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An analog of Titchmarsh's theorem for the Bessel transform in the space $L_{p,\alpha}(\mathbb{R}_+)$

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Abstract

Using a Bessel generalized translation, we obtain an analog of Titchmarsh's theorem for the Bessel transform for functions satisfying the Lipschitz condition in the space $L_{p,\alpha}(\mathbb{R}_+)$, where $\alpha > -\frac{1}{2}$ and $1 < p \leq 2$.

Keywords: Bessel operator; Bessel transform; Bessel generalized translation.
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1. Introduction and preliminaries

Integral transforms and their inverses (e.g., the Bessel transform) are widely used to solve various problems in calculus, mechanics, mathematical physics, and computational mathematics (see [3, 4, 5, 6, 7]).

E. C. Titchmarsh ([4], Theorem 85) proved that if $f(x)$ in the space $L^2(\mathbb{R})$ such that $\|f(x+h) - f(x)\|_{L^2(\mathbb{R})} = O(h^\alpha)$ as $h \longrightarrow 0$ and $\alpha \in (0,1)$ if, and only if $\int_{|\lambda|\geq r} |\widehat{f}(\lambda)|^2 d\lambda = O(r^{-2\alpha})$ as $r \longrightarrow +\infty$, where $\widehat{f}$ stands for the Fourier transform of f.

In this paper we try, among other things, to explore the validity of this theorem in case of the Bessel transform in the space $L_{p,\alpha}(\mathbb{R}_+)$.

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Let $L_{p,\alpha} = L_{p,\alpha}(\mathbb{R}_+)$; $(\alpha > -\frac{1}{2},\ 1 < p \leq 2)$, is the Banach space of measurable functions $f(t)$ on $\mathbb{R}_+$ with the norm

$$\|f\|_{p,\alpha} = \left(\int_0^\infty |f(t)|^p t^{2\alpha+1} dt\right)^{1/p}.$$

Let

$$B = \frac{d^2}{dt^2} + \frac{(2\alpha+1)}{t}\frac{d}{dt}$$

be the Bessel differential operator. For $\alpha \geq -\frac{1}{2}$, we introduce the Bessel normalized function of the first kind j_α defined by

$$j_\alpha(z) = \Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n+\alpha+1)}\left(\frac{z}{2}\right)^{2n},$$

where Γ is the gamma-function (see[2]).

The function $y = j_\alpha(z)$ satisfies the differential equation

$$By + y = 0$$

with the initial conditions $y(0) = 1$ and $y'(0) = 0$. $j_\alpha(z)$ is a function infinitely differentiable, even, and, moreover entire analytic.

Lemma 1.1. *The following inequality is true*

$$|1 - j_\alpha(x)| \geq c,$$

with $x \geq 1$, where $c > 0$ is a certain constant.

Proof . The asymptotic formulas for the Bessel function imply that $j_\alpha(x) \longrightarrow 0$ as $x \longrightarrow \infty$. Consequently, a number $x_0 > 0$ exists such that with $x \geq x_0$ the inequality $|j_\alpha(x)| \leq \frac{1}{2}$ is true. Let $m = \min_{x\in[1,x_0]} |1 - j_\alpha(x)|$. With $x \geq 1$, we get the inequality

$$|1 - j_\alpha(x)| \geq c,$$

where $c = \min(\frac{1}{2}, m)$. $\square$

In $L_{p,\alpha}$, consider the Bessel generalized translation T_h (see [2]) defined by

$$T_h f(t) = c_\alpha \int_0^\pi f(\sqrt{t^2 + h^2 - 2th\cos\varphi})\sin^{2\alpha}\varphi d\varphi,$$

where

$$c_\alpha = \left(\int_0^\pi \sin^{2\alpha}\varphi d\varphi\right)^{-1} = \frac{\Gamma(\alpha+1)}{\Gamma(\frac{1}{2})\Gamma(\alpha+\frac{1}{2})}.$$

It is easy to see that

$$T_0 f(t) = f(t).$$

The Bessel transform is defined by the following integral transform [1, 2, 8]

$$\mathcal{F}_B(f)(\lambda) = \int_0^\infty f(t) j_\alpha(\lambda t) t^{2\alpha+1} dt, \quad \lambda \in \mathbb{R}_+.$$

The inverse Bessel transform is given by the formula

$$f(t) = (2^\alpha\Gamma(\alpha+1))^{-2}\int_0^\infty \mathcal{F}_B(f)(\lambda) j_\alpha(\lambda t)\lambda^{2p+1} d\lambda.$$

We now formulate some properties of the Bessel generalized translation (see [1, 2]):

1.

$$\mathrm{T}_h j_\alpha(\lambda t) = j_\alpha(\lambda h) j_\alpha(\lambda t)$$

2. T_h is selfadjoint: if $f(t)$ is a continuous function in $\mathrm{L}_{1,\alpha}$ and $g(t)$ is a continuous, even, and bounded function on $\mathbb{R}$ then

$$\int_0^\infty (\mathrm{T}_h f(t)) g(t) t^{2\alpha+1} dt = \int_0^\infty f(t)(\mathrm{T}_h g(t)) t^{2\alpha+1} dt,$$

$$\mathrm{T}_h f(t) = \mathrm{T}_t f(h).$$

The following relation connects the Bessel generalized translation and Bessel transform

$$\mathcal{F}_{\mathrm{B}}(\mathrm{T}_h f)(\lambda) = j_\alpha(\lambda h)\mathcal{F}_{\mathrm{B}}(f)(\lambda).$$

We have the Hausdorff-Young inequality

$$\|\mathcal{F}_{\mathrm{B}}(f)\|_{q,\alpha} \leq C\|f\|_{p,\alpha}, \tag{1.1}$$

where C is a positive constant and $\frac{1}{p} + \frac{1}{q} = 1$.

2. Main Result

In this section we give the main result of this paper. We need first to define (β, γ, p)-Bessel Lipschitz class.

Definition 2.1. Let $\beta > 0$ and $\gamma > 0$. A function $f \in \mathrm{L}_{p,\alpha}$ is said to be in the (β, γ, p)-Bessel Lipschitz class, denoted by $Lip(\beta, \gamma, p)$, if

$$\|\mathrm{T}_h f(t) - f(t)\|_{p,\alpha} = O\left(\frac{h^\beta}{(\log \frac{1}{h})^\gamma}\right), \quad as\ h \longrightarrow 0.$$

Theorem 2.2. *Let $f(x)$ belong to $Lip(\beta, \gamma, p)$. Then*

$$\int_r^\infty |\mathcal{F}_{\mathrm{B}}(f)(\lambda)|^q \lambda^{2\alpha+1} d\lambda = O\left(\frac{r^{-q\beta}}{(\log \frac{1}{h})^{q\gamma}}\right) \ as\ r \longrightarrow +\infty,$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof . Suppose that $f \in Lip(\beta, \gamma, p)$. Then we have

$$\|\mathrm{T}_h f(t) - f(t)\|_{p,\alpha} = O\left(\frac{h^\beta}{(\log \frac{1}{h})^\gamma}\right), \quad as\ h \longrightarrow 0.$$

If $\lambda \in [\frac{1}{h}, \frac{2}{h}]$ then $\lambda h \geq 1$ and Lemma 1.1 implies that

$$1 \leq \frac{1}{c^q}|1 - j_\alpha(\lambda h)|^q.$$

According to Lemma 1.1, we obtain that

$$\begin{aligned}
\int_{1/h}^{2/h} |\mathcal{F}_{\mathrm{B}}(f)(\lambda)|^q \lambda^{2\alpha+1} d\lambda &\leq \frac{1}{c^q} \int_{1/h}^{2/h} |1 - j_\alpha(\lambda h)|^q |\mathcal{F}_{\mathrm{B}}(f)(\lambda)|^q \lambda^{2\alpha+1} d\lambda \\
&\leq \frac{1}{c^q} \int_{0}^{\infty} |1 - j_\alpha(\lambda h)|^q |\mathcal{F}_{\mathrm{B}}(f)(\lambda)|^q \lambda^{2\alpha+1} d\lambda \\
&\leq K \|\mathrm{T}_h f(x) - f(x)\|_{p,\alpha}^q \\
&= O\left(\frac{h^{q\beta}}{(\log \frac{1}{h})^{q\gamma}}\right)
\end{aligned}$$

for all $r > 0$. Thus there exists $C_1 > 0$ such that

$$\int_{r}^{2r} |\mathcal{F}_{\mathrm{B}}(f)(\lambda)|^q \lambda^{2\alpha+1} d\lambda \leq C_1 \frac{r^{-q\beta}}{(\log r)^{q\gamma}}.$$

Furthermore, we have

$$\begin{aligned}
\int_{r}^{\infty} |\mathcal{F}_{\mathrm{B}}(f)(\lambda)|^q \lambda^{2\alpha+1} d\lambda &= \sum_{i=0}^{\infty} \int_{2^i r}^{2^{i+1} r} |\mathcal{F}_{\mathrm{B}}(f)(\lambda)|^q \lambda^{2\alpha+1} d\lambda \\
&\leq C_1 \sum_{i=0}^{\infty} \frac{(2^i r)^{-q\beta}}{(\log 2^i r)^{q\gamma}} \\
&\leq C_1 \sum_{i=0}^{\infty} \frac{(2^i r)^{-q\beta}}{(\log r)^{q\gamma}} \\
&\leq C_2 \frac{r^{-q\beta}}{(\log r)^{q\gamma}}.
\end{aligned}$$

This proves that

$$\int_{r}^{\infty} |\mathcal{F}_{\mathrm{B}}(f)(\lambda)|^q \lambda^{2\alpha+1} d\lambda = O\left(\frac{r^{-q\beta}}{(\log r)^{q\gamma}}\right) \text{ as } r \longrightarrow \infty,$$

which proves the theorem. □

Definition 2.3. A function $f \in \mathrm{L}_{p,\alpha}$ is said to be in the ψ-Dini Lipschitz class, denoted by $Lip(p, \psi)$, if

$$\|\mathrm{T}_h f(x) - f(x)\|_{p,\alpha} = O\left(\frac{\psi(h)}{(\log \frac{1}{h})^{\gamma}}\right) \ \gamma > 0, \ as \ h \longrightarrow 0,$$

where

1. $\psi(t)$ is a continuous increasing function on $[0, \infty)$,
2. $\psi(ts) \leq \psi(t)\psi(s)$ for all $s, t \in [0, \infty)$.

Theorem 2.4. *Let $f \in \mathrm{L}_{p,\alpha}$ and let ψ be a fixed function satisfying the conditions of Definition 2.3, if $f(x)$ belong to $Lip(p, \psi)$. Then*

$$\int_{r}^{\infty} |\mathcal{F}_{\mathrm{B}}(f)(\lambda)|^q \lambda^{2\alpha+1} d\lambda = O(\psi(r^{-q})(\log r)^{-q\gamma}) \ as \ r \longrightarrow +\infty.$$

Proof . Assume that $f \in Lip(p,\psi)$. Then we have

$$\|\mathrm{T}_h f(x) - f(x)\|_{p,\alpha} = O\left(\frac{\psi(h)}{(\log\frac{1}{h})^{\gamma}}\right) \ as\ h \longrightarrow 0.$$

If $\lambda \in [\frac{1}{h}, \frac{2}{h}]$ then $h\lambda \geq 1$, then the following inequalities can be derived from (1.1) and from similar reasoning as in the proof of Theorem 2.2, so that we obtain

$$\begin{aligned}
\int_{1/h}^{2/h} |\mathcal{F}_{\mathrm{B}}(f)(\lambda)|^q \lambda^{2\alpha+1} d\lambda &\leq \frac{1}{c^q}\int_{1/h}^{2/h} |1 - j_\alpha(h\lambda)|^q |\mathcal{F}_{\mathrm{B}}(f)(\lambda)|^q \lambda^{2\alpha+1} d\lambda \\
&\leq \frac{1}{c^q}\int_{0}^{+\infty} |1 - j_\alpha(h\lambda)|^q |\mathcal{F}_{\mathrm{B}}(f)(\lambda)|^q \lambda^{2\alpha+1} d\lambda \\
&= \frac{1}{c^q}\|\mathrm{T}_h f(x) - f(x)\|_{p,\alpha}^q \\
&= O\left(\frac{\psi(h^q)}{(\log\frac{1}{h})^{q\gamma}}\right).
\end{aligned}$$

Thus there exists then a positive constant C_1 such that

$$\int_r^{2r} |\mathcal{F}_{\mathrm{B}}(f)(\lambda)|^q \lambda^{2\alpha+1} d\lambda \leq C_1 \frac{\psi(r^{-q})}{(\log r)^{q\gamma}}.$$

So that

$$\begin{aligned}
\int_r^{\infty} |\mathcal{F}_{\mathrm{B}}(f)(\lambda)|^q \lambda^{2\alpha+1} d\lambda &= \left[\int_r^{2r} + \int_{2r}^{4r} + \int_{4r}^{8r} + \cdots\right] |\mathcal{F}_{\mathrm{B}}(f)(\lambda)|^q \lambda^{2\alpha+1} d\lambda \\
&\leq C_1\frac{\psi(r^{-q})}{(\log r)^{q\gamma}} + C_1\frac{\psi((2r)^{-q})}{(\log 2r)^{q\gamma}} + C_1\frac{\psi((4r)^{-q})}{(\log 4r)^{q\gamma}} + \cdots \\
&\leq C_1\frac{\psi(r^{-q})}{(\log r)^{q\gamma}} + C_1\frac{\psi((2r)^{-q})}{(\log r)^{q\gamma}} + C_1\frac{\psi((4r)^{-q})}{(\log r)^{q\gamma}} + \cdots \\
&\leq C_1\frac{\psi(r^{-q})}{(\log r)^{q\gamma}}(1 + \psi(2^{-q}) + (\psi(2^{-q}))^2 + (\psi(2^{-q}))^3 + \cdots \\
&\leq C_1 K_1 \frac{\psi(r^{-q})}{(\log r)^{q\gamma}},
\end{aligned}$$

where $K_1 = (1-\psi(2^{-q}))^{-1}$ since Definition 2.3 it follows that $\psi(2^{-q}) < 1$. Then

$$\int_r^{+\infty} |\mathcal{F}_{\mathrm{B}}(f)(\lambda)|^q \lambda^{2\alpha+1} d\lambda = O\left(\psi(r^{-q})(\log r)^{-q\gamma}\right) \ as\ r \longrightarrow +\infty$$

which proves the theorem. □

References

[1] I.A. Kipriyanov, *Singular Elliptic Boundary Value Problems*, Nauka, Moscow, 1997.

[2] B.M. Levitan, *Expansion in Fourier series and Integrals over Bessel Functions*, Uspekhi Mat. Nauk 6 (1951) 102–143.

[3] A.N. Tikhonov and A.A. Samarskii, *Equations of Mathematical Physics*, Pergamon Press, Oxford, 1964; Gostekhteorizdat, Moscow, 1972.

[4] E.C. Titchmarsh, *Introduction to the theory of Fourier Integrals* , Clarendon, Oxford, 1948; Komkniga, Moscow, 2005.

[5] A.G. Sveshnihov, A.N. Bogolyubov, and V.V. Kravtsov, *Lectures on Mathematical Physics*, Nauka, Moscow, 1949; Cambridge, Univ. Press, Cambridge, 1952.

[6] V.S. Vladimirov, *Equations of Mathematical Physics*, Marcel Dekker, New York, 1971; Nauka, Moscow, 1976.

[7] G.N. Watson, *Treatise on the Theory of Bessel Functions*, Inostrannaya Litetatura, Moscow, 1949; Cambridge Univ. Press, Cambridge, 1952.

[8] K. Trimèche, *Transmutation operators and mean-periodic functions associated with differential operators*, Math. Rep. 4 (1988) 1–282.