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Stability for certain subclasses of harmonic univalent functions

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Abstract

In this paper, the problem of stability for certain subclasses of harmonic univalent functions is investigated. Some lower bounds for the radius of stability of these subclasses are found.

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1. Introduction and preliminaries

A complex-valued harmonic function $F = u + iv$ in the unit disk $\mathbb{D} = \{z \; : \; |z| < 1\}$ admits the decomposition $F = h + \overline{g}$, where both h and g are analytic in $\mathbb{D}$ (see [9]). Here h and g are referred to as analytic and co-analytic parts of f. A complex-valued harmonic function $F(z) = h(z) + \overline{g(z)}$ is locally univalent if and only if the Jacobian $J_F(z) = |h'(z)|^2 - |g'(z)|^2$ is non-vanishing in $\mathbb{D}$. The reader is referred to [9, 11] for the properties of harmonic functions.

Let $\mathcal{H}$ be the class of complex-valued harmonic functions in $\mathbb{D}$ such that $F(0) = 0$ and $F_z(0) = 1$. Then every function $F \in \mathcal{H}$ can be expressed as the form:

$$F(z) = h(z) + \overline{g(z)} = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}. \tag{1.1}$$

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The class of functions $F \in \mathcal{H}$ that are sense-preserving and univalent in $\mathbb{D}$ is denoted by $\mathcal{S}_H$. Also, let

$$\mathcal{S}_H^* = \{F \in \mathcal{S}_H \ : \ F(\mathbb{D}) \text{ is a starlike domain with respect to the origin}\}.$$

Functions in $\mathcal{S}_H^*$ are called starlike functions. In the sequel, we also need

$$\mathcal{H}_1 = \{F \in \mathcal{H} \ : \ b_1 = F_{\overline{z}}(0) = 0\}, \quad \mathcal{S}_H^0 = \{F \in \mathcal{S}_H \ : \ F_{\overline{z}}(0) = 0\}, \quad \mathcal{S}_H^{*0} = \{F \in \mathcal{S}_H^* \ : \ F_{\overline{z}}(0) = 0\}.$$

Harmonic starlikeness is not a hereditary property, because it is possible that for $f \in \mathcal{S}_H^*$, $f(|z| < r)$ is not necessarily starlike for each $r < 1$ (see [11]).

Definition 1.1. A harmonic mapping $f \in \mathcal{H}$ is said to be *fully starlike* (resp. *fully convex*) if each $|z| < r$ is mapped onto a starlike (resp. convex) domain (see [8]).

Fully convex mappings are known to be fully starlike but not the converse as the function $f(z) = z + (1/n)\overline{z}^n$ $(n \geq 2)$ shows.

It is easy to see that the harmonic koebe function K with the dilation $w(z) = z$ is not fully starlike, although $K = H + \overline{G} \in \mathcal{S}_H^{*0}$, where

$$H(z) = \frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}, \quad G(z) = \frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}.$$

For further details, we refer to [8].

Let C_H^0 denote the class of harmonic,univalent, convex functions F of the form (1.1) with $b_1 = 0$. It is known [9] that the below sharp inequalities hold:

$$|a_n| \leq \frac{n+1}{2}, \quad |b_n| \leq \frac{n-1}{2}. \tag{1.2}$$

In the sequel, we need

$$\mathcal{FS}_H^{*0} = \{F \in \mathcal{S}_H^0 \ : \ \text{F is fully starlike in } \mathbb{D}\}, \quad \mathcal{C}_H^1 = \{F \in \mathcal{S}_H \ : \ Re\ F_z(z) > |F_{\overline{z}}(z)| \ in\ \mathbb{D}\}.$$

Definition 1.2. Let $0 \leq \lambda \leq 1$. A function $F \in \mathcal{H}_1$ with the form (1.1) is said to be in the class $HS^0(\lambda)$ if

$$\sum_{n=2}^{\infty} n(\lambda n + 1 - \lambda)(|a_n| + |b_n|) \leq 1.$$

The class $HS^0(\lambda)$ is a special case of the class $HS_p^0(\lambda)$ of polyharmonic mappings (see [7]). If $\lambda = 0$ or $\lambda = 1$, then the class $HS^0(\lambda)$ reduces to HS^0 or HC^0, respectively. The classes HS^0 and HC^0 introduced by Avci and Złotkiewicz [3].

If

$$F(z) = z + \sum_{n=2}^{\infty} \left(a_n z^n + \overline{b_n} \overline{z^n}\right),$$

and

$$G(z) = z + \sum_{n=2}^{\infty} \left(A_n z^n + \overline{B_n} \overline{z^n}\right),$$

then the *convolution* $F * G$ is defined to be the function

$$(F * G)(z) = z + \sum_{n=2}^{\infty} \left(a_n A_n z^n + \overline{b_n B_n z^n} \right), \tag{1.3}$$

while the *integral convolution* is defined by

$$(F \diamond G)(z) = z + \sum_{n=2}^{\infty} \left(\frac{a_n A_n}{n} z^n + \frac{\overline{b_n B_n}}{n} \overline{z^n} \right). \tag{1.4}$$

See [10] for similar operators defined on the class of analytic functions.

For $V \subset \mathcal{H}_1$, its dual V^* is defined as

$$\mathcal{V}^* = \left\{ G \in \mathcal{H}_1 : (F * G)(z) \neq 0, \text{ for all } z \in \check{\mathbb{D}},\ f \in \mathcal{V} \right\},$$

where $\check{\mathbb{D}} = \mathbb{D} \setminus \{0\}$. We say that V is a dual class if $V = W^*$ for some $W \subset \mathcal{H}_1$ (see [2]). Denote by Σ the dual set of $\mathcal{S}_H^{*0}$. Then for $F \in \mathcal{H}_1$, we have

$$F \in \mathcal{S}_H^{*0} \Longleftrightarrow (F * H)(z) \neq 0, \forall H \in \Sigma, \forall z \in \check{\mathbb{D}}.$$

Following Goodman [12] and Ruscheweyeh [13], we define the set $\delta-$neighborhood of $F = h + \overline{g} \in \mathcal{H}_1$ by

$$N_\delta(F) = \left\{ G(z) \ : \ G(z) = z + \sum_{n=2}^{\infty} \left(A_n z^n + \overline{B_n} \overline{z^n} \right), \ \sum_{n=2}^{\infty} n(|a_n - A_n| + |b_n - B_n|) \leq \delta, \ \delta \geq 0 \right\}$$

(see [14]). Also, let

$$\widetilde{N}_\delta(F) = \left\{ G(z) \ : \ G(z) = z + \sum_{n=2}^{\infty} \left(A_n z^n + \overline{B_n} \overline{z^n} \right), \ \sum_{n=2}^{\infty} n^2(|a_n - A_n| + |b_n - B_n|) \leq \delta, \ \delta \geq 0 \right\}.$$

Clearly, we have $\widetilde{N}_\delta(F) \subset N_\delta(F)$.

By $N_\delta(A)$, $A \subset \mathcal{H}_1$, we denote the union of all neighborhoods $N_\delta(F)$ with F ranging over the class A. And similarly, define $\widetilde{N}_\delta(A) = \cup_{F \in A} \widetilde{N}_\delta(F)$.

Assume that A, B are subclasses of the class $\mathcal{H}_1$. Then the set of all functions $F * G$ and $F \diamond G$, where $F \in A$ and $G \in B$, will be denoted by $A * B$ and $A \diamond B$, respectively. Let $A * B \subset C$, the *convolution* is called stabel on the pair of classes (A, B) if there exists $\delta > 0$ such that $N_\delta(A) * N_\delta(B) \subset C$ and unstable otherwise. Stability of the *integral convolution* is defined in a similar way.

The constant δ which characterizes the stability of the *convolution* or *integral convolution* is called the radius of stability and it is defined as follows.

Definition 1.3. Let A, B, C be the subclasses of the class $\mathcal{H}_1$ and $A * B \subset C$. Then a constant $\delta(A * B, C)$, such that

$$\delta(A * B, C) = \sup\{\delta : N_\delta(A) * N_\delta(B) \subset C\},$$

is called the radius of stability of the *convolution* on the pair $(A,\ B)$. And a constant $\delta(A \diamond B, C)$, such that

$$\delta(A \diamond B, C) = \sup\{\delta : N_\delta(A) \diamond N_\delta(B) \subset C\},$$

is called the radius of stability of the *integral convolution* on the pair $(A,\ B)$.

Remark 1.4. In a same way as in the above we have

$$\widetilde{\delta}(A * B, C) = \sup\{\delta : \widetilde{N}_\delta(A) * \widetilde{N}_\delta(B) \subset C\}$$

$$\widetilde{\delta}(A \diamond B, C) = \sup\{\delta : \widetilde{N}_\delta(A) \diamond \widetilde{N}_\delta(B) \subset C\}.$$

Recently, in [1, 4, 5], the authors investigated the problem of stability for certain classes of analytic functions. In this paper, we investigate the problem of stability for certain classes of harmonic univalent functions. We find the lower bounds for the radius of stability of these classes.

2. main results

In order to establish our main theorems, we shall require the following lemmas.

Lemma 2.1. (see [6]) Let $F = h + \overline{g} \in \mathcal{S}_H^0$. Then F is fully starlike in $\mathbb{D}$ if and only if

$$h(z) * A(z) - \overline{g(z)} * \overline{B(z)} \neq 0 \quad for\ |\zeta| = 1,\ 0 < |z| < 1,$$

where

$$A(z) = \frac{z + ((\zeta - 1)/2)z^2}{(1-z)^2} \quad and \quad B(z) = \frac{\overline{\zeta} z - ((\overline{\zeta} - 1)/2)z^2}{(1-z)^2}. \tag{2.1}$$

Corollary 2.2. *Let $F = h + \overline{g} \in \mathcal{S}_H^0$. Then $F \in \mathcal{FS}_H^{*0}$ if and only if $(F * H)(z) \neq 0$ for $|\zeta| = 1,\ z \in \check{\mathbb{D}}$, where $H(z) = A(z) - \overline{B(z)}$ and $A(z),\ B(z)$ are given by (2.1).*

Proof . From Lemma 2.1 and the definition of the convolution of harmonic functions, immediately, the result follows. □

Corollary 2.3. *Suppose that*

$$\Sigma = \left\{ H(z) \in \mathcal{H}_1 \ : \ H(z) = A(z) - \overline{B(z)} \right\},$$

where $A(z)$ and $B(z)$ are given by (2.1). Then $\mathcal{FS}_H^{*0} = \Sigma^*$.

Proof . The proof is obvious. In view of the definition of dual set and Corollary 2.2 , we can easily obtain the result. □

Lemma 2.4. *Let $H(z) = z + \sum_{n=2}^{\infty}(e_n z^n + \overline{f_n z^n}) \in \Sigma$. Then $|e_n| \leq n$ and $|f_n| \leq n$.*

Proof . Since $H(z) \in \Sigma$, then we have $H(z) = A(z) - \overline{B(z)}$. From the series expansion $A(z)$ and $B(z)$ we obtain

$$\begin{aligned} H(z) &= A(z) - \overline{B(z)} \\ &= \frac{z + ((\zeta - 1)/2)z^2}{(1-z)^2} - \overline{\frac{\overline{\zeta} z - ((\overline{\zeta} - 1)/2)z^2}{(1-z)^2}} \\ &= z + \sum_{n=2}^{\infty} \left(n + \frac{(n-1)(\zeta - 1)}{2} \right) z^n - \sum_{n=2}^{\infty} \left(n\zeta - \frac{(n-1)(\zeta-1)}{2} \right) \overline{z^n} \\ &= z + \sum_{n=2}^{\infty} \left(\left(n + \frac{(n-1)(\zeta - 1)}{2} \right) z^n - \left(n\zeta - \frac{(n-1)(\zeta-1)}{2} \right) \overline{z^n} \right). \end{aligned}$$

Therefore, we have

$$e_n = n + \frac{(n-1)(\zeta-1)}{2}, \ \overline{f_n} = n\zeta - \frac{(n-1)(\zeta-1)}{2}.$$

Consequently,

$$\begin{aligned}
|e_n| &= \left| n + \frac{(n-1)(\zeta-1)}{2} \right| \\
&= \left| \frac{2n + (n-1)(\zeta-1)}{2} \right| \\
&= \left| \frac{n+1+(n-1)\zeta}{2} \right| \\
&\le \frac{n+1+(n-1)|\zeta|}{2} \\
&\le \frac{n+1+n-1}{2} = n,
\end{aligned}$$

and similarly, we get $|f_n| \le n$. This completes the proof. □

Lemma 2.5. (see [6]) Let $F = h + \overline{g}$ be a harmonic function of the form (1.1) with $b_1 = g'(0) = 0$. If

$$\sum_{n=2}^{\infty} n(|a_n| + |b_n|) \le 1,$$

then $F \in \mathcal{C}_H^1 \cap \mathcal{S}_H^{*0}$. Moreover, F is fully starlike in $\mathbb{D}$. Consequently, $F \in \mathcal{FS}_H^{*0}$.

Lemma 2.6. *Let* $F = h + \overline{g} \in HS^0(\lambda)$ *be of the form* (1.1), *then*

$$\sum_{n=2}^{\infty} n(|a_n| + |b_n|) \le \frac{1}{\lambda+1},$$

and

$$\sum_{n=2}^{\infty} n^2(|a_n| + |b_n|) \le \frac{1}{\lambda}.$$

Proof . Since $0 \le \lambda \le 1$ and $\lambda n + 1 - \lambda$ is an increasing function of n ($n \ge 2$), from the definition of the class $HS^0(\lambda)$, the result follows. □

Lemma 2.7. *Let* $F = h + \overline{g} \in HS^0(\lambda)$ *be of the form* (1.1), *then*

$$|a_n| \le \frac{1}{2(\lambda+1)}, \quad |b_n| \le \frac{1}{2(\lambda+1)}.$$

Proof . From Lemma 2.6, we obtain

$$\sum_{n=2}^{\infty} n(|a_n| + |b_n|) \le \frac{1}{\lambda+1},$$

and therefore

$$\sum_{n=2}^{\infty} n|a_n| \leq \frac{1}{\lambda+1}, \tag{2.2}$$

and

$$\sum_{n=2}^{\infty} n|b_n| \leq \frac{1}{\lambda+1}. \tag{2.3}$$

From the inequalities (2.2) and (2.3), it follows that

$$\sum_{n=2}^{\infty} |a_n| \leq \frac{1}{2(\lambda+1)},$$

and

$$\sum_{n=2}^{\infty} |b_n| \leq \frac{1}{2(\lambda+1)}.$$

The above inequalities, give the desired result. □

Lemma 2.8. (see [7]) Suppose that $G(z) = z + \sum_{n=2}^{\infty}(A_n z^n + \overline{B_n}\overline{z^n}) \in C_H^0$ and $F \in HS^0(\lambda)$. Then for $1/2 \leq \lambda \leq 1$, the convolution $F * G$ is univalent and starlike, and the integral convolution $F \diamond G$ is convex.

Corollary 2.9. For $1/2 \leq \lambda \leq 1$, we have

$$C_H^0 * HS^0(\lambda) \subseteq \mathcal{FS}_H^{*0}, \quad C_H^0 \diamond HS^0(\lambda) \subseteq C_H^0.$$

Lemma 2.10. *For $0 \leq \lambda \leq 1$, we have*
(i) $C_H^0 \diamond HS^0(\lambda) \subseteq \mathcal{FS}_H^{*0}$.
(ii) $HS^0(\lambda) * HS^0(\lambda) \subseteq \mathcal{FS}_H^{*0}$.
(iii) $HC^0 * HS^0(\lambda) \subseteq \mathcal{FS}_H^{*0}$.
(iv) $HS^0(\lambda) \diamond HS^0(\lambda) \subseteq \mathcal{FS}_H^{*0}$.
(v) $HC^0 \diamond HS^0(\lambda) \subseteq \mathcal{FS}_H^{*0}$.

Proof . We only prove the parts (i) and (ii). The other parts are proved in a similar way.
(i) Let $F(z) = z + \sum_{n=2}^{\infty}(A_n z^n + \overline{B_n}\overline{z^n}) \in C_H^0$ and $G(z) = z + \sum_{n=2}^{\infty}(a_n z^n + \overline{b_n}\overline{z^n}) \in HS^0(\lambda)$. Then for $F \diamond G$, by Lemma 2.6 and the inequalities (1.2), we obtain

$$\begin{aligned}\sum_{n=2}^{\infty} n\left(\left|\frac{a_n A_n}{n}\right| + \left|\frac{b_n B_n}{n}\right|\right) &\leq \sum_{n=2}^{\infty} n\left(\frac{n+1}{2n}|a_n| + \frac{n-1}{2n}|b_n|\right) \\ &\leq \sum_{n=2}^{\infty} n(|a_n| + |b_n|) \\ &\leq \frac{1}{\lambda+1} \leq 1.\end{aligned}$$

Now, from Lemma 2.5, the result follows.
(ii) If $F, G \in HS^0(\lambda)$, then for $F * G$, using Lemma 2.6 and Lmma 2.7, we obtain

$$\begin{aligned}\sum_{n=2}^{\infty} n\left(|a_n A_n| + |b_n B_n|\right) &\leq \frac{1}{2(\lambda+1)} \sum_{n=2}^{\infty} n\left(|a_n| + |b_n|\right)\\ &\leq \frac{1}{2} \sum_{n=2}^{\infty} n\left(|a_n| + |b_n|\right)\\ &\leq \frac{1}{2(\lambda+1)} < 1.\end{aligned}$$

Hence, by Lemma 2.5, $F * G \in \mathcal{FS}_H^{*0}$. □

Theorem 2.11. *Let $0 \leq \lambda \leq 1$. For $0 \leq \delta \leq \sqrt{2} - \dfrac{1}{\lambda+1}$, we have*

$$N_\delta(HS^0(\lambda)) * N_\delta(HS^0(\lambda)) \subset \mathcal{FS}_H^{*0}.$$

Proof . Let

$$F_0(z) = z + \sum_{n=2}^{\infty} (a_{0n} z^n + \overline{b}_{0n} \overline{z^n}) \in HS^0(\lambda),$$

$$G_0(z) = z + \sum_{n=2}^{\infty} (c_{0n} z^n + \overline{d}_{0n} \overline{z^n}) \in HS^0(\lambda)$$

and

$$\begin{aligned}F(z) &= z + \sum_{n=2}^{\infty} (a_n z^n + \overline{b}_n \overline{z^n}) \in N_\delta(F_0), G(z)\\ &= z + \sum_{n=2}^{\infty} (c_n z^n + \overline{d}_n \overline{z^n}) \in N_\delta(G_0),\end{aligned}$$

$$H(z) = z + \sum_{n=2}^{\infty} (e_n z^n + \overline{f}_n \overline{z^n}) \in \Sigma.$$

We want to show that

$$(F * G * H)(z) \neq 0 \quad (H \in \Sigma,\ z \in \check{\mathbb{D}}).$$

By the identity

$$F * G * H = F_0 * G_0 * H + F_0 * (G - G_0) * H + (F - F_0) * G_0 * H + (F - F_0) * (G - G_0) * h,$$

we obtain

$$\begin{aligned}|(F * G * H)(z)| &\geq |(F_0 * G_0 * H)(z)| - |(F_0 * (G - G_0) * H)(z)|\\ &\quad - |((F - F_0) * G_0 * H)(z)| - |((F - F_0) * (G - G_0) * H)(z)|. \end{aligned} \tag{2.4}$$

Since $G_0 \in HS^0(\lambda)$, so by Lemma 2.7 we have $|c_{0n}| \leq \dfrac{1}{2(\lambda+1)}$ and $|d_{0n}| \leq \dfrac{1}{2(\lambda+1)}$. Moreover from Lemma 2.4, $|e_n| \leq n$ and $|f_n| \leq n$. Therefore, using Lemma 2.6 ,we obtain

$$\begin{aligned}
|(F_0 * G_0 * H)(z)| &= \left| z + \sum_{n=2}^{\infty} (a_{0n} c_{0n} e_n z^n + \overline{b_{0n} d_{0n} f_{0n}} \overline{z^n}) \right| \\
&\geq |z| \left[1 - \sum_{n=2}^{\infty} (|a_{0n}||c_{0n}||e_n||z|^{n-1} + |b_{0n}||d_{0n}||f_n||z|^{n-1}) \right] \\
&> |z| \left[1 - \sum_{n=2}^{\infty} (|a_{0n}||c_{0n}||e_n| + |b_{0n}||d_{0n}||f_n|) \right] \\
&\geq |z| \left[1 - \frac{1}{2(\lambda+1)} \sum_{n=2}^{\infty} n(|a_{0n}| + |b_{0n}|) \right] \\
&\geq |z| \left[1 - \frac{1}{2(\lambda+1)^2} \right] \\
&= |z| \left[\frac{2(\lambda+1)^2 - 1}{2(\lambda+1)^2} \right]. \qquad (2.5)
\end{aligned}$$

On the other hand, from $F \in N_\delta(F_0)$ and $G \in N_\delta(G_0)$, we conclude that

$$\begin{aligned}
|(F - F_0) * G_0 * H)(z)| &= \left| \sum_{n=2}^{\infty} \left(c_{0n} e_n (a_n - a_{0n}) z^n + \overline{d_{0n} f_n (b_n - b_{0n})} \overline{z^n} \right) \right| \\
&< |z| \frac{1}{2(\lambda+1)} \sum_{n=2}^{\infty} n(|a_n - a_{0n}| + |b_n - b_{0n}|) \\
&\leq |z| \frac{\delta}{2(\lambda+1)}. \qquad (2.6)
\end{aligned}$$

Similarly, we get

$$|F_0 * (G - G_0) * H)(z)| < |z| \frac{\delta}{2(\lambda+1)}, \qquad (2.7)$$

and

$$|(F - F_0) * (G - G_0) * H)(z)| < |z| \frac{\delta^2}{2}. \qquad (2.8)$$

By virtue of (2.5),(2.6),(2.7) and (2.8), inequality (2.4) gives

$$|(F * G * H)(z)| \geq |z| \left[\frac{2(\lambda+1)^2 - 1}{2(\lambda+1)^2} - \frac{\delta}{\lambda+1} - \frac{\delta^2}{2} \right]. \qquad (2.9)$$

The right side of (2.9) is non-negative whenever

$$0 \leq \delta \leq \sqrt{2} - \frac{1}{\lambda+1}.$$

□

Corollary 2.12. For $0 \le \lambda \le 1$, we have

$$\delta(HS^0(\lambda) * HS^0(\lambda), \mathcal{FS}_H^{*0}) \ge \sqrt{2} - \frac{1}{\lambda + 1}.$$

Corollary 2.13. We have

$$\delta(HS^0 * HS^0, \mathcal{FS}_H^{*0}) \ge \sqrt{2} - 1.$$

Corollary 2.14. We have

$$\delta(HC^0 * HC^0, \mathcal{FS}_H^{*0}) \ge \frac{2\sqrt{2} - 1}{2}.$$

Theorem 2.15. *Let* $0 \le \lambda \le 1$. *For* $0 \le \delta \le \sqrt{\left[\frac{\lambda+3}{4(\lambda+1)}\right]^2 + \frac{4\lambda+3}{2(\lambda+1)}} - \frac{\lambda+3}{4(\lambda+1)}$, *we have*

$$N_\delta(HC^0) * N_\delta(HS^0(\lambda)) \subset \mathcal{FS}_H^{*0}.$$

Proof . Let

$$F_0(z) = z + \sum_{n=2}^{\infty}(a_{0n}z^n + \overline{b}_{0n}\overline{z^n}) \in HC^0,$$

$$G_0(z) = z + \sum_{n=2}^{\infty}(c_{0n}z^n + \overline{d}_{0n}\overline{z^n}) \in HS^0(\lambda)$$

and

$$F(z) = z + \sum_{n=2}^{\infty}(a_n z^n + \overline{b}_n\overline{z^n}) \in N_\delta(F_0), G(z) = z + \sum_{n=2}^{\infty}(c_n z^n + \overline{d}_n\overline{z^n}) \in N_\delta(G_0),$$

$$H(z) = z + \sum_{n=2}^{\infty}(e_n z^n + \overline{f}_n\overline{z^n}) \in \Sigma.$$

We need to show that

$$(F * G * H)(z) \ne 0 \quad (H \in \Sigma,\ z \in \check{\mathbb{D}}).$$

Using the same method as in the proof of Theorem 2.11, we obtain

$$|(F_0 * G_0 * H)(z)| > |z| \left[\frac{4\lambda+3}{4(\lambda+1)}\right],$$

$$|(F - F_0) * G_0 * H)(z)| < |z|\frac{\delta}{4},$$

$$|F_0 * (G - G_0) * H)(z)| < |z|\frac{\delta}{2(\lambda+1)},$$

and

$$|(F - F_0) * (G - G_0) * H)(z)| < |z|\frac{\delta^2}{2}.$$

The remainder of the proof is similar to that of Theorem 2.11 and we omit the details. □

Corollary 2.16. For $0 \leq \lambda \leq 1$, we have

$$\delta(HC^0 * HS^0(\lambda), \mathcal{FS}_H^{*0}) \geq \sqrt{\left[\frac{\lambda+3}{4(\lambda+1)}\right]^2 + \frac{4\lambda+3}{2(\lambda+1)}} - \frac{\lambda+3}{4(\lambda+1)}.$$

Corollary 2.17. We have

$$\delta(HC^0 * HS^0, \mathcal{FS}_H^{*0}) \geq \frac{\sqrt{33}-3}{4}.$$

Using the same techniques as in the proof of Theorems 2.11 and 2.15, we obtain the following theorems and we omit the details.

Theorem 2.18. Let $0 \leq \lambda \leq 1$. For $0 \leq \delta \leq 2 - \dfrac{1}{\lambda+1}$, we have

$$N_\delta(HS^0(\lambda)) \diamond N_\delta(HS^0(\lambda)) \subset \mathcal{FS}_H^{*0}.$$

Theorem 2.19. Let $0 \leq \lambda \leq 1$. For $0 \leq \delta \leq \sqrt{\left[\dfrac{\lambda+3}{4(\lambda+1)}\right]^2 + \dfrac{8\lambda+7}{2(\lambda+1)}} - \dfrac{\lambda+3}{4(\lambda+1)}$, we have

$$N_\delta(HC^0) \diamond N_\delta(HS^0(\lambda)) \subset \mathcal{FS}_H^{*0}.$$

From Theorems 2.18 and 2.19, we obtain the following results.

Corollary 2.20. Let $0 \leq \lambda \leq 1$. We have

$$\delta(HS^0(\lambda) \diamond HS^0(\lambda), \mathcal{FS}_H^{*0}) \geq 2 - \frac{1}{\lambda+1}.$$

Corollary 2.21. For $0 \leq \lambda \leq 1$, we have

$$\delta(HC^0 \diamond HS^0(\lambda), \mathcal{FS}_H^{*0}) \geq \sqrt{\left[\frac{\lambda+3}{4(\lambda+1)}\right]^2 + \frac{8\lambda+7}{2(\lambda+1)}} - \frac{\lambda+3}{4(\lambda+1)}.$$

Corollary 2.22. We have

$$\delta(HS^0 \diamond HS^0, \mathcal{FS}_H^{*0}) \geq 1.$$

Corollary 2.23. We have

$$\delta(HC^0 \diamond HC^0, \mathcal{FS}_H^{*0}) \geq \frac{3}{2}.$$

Corollary 2.24. We have

$$\delta(HC^0 \diamond HS^0, \mathcal{FS}_H^{*0}) \geq \frac{\sqrt{65}-3}{4}.$$

Theorem 2.25. *Let* $1/2 \leq \lambda \leq 1$. *For* $0 \leq \delta \leq \sqrt{\left[\frac{2\lambda+3}{\lambda+1}\right]^2 + \frac{2(2\lambda^2-1)}{\lambda(\lambda+1)}} - \frac{2\lambda+3}{\lambda+1}$, *we have*

$$\widetilde{N}_\delta(C_H^0) * \widetilde{N}_\delta(HS^0(\lambda)) \subset \mathcal{FS}_H^{*0}.$$

Proof . Let

$$F_0(z) = z + \sum_{n=2}^{\infty}(a_{0n}z^n + \overline{b}_{0n}\overline{z^n}) \in C_H^0,$$

$$G_0(z) = z + \sum_{n=2}^{\infty}(c_{0n}z^n + \overline{d}_{0n}\overline{z^n}) \in HS^0(\lambda)$$

and

$$F(z) = z + \sum_{n=2}^{\infty}(a_n z^n + \overline{b}_n\overline{z^n}) \in N_\delta(F_0), G(z) = z + \sum_{n=2}^{\infty}(c_n z^n + \overline{d}_n\overline{z^n}) \in N_\delta(G_0),$$

$$H(z) = z + \sum_{n=2}^{\infty}(e_n z^n + \overline{f_n}\overline{z^n}) \in \Sigma.$$

We need to prove that

$$(F * G * H)(z) \neq 0 (H \in \Sigma,\ z \in \check{\mathbb{D}}).$$

From Lemmas 2.4 and 2.6 and the relation (1.2), we have

$$\begin{aligned}
|(F_0 * G_0 * H)(z)| &= \left| z + \sum_{n=2}^{\infty}(a_{0n}c_{0n}e_n z^n + \overline{b_{0n}d_{0n}f_{0n}}\overline{z^n}) \right| \\
&\geq |z| \left[1 - \sum_{n=2}^{\infty}(|a_{0n}||c_{0n}||e_n||z|^{n-1} + |b_{0n}||d_{0n}||f_n||z|^{n-1}) \right] \\
&> |z| \left[1 - \sum_{n=2}^{\infty}(|a_{0n}||c_{0n}||e_n| + |b_{0n}||d_{0n}||f_n|) \right] \\
&\geq |z| \left[1 - \sum_{n=2}^{\infty} n(\frac{n+1}{2}|c_{0n}| + \frac{n-1}{2}|d_{0n}|) \right] \\
&\geq |z| \left[1 - \frac{n(n+1)}{2}\sum_{n=2}^{\infty}(|c_{0n}| + |d_{0n}|) \right] \\
&= |z| \left[1 - \frac{1}{2}\sum_{n=2}^{\infty} n^2(|c_{0n}| + |d_{0n}| + \sum_{n=2}^{\infty} n(|c_{0n}| + |d_{0n}|) \right] \\
&\geq |z| \left[1 - \frac{1}{2}\left[\frac{1}{\lambda} + \frac{1}{\lambda+1}\right] \right] \\
&= |z| \left[\frac{2\lambda^2 - 1}{2\lambda(\lambda+1)} \right].
\end{aligned}$$

In the same way as in the proof of Theorem 2.11, we get

$$|(F - F_0) * G_0 * H)(z)| < \frac{\delta|z|}{2(\lambda+1)},$$

$$|F_0 * (G - G_0) * H)(z)| < \delta|z|,$$

and

$$|(F - F_0) * (G - G_0) * H)(z)| < \frac{\delta^2|z|}{4}.$$

The remainder of the proof is similar to that of Theorem 2.11. □

Corollary 2.26. For $1/2 \leq \lambda \leq 1$, we have

$$\widetilde{\delta}(C_H^0 * HS^0(\lambda), \mathcal{FS}_H^{*0}) \geq \sqrt{\left[\frac{2\lambda+3}{\lambda+1}\right]^2 + \frac{2(2\lambda^2-1)}{\lambda(\lambda+1)}} - \frac{2\lambda+3}{\lambda+1}.$$

Corollary 2.27. We have

$$\widetilde{\delta}(C_H^0 * HC^0, \mathcal{FS}_H^{*0}) \geq \frac{\sqrt{29}-5}{2}.$$

Using the same techniques as in the proof of Theorems 2.25, we obtain the following theorem and we omit the details.

Theorem 2.28. Let $0 \leq \lambda \leq 1$. For $0 \leq \delta \leq \sqrt{\left[\frac{8\lambda+9}{\lambda+1}\right]^2 + \frac{4(4\lambda+1)}{\lambda+1}} - \frac{8\lambda+9}{\lambda+1}$, we have

$$\widetilde{N}_\delta(C_H^0) \diamond \widetilde{N}_\delta(HS^0(\lambda)) \subset \mathcal{FS}_H^{*0}.$$

Corollary 2.29. For $0 \leq \lambda \leq 1$, we have

$$\widetilde{\delta}(C_H^0 \diamond HS^0(\lambda), \mathcal{FS}_H^{*0}) \geq \sqrt{\left[\frac{8\lambda+9}{\lambda+1}\right]^2 + \frac{4(4\lambda+1)}{\lambda+1}} - \frac{8\lambda+9}{\lambda+1}.$$

Corollary 2.30. We have

$$\widetilde{\delta}(C_H^0 \diamond HS^0, \mathcal{FS}_H^{*0}) \geq \sqrt{85} - 9.$$

Corollary 2.31. We have

$$\widetilde{\delta}(C_H^0 \diamond HC^0, \mathcal{FS}_H^{*0}) \geq \frac{\sqrt{329}-17}{2}.$$

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