Int. J. Nonlinear Anal. Appl. 7 (2016) No. 1, 295-300
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2015.341

On intermediate value theorem in ordered Banach spaces for noncompact and discontinuous mappings

Ali Farajzadeh

Department of Mathematics, Razi University, Kermanshah, 67149, Iran

(Communicated by M.B. Ghaemi)

Abstract

In this paper, a vector version of the intermediate value theorem is established. The main theorem of this article can be considered as an improvement of the main results have been appeared in [*On fixed point theorems for monotone increasing vector valued mappings via scalarizing*, Positivity, 19 (2) (2015) 333-340] with containing the uniqueness, convergent of each iteration to the fixed point, relaxation of the relatively compactness and the continuity on the map with replacing topological interior of the cone by the algebraic interior. Moreover, by applying Ascoli-Arzela's theorem an example in order to show that the main theorem of the paper [*An intermediate value theorem for monotone operators in ordered Banach spaces*, Fixed point theory and applications, 2012 (1) (2012) 1-4] may fail, is established.

Keywords: intermediate value theorem; fixed point; increasing mapping; algebraic interior; normal cone.
2010 MSC: Primary 26A25; Secondary 39B62.

1. Introduction and preliminaries

As we know that the intermediate value theorem in real analysis (especially in calculus) is one of the most important and applicable theorem and one can apply to prove fixed points of a real mapping. We recall that if $f : [a, b] \to \mathbb{R}$ is continuous mapping with $f(a) < a$ and $b < f(b)$ then there exists $c \in [a, b]$ such that $f(c) = c$. Now a natural question will arise as that how can extend this fact when the real $\mathbb{R}$ replaced by a Banach space and the interval $[a, b]$ of the real line by ordered interval of X. The aim of this note is to answer to the question. It is worth noting that the result of this paper can be viewed as an improvement of the main theorem given in [4].

Email address: `farajzadehali@gmail.com` (Ali Farajzadeh)

Received: March 2015 *Revised:* October 2015

The rest of this section deals with some definitions and basic results which we need in the next section.

Definition 1.1. ([3, 6]) Let X be a real vector space with its zero vector θ and $P \subseteq X$. P is called

(a) **Cone** if it is closed under nonnegative scalar multiplication, i.e., $tx \in P$ for all $x \in P$ and $t \geq 0$.

(b) **Convex** if $tx + (1-t)y \in P, \qquad \forall (x, y, t) \in P \times P \times [0, 1].$

(c) **Pointed** if $P \cap -P = \{\theta\}$.

Let X be a real vector space and P be a convex pointed cone of it. P induces an ordering on X as follows, $x, y \in X$,

$$x \preceq_P y \Leftrightarrow y - x \in P.$$

It is easy to verify that the ordering $\preceq_P$ is a partial ordering; that is reflexive, antisymmetric and transitive. Also if X is a topological vector space and P is a convex pointed cone of X with nonempty interior (that is $intP \neq \emptyset$) then we can define an ordering on X using the interior of P as follows, $x, y \in X$,

$$x \ll_{intP} y \Leftrightarrow y - x \in intP.$$

Remark that the ordering induced by $intP$ is not necessarily a partial ordering on X. If the cone P is known, for simplicity, we replace $\preceq_P$ and $\ll_{intP}$ by $\preceq$ and $\ll$, respectively.

Definition 1.2. ([6]) Assume that X is a real ordered vector space by a convex cone P. The cone P is said to be minihedral if $\sup\{x, y\}$ (sup means the least upper bound) exists for each x, y in X and is said to be strongly minihedral if $\sup D$ exists for each nonempty bounded subset $D \subseteq X$.

Definition 1.3. ([1, 2]) A convex cone P of a normed space is called normal if and only if there exists a constant $k \geq 1$ such that, for all $x, y \in X$,

$$\theta \preceq x \preceq y \Rightarrow \|x\| \leq k\|y\|.$$

Definition 1.4. ([3, 4]) The mapping $T : X \to Y$ acting in partially ordered real vector linear spaces X and Y is called increasing if $x \preceq y$ implies $T(x) \preceq T(y)$.

Note that if we take $X = Y = \mathbb{R}$ then Definition 1.4 collapses to the usual definition of an increasing mapping.

Definition 1.5. ([2, 4]) Let X and Y be two topological spaces. The mapping $T : X \to Y$ is called compact, if the closure of its range; that is $\overline{T(X)}$, is a compact subset of Y, where $T(X) = \cup_{x \in X} T(x)$.

We note that T is compact when T is a continuous mapping and X is compact. But there are many discontinuous mappings with non-compact domain which satisfy in Definition 1.5. For example, $T(x) = 0$, where x is a rational number and otherwise $T(x) = 1$.

Theorem 1.6. ([9]). Suppose that X is Banach space, P is a normal and convex cone, and $u_0, v_0 \in X$ with $u_0 < v_0$ (that is, $u_0 \preceq v_0$ with $u_0 \neq v_0$). Let $A : [u_0, v_0] \to X$ be an increasing mapping and let $h_0 = v_0 - u_0$. If one of the following assumptions holds:

(i) A is a convex mapping (that is $A(\lambda x + (1-\lambda)y) \preceq \lambda A(x) + (1-\lambda)A(y)$) $Au_0 \preceq u_0$ and $Av_0 \preceq v_0 - \epsilon h_0$ for some $\epsilon \in (0,1)$;

(ii) A is a concave mapping $Au_0 \geq u_0 + \epsilon h_0, Av_0 \leq v_0$, for some $\epsilon \in (0,1)$,

then A has a unique fixed point x^* in $[u_0, v_0]$. Moreover, for any $x_0 \in [u_0, v_0]$, the iterative sequence $\{x_n\}$ given by $x_n = Ax_n$ for $n = 1, 2, \ldots$ converges to x^* and satisfying $\|x_n - x^*\| \leq M(1-\epsilon)^n$, for all $n = 1, 2, \ldots$ and a positive constant M independent of x_0.

In 2012, Kostrykin and Oleynik [4] presented the following theorem which is an extension of Lemma 2.1 of [5] that plays a key role in [5]. Moreover, it can be considered as an important existence result of the unstable bumps in neural, Integral equations and operator theory (see, for instance, [5]).

Theorem 1.7. ([4]). Let X be a real Banach space with an ordered cone K satisfying:

(a) K has a nonempty interior,

(b) K is normal and minihedral.

Assume that there are two points in X, $u_- \ll u_+$ and an increasing, compact, and continuous operator $T : [u_-; u_+] \longrightarrow X$. If u_- is a strong supersolution of T and u_+ is a strong subsolution, that is,

$$Tu_- \ll u_- \text{ and } u_+ \ll Tu_+,$$

then T has a fixed point $u_* \in [u_-, u_+]$, where $[u_-, u_+]$ denotes

$$\{z \in C([u_-, u_+]) :| u_- \leq z \leq u_+\}.$$

We denote the set of all continuous mappings from $[a,b]$ into $\mathbb{R}$ (the real line) by $C([a,b])$. It is a well known fact that $(C([a,b]), d)$ is a complete metric space, where $d(f,g) = \sup_{t\in[a,b]} |f(t) - g(t)|$.

Definition 1.8. Let Ω be a nonempty subset of $C([a,b])$. The set Ω is called:

- **Pointwise bounded** if for each $x \in [a,b]$ there exists nonnegative real number m_x such that

$$f(x) \leq m_x, \qquad \forall f \in \Omega.$$

- **Equicontinuous** if for each $\epsilon > 0$ there exists $\delta > 0$ such that, for each $t, s \in [a,b]$ with $|t-s| < \delta$, we have

$$|f(t) - f(s)| < \epsilon, \qquad \forall f \in \Omega.$$

The following theorem plays a key role in the next section.

Theorem 1.9. (Arzela-Ascoli)([8]). A subset of $C([a,b])$ is compact if and only if it is pointwise bounded and equicontinuous.

2. Main results

In this section we first show, by providing an example, that the result of Theorem 1.7 may fail. Hence there are some gaps in it. Then we will try to present the correct version of Theorem 1.7 by relaxing some assumptions of it and extending it in a general space (topological vector space) by using a new proof.

The following example indicates that the result of Theorem 1.7 is not true.

Example 2.1. Let $X = C([0,1])$ and $K = \{u \in X; u(t) \geq 0,\ \forall t \in [0,1]\}$. Define $T : [u_-, u_+] \to X$:

$$(Tu)(x) = 2u(0) + \int_0^x u(t)dt - \frac{2}{3},\ \ \forall (u,x) \in [u_-, u_+] \times [0,1].$$

Let $u_- = 0, u_+ = 1$, Then $u_- << u_+$ and $T(0) = -1 << 0$, $1 << T(1)$. It is easy to check that K satisfies conditions (a) and (b) of Theorem 1.7. It follows from the inequality

$$|(Tu)(x) - (Tu)(y)| = |\int_x^y u(t)dt| \leq |y - x|, \quad \forall (x,y,u) \in [0,1] \times [0,1] \times T[u_0, u_+],$$

that the set $T[u_-, u_+]$ is equicontinuous and pointwise bounded. Consequently, the set $T[u_-, u_+]$ as a subset of X fulfils all assumptions of Theorem 1.9 and hence it is relatively compact. So T is compact. It is straightforward to verify that T is continuous and increasing. Consequently T satisfies all the assumptions of Theorem 1.6 while it does not have any fixed point in $[u_-, u_+]$, because if u is a fixed point of T then

$$2u(0) + \int_0^x u(t)dt - \frac{2}{3} = u(x) \qquad \forall x \in [0,1].$$

Hence

$$u(x) = \frac{2}{3}e^x, \qquad \forall x \in [0,1]$$

is a unique fixed point of T which

$$u \notin [u_- = 0, u_+ = 1] = \{v \in C([0,1]) : 0 \leq v(x) \leq 1,\ \forall x \in [0,1]\}.$$

Definition 2.2. Let S be a nonempty subset of a real linear space X. The set

$$cor(S) = \{\bar{x} \in S : \forall x \in X \quad \exists \lambda > 0 \quad \text{with} \ \ (\bar{x} + tx) \in S \ \ \forall t \in [0, \lambda]\},$$

is called the algebraic interior of S.

Remark 2.3. Let P be a convex cone in a linear space X with a nonempty algebraic interior. Then

(a) $cor(P) \cup \{0_X\}$ is a convex cone (see, Lemma 1.12 of [3])

(b) $cor(cor(S)) = cor(S)$, (see, Lemma 1.9 of [3]).

Note that if X is a topological vector space and S is a nonempty subset of X then the topological interior of S; that is $intS$, is a subset the algebraic interior of S. Moreover, there are some examples which show the inclusion may be strict. For instance, let

$$X = C_{00} = \{x = (x(n)) : \text{the set } \{n \in N; x(n) \neq 0\} \quad \text{is finite }\}$$

and $||x|| = \max_{n\in\mathbb{N}} x(n)$, for all $x = (x(n)) \in C_{00}$. It is easy to check that $(C_{00}, ||.||)$ is a normed space. Put

$$P = \{x = (x(n)) \in C_{00} : x(n) \leq \frac{1}{n}, (\forall n)\}.$$

One can verify that $intC = \emptyset$ while $(\alpha, 0, 0, \ldots) \in cor(C)$, where $0 < \alpha < 1$.

Hence the example shows that the algebraic interior is a suitable replacement of the topological interior for the case where it is empty. Further, we can relax the topological structure when we use of the algebraic interior.

The next result is a correct version of Theorem 1.7 by relaxing minihedrality on the cone and replacing the topological interior of the cone by the algebraic interior. Moreover, in this case the uniqueness of the fixed point has been ensured.

Theorem 2.4. *Let X be a real Banach space and let P be a normal cone with nonempty algebraic interior (i.e., $cor(P) \neq \emptyset$). Assume that $K = cor(P) \bigcup \{\theta_X\}$, there are two points in X, $u_- \prec_{corP} u_+$, and an increasing convex mapping $T : [u_-, u_+] \to X$. If $Tu_+ \preceq_K u_+$ and $u_- \preceq_K Tu_-$, then T has a unique fixed point $x_* \in [u_-, u_+]$. Moreover each iteration $Ax_n = x_{n-1}$ for all $n = 1, 2, 3, \ldots$ with $x_0 \in [u_-, u_+]$ converges to x_*.*

Proof . By Remark 2.3 (a), the set K is a convex cone and by the assumption $u_- \prec_{corP} u_+$, we get $h_0 = u_+ - u_- \in cor(P)$. Hence by Remark 2.3 (b) we have

$$u_+ - Tu_+ \in cor(P) = cor(cor(P).$$

So there exists a positive number λ such that

$$u_+ - Tu_+ + \epsilon(u_+ - u_-) \in cor(P), \qquad \forall \epsilon \in [0, \lambda].$$

Therefore, we can choose $\epsilon \in (0, 1)$ such that

$$Tu_+ \preceq_K u_+ - \epsilon(u_+ - u_-).$$

This means that $Tu_+ \preceq_K u_+ - \epsilon h_0$. Consequently, it follows from part (i) of Theorem 1.6 that T has a unique fixed point x^* and each iteration $x_n = Tx_{n-1}$ with arbitrary $x_0 \in [u_-, u_+]$ converges to x^*. This completes the proof. □

It worth noting that in Theorem 1.7 minihedrality of the cone is essential. While it has been relaxed in Theorem 2.4. There are many cones which are not minihedral. In the following, for instance, one of them is presented. Hence we cannot apply Theorem 1.7 in this case.

Example 2.5. Assume that

$$X = C^1([-1, 1]) = \{f : [-1, 1] \to \mathbb{R}, f \text{ is continuously differentiable}\}$$

with

$$\|f\| = \|f\|_\infty + \|f'\|_\infty$$

and $P = \{f \in X : f(x) \geq 0, \forall x \in [-1, 1]\}$. It is easy to check that X is a Banach space and P is a convex cone but not minihedral, because $\sup\{x, -x\} \notin X$. Hence we cannot apply Theorem 1.7 for X.

Conclusion

The main theorem, that is, Theorem 1.7, of [4] may fall down is shown by Example 2.1. A correct version of Theorem 1.7, by relaxing some assumptions and a new proof, is presented. Some examples in order to support the results of the article are provided. Finally, It is worth noting that Theorem 2.4 is another version of Theorem 3.5 and Theorem 3.12 of [7] by relaxing relatively compactness of the range T, having nonempty interior of the cone, and continuity of T. Moreover, advantage of Theorem 2.4 to Theorem 2.1 and Theorems 3.5, 3.12 of [7] is containing iteration method; that is each iteration convergent to the unique fixed point which is important in numerical analysis. Finally, by applying part (ii) of Theorem 1.6 and suitable modification in Theorem 2.4 we can establish a similar result as Theorem 2.4 when the mapping T is concave.

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