(t)$ are square matrices of order ' n ', whose elements are real valued continuous functions of ' t ' on the interval $\mathbb{R}_+ = [0, \infty)$ and $F(t, X)$ is a continuous square matrix of order ' n ' on $\mathbb{R}_+ \times \mathbb{R}^{n \times n}$,

*Corresponding author

Email addresses: dragsk006@gmail.com (G. Suresh Kumar), bvardr2010@gmail.com (B. V. Appa Rao), drmsn2002@gmail.com (M. S. N. Murthy)

such that $F(t, O) = O$ (zero matrix), where $\mathbb{R}^{n \times n}$ denotes the space of all $n \times n$ real valued matrices. The continuity of A, B and F ensures the existence of a solution of (1.1).

Akinyele [1] introduced the notion of Ψ -stability and this concept extended to solutions of ordinary differential equations by Constantin [3]. Later Marchalo [10] introduced the concept of Ψ -(uniform) stability, Ψ -asymptotic stability of trivial solutions of linear and non-linear system of differential equations. The study of conditional asymptotic stability of differential equations was motivated by Coppel [4]. Further, the concept of Ψ -conditional asymptotic stability to non-linear Volterra integro-differential equations were studied by Diamandescu [5]. Recently, Murty and Suresh Kumar [[11], [12],[13]] extended the concept of Ψ -boundedness, Ψ -stability and Ψ -instability to Kronecker product matrix Lyapunov system associated with first order matrix Lyapunov systems.

The purpose of this paper is to provide sufficient conditions for Ψ -conditional asymptotic stability of (1.1). We investigate conditions on the two fundamental matrices of

$$X' = AX, \tag{1.3}$$

$$X' = B^T X \tag{1.4}$$

and $F(t, X)$ under which the solution of (1.1) or (1.2) are Ψ -conditionally asymptotically stable on \mathbb{R}_+ . Here, Ψ is a continuous matrix function. The introduction of the matrix function Ψ permits to obtain a mixed asymptotic behavior of the solutions.

This paper is well organized as follows. In section 2, we present some basic definitions, notations, lemmas and properties relating to Kronecker product of matrices and Ψ -conditionally asymptotically stability, which are useful for later discussion. In Section 3, we obtain necessary and sufficient conditions for Ψ -conditionally asymptotic stability of solutions of linear matrix Lyapunov system (1.2). The results of this section illustrated with suitable examples. In section 4, we obtain sufficient conditions for the Ψ -conditional asymptotic stability of (1.1).

This paper extends some of the results of Diamandescu [5] to matrix Lyapunov systems. The main tool used in this paper is Kronecker product of matrices.

2. Priliminaries

In this section we present some basic definitions, notations and results which are useful for later discussion.

Let \mathbb{R}^n be the Euclidean n -dimensional space. Elements in this space are column vectors, denoted by $u = (u_1, u_2, u_3, \dots, u_n)^T$ (T denotes transpose) and their norm defined by

$$\|u\| = \max\{|u_1|, |u_2|, |u_3|, \dots, |u_n|\}.$$

For $A = [a_{ij}] \in \mathbb{R}^{n \times n}$, we define the norm $|A| = \sup_{\|u\| \leq 1} \|Au\|$. It is well-known that

$$|A| = \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^n |a_{ij}| \right\}.$$

O_n denote the zero matrix of order $n \times n$ and 0_n is the zero vector of order n .

Definition 2.1. [8] Let $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{p \times q}$ then the Kronecker product of A and B written $A \otimes B$ is defined to be the partitioned matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdot & \cdot & \cdot & a_{1n}B \\ a_{21}B & a_{22}B & \cdot & \cdot & \cdot & a_{2n}B \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1}B & a_{m2}B & \cdot & \cdot & \cdot & a_{mn}B \end{pmatrix}$$

is an $mp \times nq$ matrix and is in $\mathbb{R}^{mp \times nq}$.

Definition 2.2. [8] Let $A = [a_{ij}] \in \mathbb{R}^{m \times n}$, then the vectorization operator $Vec : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{mn}$, defined and denote by

$$\hat{A} = VecA = \begin{pmatrix} A_{.1} \\ A_{.2} \\ \vdots \\ A_{.n} \end{pmatrix}, \text{ where } A_{.j} = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix} \quad (1 \leq j \leq n).$$

Lemma 2.3. [6] The vectorization operator $Vec : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n^2}$, is a linear and one-to-one operator. In addition, Vec and Vec^{-1} are continuous operators.

Regarding properties and rules for vectorization operator and Kronecker product of matrices we refer to [8].

Let $\Psi_k : \mathbb{R}_+ \rightarrow (0, \infty)$, $k = 1, 2, \dots, n$, be continuous functions, and let

$$\Psi = \text{diag}[\Psi_1, \Psi_2, \dots, \Psi_n].$$

Then the matrix $\Psi(t)$ is an invertible square matrix of order n , for all $t \in \mathbb{R}_+$.

Definition 2.4. [5] A function $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ is said to be Ψ -bounded on \mathbb{R}_+ if $\Psi(t)\phi(t)$ is bounded on \mathbb{R}_+ .

Extend this definition for matrix functions.

Definition 2.5. [6] A matrix function $F : \mathbb{R}_+ \rightarrow \mathbb{R}^{n \times n}$ is said to be Ψ bounded on \mathbb{R}_+ if the matrix function ΨF is bounded on \mathbb{R}_+

$$\left(\text{i.e., } \sup_{t \geq 0} |\Psi(t)F(t)| < \infty \right).$$

Definition 2.6. [5] The solution of the vector differential equation $x' = f(t, x)$ (where $x \in \mathbb{R}^n$ and f is a continuous n vector function) is said to be Ψ -stable on \mathbb{R}_+ , if for every $\epsilon > 0$ and any $t_0 \in \mathbb{R}_+$, there exists a $\delta = \delta(\epsilon, t_0) > 0$ such that any solution \tilde{x} of $x' = f(t, x)$, which satisfies the inequality $\|\Psi(t_0)(\tilde{x}(t_0) - x(t_0))\| < \delta(\epsilon, t_0)$ exists and satisfies the inequality $\|\Psi(t)(\tilde{x}(t) - x(t))\| < \epsilon$, for all $t \geq t_0$. Otherwise, is said that the solution $x(t)$ is Ψ -unstable on \mathbb{R}_+ .

Extend this definition for matrix differential equations.

Definition 2.7. The solution of the matrix differential equation $X' = F(t, X)$ (where $X \in \mathbb{R}^{n \times n}$ and F is a continuous $n \times n$ matrix function) is said to be Ψ -stable on \mathbb{R}_+ , if for every $\epsilon > 0$ and any $t_0 \in \mathbb{R}_+$, there exists a $\delta = \delta(\epsilon, t_0) > 0$ such that any solution \tilde{X} of $X' = F(t, X)$, which satisfies the inequality $\|\Psi(t_0)(\tilde{X}(t_0) - X(t_0))\| < \delta(\epsilon, t_0)$ exists and satisfies the inequality $\|\Psi(t)(\tilde{X}(t) - X(t))\| < \epsilon$, for all $t \geq t_0$. Otherwise, is said that the solution $X(t)$ is Ψ -unstable on \mathbb{R}_+ .

Definition 2.8. [5] The solution of the vector differential equation $x' = f(t, x)$ is said to be Ψ -conditionally stable on \mathbb{R}_+ if it is not Ψ -stable on \mathbb{R}_+ but there exists a sequence $\{x_m(t)\}$ of solutions of $x' = f(t, x)$ defined for all $t \geq 0$ such that

$$\lim_{m \rightarrow \infty} \Psi(t)x_m(t) = \Psi(t)x(t), \text{ uniformly on } \mathbb{R}_+.$$

If the sequence $\{x_m(t)\}$ can be chosen so that

$$\lim_{t \rightarrow \infty} \Psi(t)(x_m(t) - x(t)) = 0_n, \text{ for } m = 1, 2, 3, \dots,$$

then $x(t)$ is said to be Ψ -conditionally asymptotically stable on \mathbb{R}_+ .

We can easily extend this definition for matrix differential equations.

Definition 2.9. *The solution of the matrix differential equation $X' = F(t, X)$ is said to be Ψ -conditionally stable on \mathbb{R}_+ if it is not Ψ -stable on \mathbb{R}_+ but there exists a sequence $\{X_m(t)\}$ of solutions of $X' = F(t, X)$ defined for all $t \geq 0$ such that*

$$\lim_{m \rightarrow \infty} \Psi(t)X_m(t) = \Psi(t)X(t), \text{ uniformly on } \mathbb{R}_+.$$

If the matrix sequence $\{X_m(t)\}$ can be chosen so that

$$\lim_{t \rightarrow \infty} \Psi(t)(X_m(t) - X(t)) = O_n, \text{ for } m = 1, 2, 3, \dots,$$

then $X(t)$ is said to be Ψ -conditionally asymptotically stable on \mathbb{R}_+ .

Remark 2.10. *It is easy to see that if $|\Psi(t)|$ and $|\Psi^{-1}(t)|$ are bounded on \mathbb{R}_+ , then the Ψ -stability, Ψ -bounded and Ψ -conditionally asymptotically stability implies classical stability, boundedness and conditional asymptotic stability.*

The following lemmas play a vital role in the proof of main result.

Lemma 2.11. *[?] For any matrix function $F \in \mathbb{R}^{n \times n}$, we have*

$$\frac{1}{n} |\Psi(t)F(t)| \leq \|(I_n \otimes \Psi(t))\hat{F}(t)\| \leq |\Psi(t)F(t)|, \text{ for all } t \in \mathbb{R}_+.$$

Lemma 2.12. *[7] The matrix function $X(t)$ is a solution of (1.1) on the interval $J \subset \mathbb{R}_+$ if and only if the vector valued function $\hat{X}(t) = \text{Vec}X(t)$ is a solution of the differential system*

$$\hat{X}'(t) = (B^T \otimes I_n + I_n \otimes A)\hat{X}(t) + G(t, \hat{X}(t)), \quad (2.1)$$

where $G(t, \hat{X}) = \text{Vec}F(t, X)$, on the same interval J .

Definition 2.13. *[7] The above system (2.1) is called the corresponding Kronecker product system associated with (1.1).*

The linear system corresponding to (2.1) is

$$\hat{X}'(t) = (B^T \otimes I_n + I_n \otimes A)\hat{X}(t). \quad (2.2)$$

Lemma 2.14. *The solution of the system (1.1) is Ψ -unbounded on \mathbb{R}_+ if and only if the solution of the corresponding Kronecker product system (2.1) is $I_n \otimes \Psi$ -unbounded on \mathbb{R}_+ .*

Proof . It is easily seen from Lemma 5 of [6] and Lemma 2.12. \square

Lemma 2.15. *The solution of the system (1.1) is Ψ -unstable on \mathbb{R}_+ if and only if the corresponding Kronecker product system (2.1) is $I_n \otimes \Psi$ -unstable on \mathbb{R}_+ .*

Proof . It is easily seen from Lemma 7 of [7]. \square

Lemma 2.16. *The solution of the system (1.1) is Ψ -conditionally asymptotically stable on \mathbb{R}_+ if and only if the corresponding Kronecker product system (2.1) is $I_n \otimes \Psi$ -conditionally asymptotically stable on \mathbb{R}_+ .*

Proof . Suppose that the solution of the system (1.1) is Ψ -conditionally asymptotically stable on \mathbb{R}_+ . From Definition 2.9, we have that the solution $X(t)$ of (1.1) is Ψ -unstable and there exists a sequence of solutions $X_n(t)$ of (1.1) on \mathbb{R}_+ such that

$$\lim_{m \rightarrow \infty} \Psi(t)X_m(t) = \Psi(t)X(t), \text{ uniformly on } \mathbb{R}_+ \tag{2.3}$$

and

$$\lim_{t \rightarrow \infty} \Psi(t) (X_m(t) - X(t)) = O_n, \text{ for } m = 1, 2, 3, \dots \tag{2.4}$$

Since $X(t)$ is a Ψ -unstable solution of (1.1), from Lemmas 2.12 and 2.15, we have that $\hat{X}(t)$ is $I_n \otimes \Psi$ -unstable solution of (2.1) on \mathbb{R}_+ . Now applying vectorization(Vec) operator to (2.3) and (2.4), we have

$$\lim_{m \rightarrow \infty} (I_n \otimes \Psi(t)) \hat{X}_m(t) = (I_n \otimes \Psi(t)) \hat{X}(t), \text{ uniformly on } \mathbb{R}_+ \tag{2.5}$$

and

$$\lim_{t \rightarrow \infty} (I_n \otimes \Psi(t)) \left(\hat{X}_m(t) - \hat{X}(t) \right) = 0_{n^2}, \text{ for } m = 1, 2, 3, \dots \tag{2.6}$$

From Definition 2.8, $\hat{X}(t)$ is $I_n \otimes \Psi$ -conditionally asymptotically stable on \mathbb{R}_+ .

Conversely suppose that, the solution of (2.1) is $I_n \otimes \Psi$ -conditionally asymptotically stable on \mathbb{R}_+ . From Definition 2.8, we have that the solution $\hat{X}(t)$ of (2.1) is $I_n \otimes \Psi$ -unstable and there exists a sequence of solutions $\hat{X}_m(t)$ of (2.1) on \mathbb{R}_+ , which satisfies (2.5) and (2.6). Since $\hat{X}(t)$ is a $I_n \otimes \Psi$ -unstable solution of (2.1), again from Lemmas 2.12 and 2.15, we have that $X(t) = \text{Vec}^{-1} \hat{X}(t)$ is a Ψ -unstable solution of (1.1) on \mathbb{R}_+ . By applying Vec^{-1} operator to (2.5) and (2.6), we have that the sequence of solutions $X_m(t) = \text{Vec}^{-1} \hat{X}_m(t)$ of (1.1) satisfying (2.3) and (2.4). Thus, from Definition 2.9 the solution $X(t)$ of (1.1) is Ψ -conditionally asymptotically stable on \mathbb{R}_+ . \square

Lemma 2.17. *Let $Y(t)$ and $Z(t)$ be the fundamental matrices for the systems (1.3) and (1.4) respectively. Then the matrix $Z(t) \otimes Y(t)$ is a fundamental matrix of (2.2).*

Proof . It is easily seen from Lemma 2.4 of [13]. \square

Theorem 2.18. *Let $A(t)$, $B(t)$ and $F(t, X)$ be continuous matrix functions on \mathbb{R}_+ . If $Y(t)$, $Z(t)$ are the fundamental matrices for the systems (1.3), (1.4) respectively and P_1, P_2 are non-zero supplementary projections, then*

$$\begin{aligned} \hat{X}(t) &= \int_0^t (Z(t) \otimes Y(t)) P_1 (Z^{-1}(s) \otimes Y^{-1}(s)) G(s, \hat{X}(s)) ds \\ &+ \int_t^\infty (Z(t) \otimes Y(t)) P_2 (Z^{-1}(s) \otimes Y^{-1}(s)) G(s, \hat{X}(s)) ds \end{aligned} \tag{2.7}$$

is a solution of (2.1) on \mathbb{R}_+ .

Proof . It is easily seen that $\hat{X}(t)$ is the solution of (2.1) on \mathbb{R}_+ . \square

3. Linear Matrix Lyapunov Systems

In this section, we prove necessary and sufficient conditions for the Ψ -conditional asymptotic stability of the linear matrix Lyapunov system (1.2). The results of this section are illustrated with suitable examples.

Theorem 3.1. *The linear matrix Lyapunov system (1.2) is Ψ -conditionally asymptotically stable on \mathbb{R}_+ if and only if it has a Ψ -unbounded solution and a non-trivial solution $W(t)$ such that*

$$\lim_{t \rightarrow \infty} (I_n \otimes \Psi(t))\hat{W}(t) = 0_{n^2}. \quad (3.1)$$

Proof . Suppose that the solution of linear matrix Lyapunov system (1.2) is Ψ -conditionally asymptotically stable on \mathbb{R}_+ . From Lemmas 2.12 and 2.16 with $F = O_n$, it follows that the solution of (2.2) is $I_n \otimes \Psi$ -conditionally asymptotically stable on \mathbb{R}_+ . From Theorem 3.1 of [5], we have that the linear system (2.2) has an $I_n \otimes \Psi$ -unbounded solution and a non-trivial solution $\hat{W}(t)$ such that (3.1) satisfied. Since (2.2) has a $I_n \otimes \Psi$ -unbounded solution and from Lemmas 2.12 and 2.14, the linear system (1.2) has a Ψ -unbounded solution. Since $\hat{W}(t)$ is a non-trivial solution of (2.2), then $W(t) = \text{Vec}^{-1}\hat{W}(t)$ is the corresponding non-trivial solution of (1.2).

Conversely suppose that (1.2) has at least one Ψ -unbounded solution on \mathbb{R}_+ and at least one non-trivial solution $W(t)$ exists and satisfies (3.1). From Lemma 2.14 and Theorem 3.1 of [5], it follows that the solution $\hat{W}(t)$ of (2.2) is $I_n \otimes \Psi$ -conditionally asymptotically stable on \mathbb{R}_+ . Again from Lemmas 2.12 and 2.16, it follows that the solution of (1.2) is Ψ -conditionally asymptotically stable on \mathbb{R}_+ . \square

Example 3.2. *Consider the linear matrix Lyapunov matrix system (1.2) with*

$$A = \begin{pmatrix} \frac{1}{t+1} & 0 \\ 0 & -\frac{1}{t+1} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}.$$

Then the fundamental matrices of (1.3) and (1.4) are

$$Y(t) = \begin{pmatrix} t+1 & 0 \\ 0 & \frac{1}{t+1} \end{pmatrix} \quad \text{and} \quad Z(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-2t} \end{pmatrix}$$

Let $\Psi(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$. Clearly,

$$X(t) = \begin{pmatrix} (t+1)e^t & (t+1)e^{-2t} \\ \frac{e^t}{t+1} & \frac{e^{-2t}}{t+1} \end{pmatrix}$$

is a solution of (1.2) and $|\Psi(t)X(t)| = (t+1)e^{2t}$, $t \geq 0$. Therefore, $X(t)$ is a Ψ -unbounded solution of (1.2). Let $W(t) = \begin{pmatrix} 0 & (t+1)e^{-2t} \\ \frac{e^t}{t+1} & \frac{e^{-2t}}{t+1} \end{pmatrix}$. Clearly, $W(t)$ is a non-trivial solution of (1.2) and

$$(I_2 \otimes \Psi(t))\hat{W}(t) = \begin{pmatrix} 0 \\ \frac{1}{t+1} \\ (t+1)e^{-t} \\ \frac{e^{-3t}}{t+1} \end{pmatrix}.$$

Also, $\lim_{t \rightarrow \infty} (I_2 \otimes \Psi(t))\hat{W}(t) = 0_4$. From Theorem (3.1), the linear system (1.2) is Ψ -conditionally asymptotically stable on \mathbb{R}_+ .

The conditions for Ψ -conditional asymptotic stability of (1.2) can be expressed in terms of fundamental matrices of (1.3) and (1.4) in the following theorems.

Theorem 3.3. *Let $Y(t)$ and $Z(t)$ be the fundamental matrices of (2.4) and (2.5). Then the linear matrix Lyapunov system (1.2) is Ψ -conditionally asymptotically stable on \mathbb{R}_+ if and only if the following conditions are satisfied;*

- (i) *there exists a projection P_1 , such that $(Z(t) \otimes \Psi(t)Y(t))P_1$ is unbounded on \mathbb{R}_+ .*
- (ii) *there exists a projection $P_2 \neq O_{n^2}$ such that*

$$\lim_{t \rightarrow \infty} (Z(t) \otimes \Psi(t)Y(t))P_2 = O_{n^2}.$$

Proof . Suppose that the linear system (1.2) is Ψ -conditional asymptotic stable on \mathbb{R}_+ . From Lemmas 2.12 and 2.16 with $F = O_n$, the Kronecker product system (2.2) is $I_n \otimes \Psi$ -conditionally asymptotically stable on \mathbb{R}_+ . From Lemma 2.17 and Theorem 3.2 of [5], it follows that the fundamental matrix $S(t) = Z(t) \otimes Y(t)$ of (2.2) satisfies the following conditions;

1. there exists a projection P_1 such that $(I_n \otimes \Psi(t))S(t)P_1$ is unbounded on \mathbb{R}_+ .
2. there exists a projection $P_2 \neq O_{n^2}$ such that

$$\lim_{t \rightarrow \infty} (I_n \otimes \Psi(t))S(t)P_2 = O_{n^2}.$$

Substitute $S(t) = Z(t) \otimes Y(t)$ in (1) and (2) and simplifying with the use of Kronecker product properties, we have that the fundamental matrices of (1.3) and (1.4) satisfies conditions (i) and (ii).

Conversely suppose that, the fundamental matrices of (1.3) and (1.4) satisfies the conditions (i) and (ii). From Theorem 3.2 of [5], Lemma 2.12 and properties of Kronecker products, the corresponding Kronecker product system (2.2) is $I_n \otimes \Psi$ -conditionally asymptotically stable on \mathbb{R}_+ . Again from Lemmas 2.12 and 2.16, the linear system (1.2) is Ψ -conditionally asymptotically stable on \mathbb{R}_+ . \square

Example 3.4. *In Example 3.2, taking*

$$\Psi(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix}.$$

There exists two non-zero projections

$$P_1 = \begin{pmatrix} I_2 & O_2 \\ O_2 & O_2 \end{pmatrix} \text{ and } P_2 = \begin{pmatrix} O_2 & O_2 \\ O_2 & I_2 \end{pmatrix}$$

such that

$$(Z(t) \otimes \Psi(t)Y(t))P_1 = \begin{pmatrix} t+1 & 0 & 0 & 0 \\ 0 & \frac{e^{2t}}{t+1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$(Z(t) \otimes \Psi(t)Y(t))P_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-3t}(t+1) & 0 \\ 0 & 0 & 0 & \frac{e^{-t}}{t+1} \end{pmatrix}$$

Clearly, $(Z(t) \otimes \Psi(t)Y(t))P_1$ is unbounded on \mathbb{R}_+ and $(Z(t) \otimes \Psi(t)Y(t))P_2 \rightarrow O_4$ as $t \rightarrow \infty$. Therefore, from Theorem 3.3 the system (1.2) is Ψ -conditionally asymptotically stable on \mathbb{R}_+ .

A sufficient condition for Ψ -conditional asymptotically stability is given by the following theorem.

Theorem 3.5. *If there exist two supplementary projections P_1, P_2 ($P_i \neq O_{n^2}$, $i=1, 2$) and a positive constant L such that the fundamental matrices $Y(t)$ and $Z(t)$ of (1.3) and (1.4) satisfies the condition*

$$\int_0^t |(Z(t) \otimes \Psi(t)Y(t))P_1(Z^{-1}(s) \otimes Y^{-1}(s)\Psi^{-1}(s))|ds + \int_t^\infty |(Z(t) \otimes \Psi(t)Y(t))P_2(Z^{-1}(s) \otimes Y^{-1}(s)\Psi^{-1}(s))|ds \leq L \quad (3.2)$$

for all $t \geq 0$, then, the linear equation (1.2) is Ψ -conditionally asymptotically stable on \mathbb{R}_+ .

Proof . From Theorem 3.2, Theorem 3.1 and Lemma 2.2 of [12], we have that the conditions in Theorem 3.3 are satisfied. Therefore, the Kronecker product system (2.2) is $I_n \otimes \Psi$ -conditionally asymptotically stable on \mathbb{R}_+ . From Lemmas 2.12 and 2.16 with $F = O_n$, the linear system (1.2) is Ψ -conditionally asymptotically stable on \mathbb{R}_+ . \square

Example 3.6. *Consider the linear matrix Lyapunov system (1.2) with $A = I_2$ and $B = -I_2$, then the fundamental matrices of (1.3) and (1.4) are $Y(t) = e^t I_2$ and $Z(t) = e^{-t} I_2$. Let*

$$\Psi(t) = \begin{pmatrix} \frac{e^t}{t+1} & 0 \\ 0 & e^{-t} \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now

$$|(Z(t) \otimes \Psi(t)Y(t))P_1(Z^{-1}(s) \otimes Y^{-1}(s)\Psi^{-1}(s))| = e^{s-t}$$

and

$$|(Z(t) \otimes \Psi(t)Y(t))P_2(Z^{-1}(s) \otimes Y^{-1}(s)\Psi^{-1}(s))| = \left(\frac{s+1}{t+1}\right) e^{t-s}.$$

Therefore, the condition (3.2) satisfied with $L = 2$. Thus, from Theorem 3.5, the linear system (1.2) is Ψ -conditionally asymptotically stable on \mathbb{R}_+ .

4. Non-Linear Matrix Lyapunov Systems

In this section, we prove sufficient conditions for the Ψ -conditional asymptotic stability of the non-linear matrix Lyapunov system (1.1).

Theorem 4.1. *Suppose that:*

1. *There exist supplementary projections P_1, P_2 ($P_i \neq O_{n^2}$, $i=1, 2$) and a constant $L > 0$ such that the fundamental matrices $Y(t), Z(t)$ of (1.3), (1.4) satisfies the condition (3.2).*
2. *The function $F(t, X)$ satisfies the inequality*

$$|\Psi(t) (F(t, X(t)) - F(t, Y(t)))| \leq \xi(t) |\Psi(t) (X(t) - Y(t))|,$$

for $t \geq 0$ and for all continuous and Ψ -bounded matrix functions $X, Y : \mathbb{R}_+ \rightarrow \mathbb{R}^{n \times n}$, where $\xi(t)$ is a continuous nonnegative bounded function on \mathbb{R}_+ such that

$$|\xi(t)| \leq M, \quad \text{for all } t \geq 0,$$

where M is a positive constant.

3. $p = nML < 1$.

Then, all Ψ -bounded solutions of (1.1) are Ψ -conditionally asymptotically stable on \mathbb{R}_+ .

Proof . Let $X(t)$ be the solution of (1.1) with $X(t_0) = X_0$, then by Lemma 2.12, $\hat{X}(t)$ is the unique solution of Kronecker product system (2.1) with $\hat{X}(t_0) = \hat{X}_0$.

We put

$$\mathbb{S} = \left\{ \hat{X} : \mathbb{R}_+ \rightarrow \mathbb{R}^{n^2} : \hat{X} \text{ is continuous and } I_n \otimes \Psi - \text{ bounded on } \mathbb{R}_+ \right\}.$$

Define a norm on the set \mathbb{S} by

$$\|\hat{X}\|_{\mathbb{S}} = \sup_{t \geq 0} \|(I_n \otimes \Psi(t))\hat{X}(t)\|.$$

It is well-known that $(\mathbb{S}, \|\cdot\|_{\mathbb{S}})$ is a Banach space. For $\hat{X} \in \mathbb{S}$, we define

$$\begin{aligned} (T\hat{X})(t) &= \int_0^t (Z(s) \otimes Y(s))P_1(Z^{-1}(s) \otimes Y^{-1}(s))G(s, \hat{X}(s))ds \\ &\quad - \int_t^\infty (Z(s) \otimes Y(s))P_2(Z^{-1}(s) \otimes Y^{-1}(s))G(s, \hat{X}(s))ds, \quad \forall t \geq 0. \end{aligned}$$

From Lemma 2.11 and hypothesis (2), it follows that

$$\begin{aligned} \|(I_n \otimes \Psi(t))G(t, \hat{X})\| &= \|(I_n \otimes \Psi(t))\hat{F}(t, X)\| \\ &\leq |\Psi(t)F(t, X)| \leq \xi(t)|\Psi(t)X(t)| \\ &\leq nM\|(I_n \otimes \Psi(t))\hat{X}(t)\|, \quad \forall t \in \mathbb{R}_+ \text{ and } \hat{X} \in \mathbb{R}^{n^2}. \end{aligned}$$

For $0 \leq t \leq v$, we have

$$\begin{aligned} &\left\| \int_t^v (Z(s) \otimes Y(s))P_2(Z^{-1}(s) \otimes Y^{-1}(s))G(s, \hat{X}(s))ds \right\| \\ &\leq |I_n \otimes \Psi^{-1}(t)| \int_t^v \|(I_n \otimes \Psi(s))(Z(s) \otimes Y(s))P_2(Z^{-1}(s) \otimes Y^{-1}(s)) \\ &\quad (I_n \otimes \Psi^{-1}(s))(I_n \otimes \Psi(s))G(s, \hat{X}(s))\|ds \\ &\leq |\Psi^{-1}(t)| \int_t^v |(Z(s) \otimes \Psi(s)Y(s))P_2(Z^{-1}(s) \otimes Y^{-1}(s)\Psi^{-1}(s))| \\ &\quad \|(I_n \otimes \Psi(s))G(s, \hat{X}(s))\|ds \\ &\leq nM|\Psi^{-1}(t)| \int_t^v |(Z(s) \otimes \Psi(s)Y(s))P_2(Z^{-1}(s) \otimes Y^{-1}(s)\Psi^{-1}(s))| \\ &\quad \|(I_n \otimes \Psi(s))\hat{X}(s)\|ds \end{aligned}$$

$$\leq pL^{-1}|\Psi^{-1}(t)| \sup_{t \geq 0} \|(I_n \otimes \Psi(t))\hat{X}(t)\|$$

$$\int_t^v |(Z(t) \otimes \Psi(t)Y(t))P_2(Z^{-1}(s) \otimes Y^{-1}(s)\Psi^{-1}(s))| ds.$$

From the hypothesis (1), the integral

$$\int_t^\infty |(Z(t) \otimes \Psi(t)Y(t))P_2(Z^{-1}(s) \otimes Y^{-1}(s)\Psi^{-1}(s))| ds$$

is convergent. Thus, the operator $(T\hat{X})(t)$ exists and is continuous for $t \geq 0$. For $\hat{X} \in \mathbb{S}$ and $t \geq 0$, we have

$$\begin{aligned} & \|(I_n \otimes \Psi(t))(T\hat{X})(t)\| \\ & \leq \left\| \int_0^t (I_n \otimes \Psi(t))(Z(t) \otimes Y(t))P_1(Z^{-1}(s) \otimes Y^{-1}(s)) \right. \\ & \quad \left. (I_n \otimes \Psi^{-1}(s))(I_n \otimes \Psi(s))G(s, \hat{X}(s)) ds \right\| \\ & + \left\| \int_t^\infty (I_n \otimes \Psi(t))(Z(t) \otimes Y(t))P_2(Z^{-1}(s) \otimes Y^{-1}(s)) \right. \\ & \quad \left. (I_n \otimes \Psi^{-1}(s))(I_n \otimes \Psi(s))G(s, \hat{X}(s)) ds \right\| \\ & \leq \int_0^t |(Z(t) \otimes \Psi(t)Y(t))P_1(Z^{-1}(s) \otimes Y^{-1}(s)\Psi^{-1}(s))| \\ & \quad \|(I_n \otimes \Psi(s))G(s, \hat{X}(s))\| ds \\ & + \int_t^\infty |(Z(t) \otimes \Psi(t)Y(t))P_2(Z^{-1}(s) \otimes Y^{-1}(s)\Psi^{-1}(s))| \\ & \quad \|(I_n \otimes \Psi(s))G(s, \hat{X}(s))\| ds \\ & \leq nM \int_0^t |(Z(t) \otimes \Psi(t)Y(t))P_1(Z^{-1}(s) \otimes Y^{-1}(s)\Psi^{-1}(s))| \\ & \quad \|(I_n \otimes \Psi(s))\hat{X}(s)\| ds \\ & + nM \int_t^\infty |(Z(t) \otimes \Psi(t)Y(t))P_2(Z^{-1}(s) \otimes Y^{-1}(s)\Psi^{-1}(s))| \\ & \quad \|(I_n \otimes \Psi(s))\hat{X}(s)\| ds \\ & \leq p \sup_{t \geq 0} \|(I_n \otimes \Psi(t))\hat{X}(t)\|. \end{aligned}$$

Therefore,

$$\|T\hat{X}\|_{\mathbb{S}} \leq p\|\hat{X}\|_{\mathbb{S}}.$$

Thus, $T\mathbb{S} \subseteq \mathbb{S}$. On the other hand, for $\hat{U}, \hat{V} \in \mathbb{S}$ and $t \geq 0$, we have

$$\begin{aligned} & \| (I_n \otimes \Psi(t))[(T\hat{U})(t) - (T\hat{V})(t)] \| \\ & \leq \int_0^t |(Z(t) \otimes \Psi(t)Y(t))P_1(Z^{-1}(s) \otimes Y^{-1}(s)\Psi^{-1}(s))| \\ & \qquad \| (I_n \otimes \Psi(s))[G(s, \hat{U}(s)) - G(s, \hat{V}(s))] \| ds \\ & + \int_t^\infty |(Z(t) \otimes \Psi(t)Y(t))P_2(Z^{-1}(s) \otimes Y^{-1}(s)\Psi^{-1}(s))| \\ & \qquad \| (I_n \otimes \Psi(s))[G(s, \hat{U}(s)) - G(s, \hat{V}(s))] \| ds \\ & \leq nM \int_0^t |(Z(t) \otimes \Psi(t)Y(t))P_1(Z^{-1}(s) \otimes Y^{-1}(s)\Psi^{-1}(s))| \\ & \qquad \| (I_n \otimes \Psi(s))[\hat{U}(s) - \hat{V}(s)] \| ds \\ & + nM \int_t^\infty |(Z(t) \otimes \Psi(t)Y(t))P_2(Z^{-1}(s) \otimes Y^{-1}(s)\Psi^{-1}(s))| \\ & \qquad \| (I_n \otimes \Psi(s))[\hat{U}(s) - \hat{V}(s)] \| ds \\ & \leq nM \left(\sup_{t \geq 0} \| (I_n \otimes \Psi(t))[\hat{U}(t) - \hat{V}(t)] \| \right) \left\{ \int_0^t |(Z(t) \otimes \Psi(t)Y(t))P_1 \right. \\ & (Z^{-1}(s) \otimes Y^{-1}(s)\Psi^{-1}(s))| ds \\ & \left. + \int_t^\infty |(Z(t) \otimes \Psi(t)Y(t))P_2(Z^{-1}(s) \otimes Y^{-1}(s)\Psi^{-1}(s))| ds \right\} \\ & \leq p \sup_{t \geq 0} \| (I_n \otimes \Psi(t))[\hat{U}(t) - \hat{V}(t)] \|. \end{aligned}$$

It follows that

$$\sup_{t \geq 0} \| (I_n \otimes \Psi(t))[(T\hat{U})(t) - (T\hat{V})(t)] \| \leq p \sup_{t \geq 0} \| (I_n \otimes \Psi(t))[\hat{U}(t) - \hat{V}(t)] \|.$$

Thus, we have

$$\|T\hat{U} - T\hat{V}\|_{\mathbb{S}} \leq p\|\hat{U} - \hat{V}\|_{\mathbb{S}}.$$

Therefore, T is a contraction mapping on $(\mathbb{S}, \|\cdot\|_{\mathbb{S}})$. Now, for any function $\hat{W} \in \mathbb{S}$, we define an operator $\mathbb{S}_{\hat{W}} : \mathbb{S} \rightarrow \mathbb{S}$, by the relation $\mathbb{S}_{\hat{W}}\hat{X}(t) = \hat{W}(t) + (T\hat{X})(t)$, $\forall t \in \mathbb{R}_+$. By Banach contraction principle $\mathbb{S}_{\hat{W}}$ has fixed point in \mathbb{S} . Therefore, for any $\hat{W} \in \mathbb{S}$, the integral equation

$$\hat{X} = \hat{W} + T\hat{X} \tag{4.1}$$

has a unique solution $\hat{X} \in \mathbb{S}$. Furthermore, by the definition of T , $\hat{X}(t) - \hat{W}(t)$ is differentiable and

$$\left(\hat{X}(t) - \hat{W}(t)\right)' = (B^T(t) \otimes I_n + I_n \otimes A(t)) \left(\hat{X}(t) - \hat{W}(t)\right) + G(t, \hat{X}(t)).$$

From (4.1), if $\hat{W}(t)$ is a $I_n \otimes \Psi$ -bounded solution of (2.2) if and only if $\hat{X}(t)$ is a $I_n \otimes \Psi$ -bounded solution of (2.1). Thus, (4.1) establishes a one-to-one correspondence between the $I_n \otimes \Psi$ -bounded solutions of (2.1) and (2.2). Now, we consider analogous equation

$$\hat{X}_0 = \hat{W}_0 + T\hat{X}_0.$$

We get

$$(1-p)\|\hat{X} - \hat{X}_0\|_{\mathbb{S}} \leq \|\hat{W} - \hat{W}_0\|_{\mathbb{S}}. \quad (4.2)$$

Now, we prove that, if $\hat{X}, \hat{W} \in \mathbb{S}$ are $I_n \otimes \Psi$ -bounded solutions of (2.1) and (2.2) respectively such that they satisfy (4.1), then

$$\lim_{t \rightarrow \infty} \|(I_n \otimes \Psi(t))(\hat{X}(t) - \hat{W}(t))\| = 0. \quad (4.3)$$

For a given $\epsilon > 0$, we can choose $t_1 \geq 0$ such that

$$p\|\hat{X}\|_{\mathbb{S}} < \frac{\epsilon}{2}, \text{ for } t \geq t_1.$$

Moreover, since $\lim_{t \rightarrow \infty} |(I_n \otimes \Psi(t))(Z(t) \otimes Y(t))P_1| = 0$, there exists a number $t_2 \geq t_1$ such that

$$pL^{-1}|(Z(t) \otimes \Psi(t)Y(t))P_1|\|\hat{X}\|_{\mathbb{S}} \int_0^{t_1} |P_1(Z^{-1}(s) \otimes Y^{-1}(s)\Psi^{-1}(s))| ds < \frac{\epsilon}{2}, t \geq t_2.$$

For $t \geq t_2$, we have

$$\begin{aligned} & \|(I_n \otimes \Psi(t))(\hat{X}(t) - \hat{W}(t))\| = \|(I_n \otimes \Psi(t))(T\hat{X})(t)\| \\ & \leq \int_0^t |(Z(t) \otimes \Psi(t)Y(t))P_1(Z^{-1}(s) \otimes Y^{-1}(s)\Psi^{-1}(s))| \\ & \quad \|(I_n \otimes \Psi(s))G(s, \hat{X}(s))\| ds \\ & + \int_t^\infty |(Z(t) \otimes \Psi(t)Y(t))P_2(Z^{-1}(s) \otimes Y^{-1}(s)\Psi^{-1}(s))| \\ & \quad \|(I_n \otimes \Psi(s))G(s, \hat{X}(s))\| ds \\ & \leq nM \int_0^t |(Z(t) \otimes \Psi(t)Y(t))P_1(Z^{-1}(s) \otimes Y^{-1}(s)\Psi^{-1}(s))| \\ & \quad \|(I_n \otimes \Psi(s))\hat{X}(s)\| ds \\ & + nM \int_t^\infty |(Z(t) \otimes \Psi(t)Y(t))P_2(Z^{-1}(s) \otimes Y^{-1}(s)\Psi^{-1}(s))| \\ & \quad \|(I_n \otimes \Psi(s))\hat{X}(s)\| ds \\ & \leq pL^{-1}|(Z(t) \otimes \Psi(t)Y(t))P_1|\|\hat{X}\|_{\mathbb{S}} \int_0^{t_1} |P_1(Z^{-1}(s) \otimes Y^{-1}(s)\Psi^{-1}(s))| ds \end{aligned}$$

$$\begin{aligned} & +nM\|\hat{X}\|_S \int_{t_1}^t |(Z(t) \otimes \Psi(t)Y(t))P_1(Z^{-1}(s) \otimes Y^{-1}(s)\Psi^{-1}(s))|ds \\ & +nM\|\hat{X}\|_S \int_t^\infty |(Z(t) \otimes \Psi(t)Y(t))P_2(Z^{-1}(s) \otimes Y^{-1}(s)\Psi^{-1}(s))|ds \\ & \leq \frac{\epsilon}{2} + nML\|\hat{X}\|_S < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Now, we prove that, if $\hat{X}(t)$ is a $I_n \otimes \Psi$ -bounded solution of (2.1), then it is $I_n \otimes \Psi$ -unstable on \mathbb{R}_+ .

Suppose that $\hat{X}(t)$ is $I_n \otimes \Psi$ -stable on \mathbb{R}_+ . From Definition 2.6, for every $\epsilon > 0$ and any $t_0 \in \mathbb{R}_+$, there exists a $\delta = \delta(\epsilon, t_0) > 0$ such that any solution $\tilde{X}(t)$ of (2.1), which satisfies the inequality $\|(I_n \otimes \Psi(t_0))(\tilde{X}(t_0) - \hat{X}(t_0))\| < \delta(\epsilon, t_0)$ exists and satisfies the inequality $\|(I_n \otimes \Psi(t))(\tilde{X}(t) - \hat{X}(t))\| < \epsilon$, for all $t \geq t_0$.

Let $u_0 \in \mathbb{R}^{n^2}$ be such that $P_1 u_0 = 0_{n^2}$ and $|(I_n \otimes \Psi(0))u_0| < \delta(\epsilon, 0)$ and let $\tilde{X}(t)$ be the solution of (2.1) with the initial condition $\tilde{X}(0) = \hat{X}(0) + u_0$. Then $\|(I_n \otimes \Psi(t))u(t)\| < \epsilon$, for all $t \geq 0$, where $u(t) = \tilde{X}(t) - \hat{X}(t)$.

Now consider the function $w(t) = u(t) - Tu(t)$, $t \geq 0$. Clearly, $w(t)$ is a $I_n \otimes \Psi$ -bounded solution of (2.2) on \mathbb{R}_+ . Without loss of generality, we can suppose that $Z(0) \otimes Y(0) = I_{n^2}$. It is easy to see that $P_1 w(0) = 0_{n^2}$. If $P_2 w(0) \neq 0_{n^2}$, then from Lemma 2.3 of [12], we have

$$\limsup_{t \rightarrow \infty} \|(I_n \otimes \Psi(t))(Z(t) \otimes Y(t))P_2 w(0)\| = \limsup_{t \rightarrow \infty} \|(I_n \otimes \Psi(t))w(t)\| = \infty,$$

which is contradiction to $w(t)$ is $I_n \otimes \Psi$ -bounded on \mathbb{R}_+ . Thus, $P_2 w(0) = 0_{n^2}$ and hence $w(t) = 0_{n^2}$, for $t \geq 0$. It follows that $u = Tu$ and $u = 0_{n^2}$ (T is linear), which is a contradiction. Thus the solution $\hat{X}(t)$ is $I_n \otimes \Psi$ -unstable on \mathbb{R}_+ .

Let $\hat{W} = \hat{X} - T\hat{X}$. From Theorem 3.5 and Definition 2.8, there exists a sequence $\{\hat{W}_m\}$ of solutions of (2.2) on \mathbb{R}_+ such that

$$\lim_{m \rightarrow \infty} (I_n \otimes \Psi(t))\hat{W}_m(t) = (I_n \otimes \Psi(t))\hat{W}(t), \text{ uniformly on } \mathbb{R}_+$$

and

$$\lim_{t \rightarrow \infty} (I_n \otimes \Psi(t)) \left(\hat{W}_m(t) - \hat{W}(t) \right) = 0_{n^2}, \text{ for } m = 1, 2, 3, \dots$$

Let $\hat{X}_m = \hat{W}_m + T\hat{X}_m$. From (4.2), it follows that the sequence $\{\hat{X}_m\}$ of solutions of (2.1) on \mathbb{R}_+ such that

$$\lim_{m \rightarrow \infty} (I_n \otimes \Psi(t))\hat{X}_m(t) = (I_n \otimes \Psi(t))\hat{X}(t), \text{ uniformly on } \mathbb{R}_+.$$

Therefore, the solution $\hat{X}(t)$ of (2.1) is $I_n \otimes \Psi$ -conditionally stable on \mathbb{R}_+ . From (4.3) and

$$\left(\hat{X}_m(t) - \hat{X}(t) \right) = \left(\hat{X}_m(t) - \hat{W}_m(t) \right) + \left(\hat{W}_m(t) - \hat{W}(t) \right) + \left(\hat{W}(t) - \hat{X}(t) \right).$$

It follows that

$$\lim_{t \rightarrow \infty} (I_n \otimes \Psi(t)) \left(\hat{X}_m(t) - \hat{X}(t) \right) = 0_{n^2}, \text{ for } m = 1, 2, 3, \dots$$

Thus, the solution $\hat{X}(t)$ of (2.1) is $I_n \otimes \Psi$ -conditionally asymptotically stable on \mathbb{R}_+ . From Lemma 2.16, the solution $X(t) = Vec^{-1}\hat{X}(t)$ of (1.1) is Ψ -conditionally asymptotically stable on \mathbb{R}_+ . Hence the system (1.1) is Ψ -conditionally asymptotically stable on \mathbb{R}_+ . \square

Example 4.2. Consider the non-linear matrix Lyapunov system (1.1) with

$$A(t) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{-1}{t+1} \end{pmatrix}, \quad B(t) = \begin{pmatrix} \frac{e^t(t-1)}{t(e^t+1)} & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$F(t, X) = \frac{1}{t+5} \begin{pmatrix} \sin x_1(t) & x_2(t) \\ x_3(t) & \sin x_4(t) \end{pmatrix}.$$

The fundamental matrices of (1.3) and (1.4) are

$$Y(t) = \begin{pmatrix} e^t & 0 \\ 0 & \frac{1}{t+1} \end{pmatrix} \quad \text{and} \quad Z(t) = \begin{pmatrix} \frac{e^t}{t+1} & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

Let

$$\Psi(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & t+1 \end{pmatrix}.$$

Then there exist two projections

$$P_1 = \begin{pmatrix} O_2 & O_2 \\ O_2 & I_2 \end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix} I_2 & O_2 \\ O_2 & O_2 \end{pmatrix}$$

such that the fundamental matrices $Y(t)$ and $Z(t)$ of (1.3) and (1.4) satisfies (3.2) with $L = 2$.

On the otherhand, condition (ii) of Theorem 4.1 is satisfied with $\xi(t) = \frac{1}{t+5}$, for $t \geq 0$ and $M = \frac{1}{5}$. Also, $p = nML = 2 \left(\frac{1}{5}\right) 2 = \frac{4}{5} < 1$. Therefore, the non-linear system (1.1) is Ψ -conditionally asymptotically stable on \mathbb{R}_+ .

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