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A new type of approximation for cubic functional equations in Lipschitz spaces

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Abstract

Let $\mathcal{G}$ be an abelian group with a metric d, $\mathcal{E}$ be a normed space and $f:\mathcal{G}\to\mathcal{E}$ be a given function. We define difference $\mathcal{C}_{3,1}f$ by the formula

$$\mathcal{C}_{3,1}f(x,y)=3f(x+y)+3f(x-y)+48f(x)-f(3x+y)-f(3x-y)$$

for every $x,y\in\mathcal{G}$. Under some assumptions about f and $\mathcal{C}_{3,1}f$, we show that if $\mathcal{C}_{3,1}f$ is Lipschitz, then there exists a cubic function $\mathcal{C}:\mathcal{G}\to\mathcal{E}$ such that $f-\mathcal{C}$ is Lipschitz with the same constant. Moreover, we study the approximation of the equality $\mathcal{C}_{3,1}f(x,y)=0$ in the Lipschitz norms.

Keywords: Approximation; **d**-Lipschitz; Left invariant mean; Cubic difference; Lipschitz norm.
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1. Introduction

The approximation of the Cauchy and Jensen functional equations in the Lipschitz norms were studied by Tabor [23, 24]. Czerwik and Dłutek [7] studied the approximation of the quadratic functional equations in Lipschitz spaces. The concept of stability in Lipschitz spaces of several functional equations such as bi-quadratic, tri-quadratic, cubic, bi-cubic and quartic functional equations has been studied by Nikoufar et al.; see for instance [6, 8] and [15]-[19].

Let $\mathcal{G}$ be an abelian group and $\mathcal{E}$ be a vector space. A cubic ($\{2,1\}$-cubic) difference and a $\{3,1\}$-cubic difference of a function $f:\mathcal{G}\to\mathcal{E}$ are defined by

$$\mathcal{C}_{2,1}f(x,y)=f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x)$$

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and

$$\mathcal{C}_{3,1}f(x,y) = f(3x+y) + f(3x-y) - 3f(x+y) - 3f(x-y) - 48f(x),$$

respectively. The equality $\mathcal{C}_{2,1}f(x,y) = 0$ is called a cubic functional equation and every solution of the last functional equation is said to be a cubic function; see [10]. The stability of cubic functional equation in Banach spaces and Lipschitz spaces were investigated by Jun and Kim [10] and Ebadian et al. [8], respectively.

Also, the equality $\mathcal{C}_{3,1}f(x,y) = 0$ is called a $\{3,1\}$-cubic functional equation. Park and Jung [22] proved that the $\{3,1\}$-cubic functional equation is equivalent to cubic functional equation and investigated the stability of $\{3,1\}$-cubic functional equation on abelian groups to a Banach space.

In this present paper, under some assumptions about f and $\mathcal{C}_{3,1}f$, we show that if $\mathcal{C}_{3,1}f$ is Lipschitz, then there exists a cubic function $\mathcal{C} : \mathcal{G} \to \mathcal{E}$ such that $f - \mathcal{C}$ is Lipschitz with the same constant. We also investigate the approximation of $\{3,1\}$-cubic functional equation in the Lipschitz norms. Finally, we bring several open problems related to this concept.

2. Main results

All over this paper, we use notations $\mathcal{G}$ and $\mathcal{E}$ for arbitrary Abelian group and vector space, unless they are otherwise specified. Let $S(\mathcal{E})$ be a given family of subsets of $\mathcal{E}$. We say that this family is linearly invariant (see [1]) if it is closed under the addition and scalar multiplication defined as usual sense and translation invariant, i.e., $x + A \in S(\mathcal{E})$, for all $x \in \mathcal{E}$ and $A \in S(\mathcal{E})$. One can easily check that due to properties of $S(\mathcal{E})$ it contains all singleton subsets of $\mathcal{E}$. In the special case, for a normed vector space $\mathcal{E}$, the family of all closed balls with center at zero is denoted by $CB(\mathcal{E})$.

By $B(\mathcal{G}, S(\mathcal{E}))$ we denote the family of all functions $f : \mathcal{G} \to \mathcal{E}$ with $\mathrm{Im} f \subset V$ for some $V \in S(\mathcal{E})$. It is easy to verify that $B(\mathcal{G}, S(\mathcal{E}))$ is a vector space.

Definition 2.1. *[23] We say that $B(\mathcal{G}, S(\mathcal{E}))$ admits a left invariant mean (LIM, in short) if the family $S(\mathcal{E})$ is linearly invariant and there exists a linear operator $M : B(\mathcal{G}, S(\mathcal{E})) \to \mathcal{E}$ such that*

(i) if $Im(f) \subset V$ for some $V \in S(\mathcal{E})$, then $M[F] \in V$;

(ii) if $f \in B(\mathcal{G}, S(\mathcal{E}))$ and $a \in \mathcal{G}$, then $M[f^a] = M[f]$, where $f^a(\cdot) = f(\cdot + a)$.

Definition 2.2. *[7, 23] Let $\mathbf{d} : \mathcal{G} \times \mathcal{G} \to S(\mathcal{E})$ be a set-valued function such that for each $x, y, a \in G$,*

$$\mathbf{d}(x+a, y+a) = \mathbf{d}(a+x, a+y) = \mathbf{d}(x,y)$$

and a function $f : \mathcal{G} \to \mathcal{E}$ is said to be $\mathbf{d}$-Lipschitz if

$$f(x) - f(y) \in \mathbf{d}(x,y).$$

Theorem 2.3. *Assume that $S(\mathcal{E})$ is a linearly invariant family such that $B(\mathcal{G}, S(\mathcal{E}))$ admits LIM. If $f : \mathcal{G} \to \mathcal{E}$ is an odd function and $\mathcal{C}_{3,1}f(.,y) : \mathcal{G} \to \mathcal{E}$ is $\mathbf{d}$-Lipschitz for every $y \in \mathcal{G}$, then there exists a cubic function $\mathcal{C} : \mathcal{G} \to \mathcal{E}$ such that $f - \mathcal{C}$ is $\frac{1}{48}$ $\mathbf{d}$-Lipschitz. Moreover, if $Im(\mathcal{C}_{3,1}f) \subset A$ for some $A \in S(\mathcal{E})$, then $Im(f - \mathcal{C}) \subset \frac{1}{48}A$.*

Proof . Let $M : B(\mathcal{G}, S(\mathcal{E})) \to \mathcal{E}$ be a LIM. We define $C^3_{(x,z)} : \mathcal{G} \to \mathcal{E}$ by

$$C^3_{(x,z)}(y) = \frac{1}{48}\mathcal{C}_{3,1}f(x,y) - \frac{1}{48}\mathcal{C}_{3,1}f(z,y)$$

for all $x, y, z \in \mathcal{G}$. Consider the function $F_a : \mathcal{G} \to \mathcal{E}$ given by

$$F_a(y) = \frac{1}{48}f(3a+y) + \frac{1}{48}f(3a-y) - \frac{1}{16}f(a+y) - \frac{1}{16}f(a-y)$$

for all $a, y \in \mathcal{G}$. By the oddness of f, we have $\frac{1}{48}\mathcal{C}_{3,1}f(0,y) = f(0)$. We will prove that $F_a \in B(\mathcal{G}, S(\mathcal{E}))$. In fact, for each $y, a \in \mathcal{G}$ we have

$$\begin{aligned} F_x(y) &= \frac{1}{48}f(3x+y) + \frac{1}{48}f(3x-y) - \frac{1}{16}f(x+y) - \frac{1}{16}f(x-y) \\ &= \frac{1}{48}\mathcal{C}_{3,1}f(0,y) - \frac{1}{48}\mathcal{C}_{3,1}f(x,y) + f(x) - f(0) \\ &= C^3_{(0,x)}(y) + f(x) - f(0). \end{aligned}$$

Since $Im(C^3_{(x,z)}) \subset \frac{1}{48}\mathbf{d}(x,z)$ for all $y, z \in \mathcal{G}$, it follows that $F_a \in B(\mathcal{G}, S(\mathcal{E}))$ for all $a \in \mathcal{G}$. Thus we may define $\mathcal{C} : \mathcal{G} \to \mathcal{E}$ by the formula

$$\mathcal{C}(x) = M(F_x), \quad (x \in \mathcal{G}).$$

We will verify that $f - \mathcal{C}$ is $\frac{1}{48}$ $\mathbf{d}$-Lipschitz. By properties of the mean M, it is easy to see that if $f : \mathcal{G} \to \mathcal{E}$ is constant, then $M(f) = Imf$. Furthermore,

$$\begin{aligned} (f(x) - \mathcal{C}(x)) - (f(z) - \mathcal{C}(z)) &= (f(x) - M(F_x)) - (f(z) - M(F_z)) \\ &= (M(f(x)) - M(F_x)) - (M(f(z)) - M(F_z)) \\ &= M((f(x)) - (F_x)) - M((f(z)) - (F_z)) \\ &= M(\frac{1}{48}\mathcal{C}_{3,1}f(x,\cdot) - \frac{1}{48}\mathcal{C}_{3,1}f(z,\cdot)) \\ &= M(C^3_{(x,z)}(\cdot)) \end{aligned}$$

for all $x, z \in \mathcal{G}$. So, $f - \mathcal{C}$ is $\frac{1}{48}$ $\mathbf{d}$-Lipschitz. Now, we show that $\mathcal{C}$ is a cubic function. We have the equalities

$$\begin{aligned} &\mathcal{C}(3x+z) + \mathcal{C}(3x-z) \\ &= M[F_{3x+z}(y)] + M[F_{3x-z}(y)] \\ &= M\left[\frac{1}{48}f(9x+3z+y) + \frac{1}{48}f(9x+3z-y) - \frac{1}{16}f(3x+y+z) - \frac{1}{16}f(3x+z-y)\right] \\ &\quad + M\left[\frac{1}{48}f(9x-3z+y) + \frac{1}{48}f(9x-3z-y) - \frac{1}{16}f(3x-z+y) - \frac{1}{16}f(3x-z-y)\right]. \end{aligned}$$

On the other hand,

$$
\begin{aligned}
&3\mathcal{C}(x+z)+3\mathcal{C}(x-z)+48\mathcal{C}(x)\\
&=3M[F_{x+z}(y)]+3M[F_{x-z}(y)]+48M[F_x(y)]\\
&=M[F_{x+z}(y+6x)]+M[F_{x+z}(y-6x)]+M[F_{x+z}(y)]\\
&\quad+M[F_{x-z}(y+6x)]+M[F_{x-z}(y-6x)]+M[F_{x-z}(y)]\\
&\quad+3M[F_x(y+4x+z)]+3M[F_x(y+4x-z)]\\
&\quad+3M[F_x(y-4x+z)]+3M[F_x(y-4x-z)]\\
&\quad+6M[F_x(y+2x+z)]+6M[F_x(y+2x-z)]\\
&\quad+6M[F_x(y-2x+z)]+6M[F_x(y-2x-z)]\\
&\quad+6M[F_x(y+z)]+6M[F_x(y-z)]\\
&=M\left[\frac{1}{48}f(9x+3z+y)+\frac{1}{48}f(-3x+3z-y)-\frac{1}{16}f(7x+z+y)-\frac{1}{16}f(-5x+z-y)\right]\\
&\quad+M\left[\frac{1}{48}f(-3x+3z+y)+\frac{1}{48}f(9x+3z-y)-\frac{1}{16}f(-5x+z+y)-\frac{1}{16}f(7x+z-y)\right]\\
&\quad+M\left[\frac{1}{48}f(3x+3z+y)+\frac{1}{48}f(3x+3z-y)-\frac{1}{16}f(x+z+y)-\frac{1}{16}f(x+z-y)\right]\\
&\quad+M\left[\frac{1}{48}f(9x-3z+y)+\frac{1}{48}f(-3x-3z-y)-\frac{1}{16}f(-5x-z-y)-\frac{1}{16}f(7x-z+y)\right]\\
&\quad+M\left[\frac{1}{48}f(-3x-3z+y)+\frac{1}{48}f(9x-3z-y)-\frac{1}{16}f(-5x-z-y)-\frac{1}{16}f(7x-z-y)\right]\\
&\quad+M\left[\frac{1}{48}f(3x-3z+y)+\frac{1}{48}f(3x-3z-y)-\frac{1}{16}f(x-z+y)-\frac{1}{16}f(x-z-y)\right]\\
&\quad+3M\left[\frac{1}{48}f(7x+y+z)+\frac{1}{48}f(-x-y-z)-\frac{1}{16}f(5x+y+z)-\frac{1}{16}f(-3x-y-z)\right]\\
&\quad+3M\left[\frac{1}{48}f(7x+y-z)+\frac{1}{48}f(-x-y+z)-\frac{1}{16}f(5x+y-z)-\frac{1}{16}f(-3x-y+z)\right]\\
&\quad+3M\left[\frac{1}{48}f(-x+y+z)+\frac{1}{48}f(7x-y-z)-\frac{1}{16}f(-3x+y+z)-\frac{1}{16}f(5x-y-z)\right]\\
&\quad+3M\left[\frac{1}{48}f(-x+y-z)+\frac{1}{48}f(7x-y+z)-\frac{1}{16}f(-3x+y-z)-\frac{1}{16}f(5x-y+z)\right]\\
&\quad+6M\left[\frac{1}{48}f(5x+y+z)+\frac{1}{48}f(x-y-z)-\frac{1}{16}f(3x+y+z)-\frac{1}{16}f(-x-y-z)\right]\\
&\quad+6M\left[\frac{1}{48}f(x+y+z)+\frac{1}{48}f(5x-y-z)-\frac{1}{16}f(-x+y+z)-\frac{1}{16}f(3x-y-z)\right]\\
&\quad+6M\left[\frac{1}{48}f(x+y-z)+\frac{1}{48}f(5x-y+z)-\frac{1}{16}f(-x+y-z)-\frac{1}{16}f(3x-y+z)\right]\\
&\quad+6M\left[\frac{1}{48}f(5x+y-z)+\frac{1}{48}f(x-y+z)-\frac{1}{16}f(3x+y-z)-\frac{1}{16}f(-x-y+z)\right]\\
&\quad+6M\left[\frac{1}{48}f(3x+y+z)+\frac{1}{48}f(3x-y-z)-\frac{1}{16}f(x+y+z)-\frac{1}{16}f(x-y-z)\right]\\
&\quad+6M\left[\frac{1}{48}f(3x+y-z)+\frac{1}{48}f(3x-y+z)-\frac{1}{16}f(x+y-z)-\frac{1}{16}f(x-y+z)\right].
\end{aligned}
$$

By properties of the linear operator M and oddness of f, one gets

$$3\mathcal{C}(x+z)+3\mathcal{C}(x-z)+48\mathcal{C}(x)=\mathcal{C}(3x+z)+\mathcal{C}(3x-z).$$

To finish the proof, assume that $Im\mathcal{C}_{3,1}(f)\subset A$ for some $A\in S(\mathcal{E})$. Then we have

$$Im(\frac{1}{48}\mathcal{C}_{3,1}(f))\subset\frac{1}{48}A.$$

So, $f(y)-\mathcal{C}(y)=M(\frac{1}{48}\mathcal{C}_{3,1}f(y,.))\in\frac{1}{48}A$, for all $y\in G$. Thus,

$$Im(f-\mathcal{C})\subset\frac{1}{48}A,$$

as required. □

Corollary 2.4. *Let $(\mathcal{E},\parallel.\parallel)$ be a normed space. Assume that $S(\mathcal{E})$ is a family of closed balls such that $B(\mathcal{G},S(\mathcal{E}))$ admits LIM. Let $f:\mathcal{G}\to\mathcal{E}$ be an odd function and $g:\mathcal{G}\to\mathbb{R}_+$ satisfy the inequality*

$$\parallel\mathcal{C}_{3,1}f(y,z)-\mathcal{C}_{3,1}f(z,x)\parallel\leq g(y-z)$$

for all $x,y,z\in\mathcal{G}$. Then there is a cubic function $\mathcal{C}:\mathcal{G}\to\mathcal{E}$ with

$$\parallel(f(x)-\mathcal{C}(x))-(f(y)-\mathcal{C}(y))\parallel\leq\frac{1}{48}g(x-y)$$

for all $x,y\in\mathcal{G}$.

Proof . We put $\mathbf{d}(x,y)=g(x-y)B(0,1)$ for all $x,y\in\mathcal{G}$, where $B(0,1)$ is the closed unit ball with center at zero, and make use of Theorem 2.3. □

3. The approximation in the Lipschitz norms

In this section, we study the approximation of the $\{3,1\}$-cubic functional equation in the Lipschitz norms. Before that, we recall the following definition.

Definition 3.1. *[23] Let $(\mathcal{E},\parallel.\parallel)$ be a normed space.*

(i) *A function $f:\mathcal{G}\to\mathcal{E}$ is called Lipschitz function if*

$$\parallel f(x)-f(y)\parallel\leq Ld(x,y)$$

for all $x,y\in\mathcal{G}$ and for a constant $L\in\mathbb{R}_+$. We denote the smallest constant possesing this property by $lip(f)$. Let $Lip(\mathcal{G},\mathcal{E})$ be the space of all bounded Lipschitz functions with the norm

$$\parallel f\parallel_{Lip}:=\parallel f\parallel_{\sup}+lip(f)$$

for all $f\in Lip(\mathcal{G},\mathcal{E})$. Moreover, by $Lip_0(\mathcal{G},\mathcal{E})$ we denote the space of all Lipschitz functions $f:\mathcal{G}\to\mathcal{E}$ with the norm defined by the formula

$$\parallel f\parallel_0=\parallel f(0)\parallel+lip(f).$$

(ii) *A function* $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ *is the module of continuity of the function* $f : \mathcal{G} \to \mathcal{E}$*, if for every* $\delta > 0$ *and* $x, y \in \mathcal{G}$*,*

$$d(x,y) \leq \delta \quad \Rightarrow \quad \| f(x) - f(y) \| \leq \omega(\delta).$$

(iii) *A group* $(\mathcal{G}, d, d')$ *is said to be a metric pair if* d *is a metric in abelian group* $\mathcal{G}$ *and* d' *is a metric in* $\mathcal{G} \times \mathcal{G}$ *such that*

$$d'((a,x),(a,y)) = d'((x,a),(y,a)) = d(x,y)$$

for all $a, x, y \in \mathcal{G}$.

Theorem 3.2. *Let* $(\mathcal{G}, d, d')$ *be a metric pair and let* $(\mathcal{E}, \| \,.\, \|)$ *be a normed space. Assume that* $S(\mathcal{E})$ *is a family of closed balls such that* $B(\mathcal{G}, S(\mathcal{E}))$ *admits* LIM*. Let* $f : \mathcal{G} \to \mathcal{E}$ *be an odd function. If* $\mathcal{C}_{3,1} f \in Lip(\mathcal{G} \times \mathcal{G}, \mathcal{E})$*, then there is a cubic function* $\mathcal{C} : \mathcal{G} \to \mathcal{E}$ *such that* $f - \mathcal{C} \in Lip(\mathcal{G}, \mathcal{E})$ *and*

$$\| f - \mathcal{C} \|_{Lip} \leq \frac{1}{48} \| \mathcal{C}_{3,1} f \|_{Lip} .$$

Furthermore, if $\mathcal{C}_{3,1} f \in Lip^0(\mathcal{G} \times \mathcal{G}, \mathcal{E})$*, then there is a cubic function* $\mathcal{C} : \mathcal{G} \to \mathcal{E}$ *such that* $f - \mathcal{C} \in Lip^0(\mathcal{G}, \mathcal{E})$ *and*

$$\| f - \mathcal{C} \|_{Lip^0} \leq \frac{1}{48} \| \mathcal{C}_{3,1} f \|_{Lip^0} .$$

Proof . Suppose that $\mathcal{C}_{3,1} f \in Lip(\mathcal{G} \times \mathcal{G}, \mathcal{E})$. Let $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by $\omega(x) := lip(\mathcal{C}_{3,1} f)x$, for all $x \in \mathcal{G}$. Since $\mathcal{C}_{3,1} f \in Lip(\mathcal{G} \times \mathcal{G}, \mathcal{E})$, it follows that

$$\| \mathcal{C}_{3,1} f(x,y) - \mathcal{C}_{3,1} f(m,k) \| \leq lip(\mathcal{C}_{3,1} f) d'((x,y),(m,k)) = \omega(d'((x,y),(m,k)))$$

for all $x, y, m, k \in \mathcal{G}$. This means that ω is the module of continuity of the function $\mathcal{C}_{3,1} f$. Define $\mathbf{d} : \mathcal{G} \times \mathcal{G} \to S(\mathcal{E})$ by the formula

$$\mathbf{d}(x,y) := \{ \inf_{r \geq d(x,y)} \omega(r) \} B(0,1)$$

for all $x, y \in \mathcal{G}$. Since ω is the module of continuity, it follows that

$$\| \mathcal{C}_{3,1} f(y,x) - \mathcal{C}_{3,1} f(z,x) \| \leq \inf_{r \geq d'((y,x),(z,x))} \omega(r) = \inf_{r \geq d(y,z)} \omega(r)$$

for all $x, y, z \in \mathcal{G}$. This means that $\mathcal{C}_{3,1} f(.,x)$ is a $\mathbf{d}$-Lipschitz function. Then by Theorem 2.3, there exists a new type cubic function $\mathcal{C} : \mathcal{G} \to \mathcal{E}$ such that

$$(f(x) - \mathcal{C}(x)) - (f(y) - \mathcal{C}(y)) \in (\frac{1}{48}) \mathbf{d}(x,y)$$

for all $x, y \in \mathcal{G}$. Consequently,

$$\begin{aligned}
\| (f(x) - \mathcal{C}(x)) - (f(y) - \mathcal{C}(y)) \| &\leq \inf_{r \geq d(x,y)} (\frac{1}{48}) \omega(r) \\
&\leq (\frac{1}{48}) \omega(d(x,y)) \\
&= (\frac{1}{48}) lip(\mathcal{C}_{3,1} f) d(x,y)
\end{aligned}$$

for all $x, y \in \mathcal{G}$. So, $f - \mathcal{C}$ is a Lipschitz function and $lip(f - \mathcal{C}) \leq (\frac{1}{48})lip(\mathcal{C}_{3,1}f)$. Since $\mathcal{C}_{3,1}f \in Lip(\mathcal{G} \times \mathcal{G}, \mathcal{E})$, we have $\mathcal{C}_{3,1}f \in B(\mathcal{G} \times \mathcal{G}, \mathcal{E})$ and we obtain

$$\| f - \mathcal{C} \|_{\sup} \leq \frac{1}{48} \| C_k f \|_{\sup} .$$

That is $f - \mathcal{C} \in Lip(\mathcal{G}, \mathcal{E})$. So,

$$\| f - \mathcal{C} \|_{Lip} \leq \frac{1}{48} \| \mathcal{C}_{3,1} f \|_{Lip} .$$

Now, suppose that $\mathcal{C}_{3,1}f \in Lip^0(\mathcal{G} \times \mathcal{G}, \mathcal{E})$. By the same way, one can obtain a new type cubic function $\mathcal{C} : \mathcal{G} \to \mathcal{E}$ such that $f - \mathcal{C}$ is Lipschitz and

$$lip(f - \mathcal{C}) \leq (\frac{1}{48})lip(\mathcal{C}_{3,1}f).$$

Since $C(0) = 0$, we have

$$\begin{aligned} \| f - \mathcal{C} \|_{Lip^0} &= \| f(0) - \mathcal{C}(0) \| + lip(f - \mathcal{C}) \\ &\leq \frac{1}{48} \| \mathcal{C}_{3,1} f(0,0) \| + (\frac{1}{48}) lip(\mathcal{C}_{3,1} f) \\ &\leq \frac{1}{48} \| \mathcal{C}_{3,1} f \|_{Lip^0}, \end{aligned}$$

as, required. □

4. Open research problems

Problem A.

A functional equation is said to be $\{m, k, r, s\}$-additive if $f : \mathcal{G} \to \mathcal{E}$ satisfies the functional equation

$$\mathcal{A}_{m,k,r,s} f(x, y) := f(mx + ky) - rf(x) - sf(y) = 0$$

for some real numbers $m > 0, k > 0, r, s$ with $m + k = r + s \neq 1$. Furthermore, the equalities $\mathcal{A}_{1,1,1,1}f(x, y) = 0$ and $\mathcal{A}_{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}}f(x, y) = 0$ are the additive and Jensen functional equations, and every solution of these functional equations are additive and Jensen functions, respectively.

The stability of the $\{m, k, r, s\}$-additive functional equation was studied by Lu and Park [14].

Tabor [23] proved that if the function $\mathcal{A}_{1,1,1,1}f$ is Lipschitz, then there exists an additive function $A : \mathcal{G} \to \mathcal{E}$ such that $f - A$ is Lipschitz with the same constant. He also proved analogous results for the Jensen functions.

Let us note that, a mapping $F : G^n \to E$ is said to be multi $\{m, k, r, s\}$-additive if it satisfies the $\{m, k, r, s\}$-additive equation in each of its n arguments, that is

$$\begin{aligned} F(x_1, ..., x_{i-1}, mx_i + ky_i, x_{i+1}, ..., x_n) = &rF(x_1, ..., x_{i-1}, x_i, x_{i+1}, ..., x_n) \\ &+ sF(x_1, ..., x_{i-1}, y_i, x_{i+1}, ..., x_n), \end{aligned}$$

where $i \in \{1, ..., n\}$, $x_1, ..., x_{i-1}, x_i, y_i, x_{i+1}, ..., x_n \in G$. In the special case, a multi $\{1, 1, 1, 1\}$-additive ($\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}$-additive) mapping is said to be a multi additive (multi Jensen) mapping.

The stability of multi additive mappings in Banach spaces and complete non-Archimedean spaces were investigated by Ciepliński [2, 3, 4].

(i) Is the Tabor's result true for the $\{m, k, r, s\}$-additive functional equation?

(ii) Is the Tabor's result true for multi Jensen mappings?

(iii) Is the Tabor's result true for multi $\{m, k, r, s\}$-additive mappings?

Problem B.

A functional equation is said to be $\{m, k\}$-quadratic if $f : \mathcal{G} \to \mathcal{E}$ satisfies the functional equation

$$\mathcal{Q}_{m,k}f(x, y) := f(mx + ky) + f(mx - ky) - 2m^2 f(x) - 2k^2 f(y) = 0$$

for any fixed integers m, k with $m, k \neq 0$. In a special case, the equality $\mathcal{Q}_{1,1}f(x, y) = 0$ is called a quadratic functional equation, and every solution of the last functional equation is said to be a quadratic function. Eshaghi and second author [9], studied the stability of the $\{m, k\}$-quadratic functional equation in Banach spaces. Czerwik and Dłutek [7] studied the stability of the quadratic functional equation in Lipschitz spaces.

A mapping $F : G^n \to E$ is said to be multi $\{m, k\}$-quadratic if it satisfies the $\{m, k\}$-quadratic equation in each of its n arguments, that is

$$\begin{aligned} &F(x_1, ..., x_{i-1}, mx_i + ky_i, x_{i+1}, ..., x_n) + F(x_1, ..., x_{i-1}, mx_i - ky_i, x_{i+1}, ..., x_n) \\ &= 2m^2 F(x_1, ..., x_{i-1}, x_i, x_{i+1}, ..., x_n) + 2k^2 F(x_1, ..., x_{i-1}, y_i, x_{i+1}, ..., x_n), \end{aligned}$$

where $i \in \{1, ..., n\}$, $x_1, ..., x_{i-1}, x_i, y_i, x_{i+1}, ..., x_n \in G$. In the special case, a multi $\{1, 1\}$-quadratic mapping is said to be a multi quadratic mapping.

Park [21] and Ciepliński [5] proved the stability of the multi quadratic mappings in Banach spaces and complete non-Archimedean spaces, respectively.

(i) Is the Czerwik and Dłutek's result true for the $\{1, k\}$-quadratic functional equation? (Yes, it is solved by Chahbi et al. [6].)

(ii) Is the Czerwik and Dłutek's result true for the $\{m, k\}$-quadratic functional equation?

(iii) Is the Czerwik and Dłutek's result true for the multi quadratic mappings? (Yes, it is solved by Nikoufar [20].)

(iv) Is the Czerwik and Dłutek's result true for the multi $\{m, k\}$-quadratic mappings?

Problem C.

A functional equation is said to be $\{m, k\}$-cubic if $f : \mathcal{G} \to \mathcal{E}$ satisfies the functional equation

$$\begin{aligned} \mathcal{C}_{m,k}f(x, y) := &f(mx + ky) + f(mx - ky) \\ &- mk^2 f(x + y) - mk^2 f(x - y) - 2m(m^2 - k^2)f(x) \\ = &0 \end{aligned}$$

for any fixed integers m, k with $m \neq \pm 1, 0,\ k \neq 0$ and $m \neq \pm k$. In a special case, the equality $\mathcal{C}_{2,1} f(x,y) = 0$ is called a cubic functional equation, and every solution of the last functional equation is said to be a cubic function; see [11, 13].

The stability of cubic functional equation in Banach spaces and Lipschitz spaces were investigated by Jun and Kim [10] and Ebadian et al. [8], respectively.

A mapping $F : G^n \to E$ is said to be multi $\{m,k\}$-cubic if it satisfies the $\{m,k\}$-cubic equation in each of its n arguments, that is

$$\begin{aligned} &F(x_1,...,x_{i-1},mx_i+ky_i,x_{i+1},...,x_n) + F(x_1,...,x_{i-1},mx_i-ky_i,x_{i+1},...,x_n)\\ &= mk^2F(x_1,...,x_{i-1},x_i+y_i,x_{i+1},...,x_n) + mk^2F(x_1,...,x_{i-1},x_i-y_i,x_{i+1},...,x_n)\\ &\quad + 2m(m^2-k^2)F(x_1,...,x_{i-1},x_i,x_{i+1},...,x_n), \end{aligned}$$

where $i \in \{1,...,n\},\ x_1,...,x_{i-1},x_i,y_i,x_{i+1},...,x_n \in G$.

(i) Is Ebadian et al.'s result true for $\{m,1\}$-cubic functional equation? (In the special case, the $\{3,1\}$-cubic functional equation, we responded positively to this question in Theorem 2.3 in this paper.)

(ii) Is Ebadian et al.'s result true for $\{m,k\}$-cubic functional equation?

(iii) Is Ebadian et al.'s result true for multi $\{m,k\}$-cubic mappings?

Problem D.

A functional equation is said to be $\{m,k\}$-quartic if $f : \mathcal{G} \to \mathcal{E}$ satisfies the functional equation

$$\begin{aligned} \mathfrak{Q}_{m,k} f(x,y) &:= f(mx+ky) + f(mx-ky) - 2m^2(m^2-k^2)f(x)\\ &\quad - (mk)^2[f(x+y)+f(x-y)] + 2k^2(m^2-k^2)f(y)\\ &= 0, \end{aligned}$$

where $m, k \neq 0, m \neq \pm k$. In a special case, the equality $\mathfrak{Q}_{2,1} f(x,y) = 0$ is called a quartic functional equation, and every solution of the last functional equation is said to be a quartic function [12].

The stability of $\{m,k\}$-quartic functional equation in Banach spaces was investigated by Kang [12]. Nikoufar [15, 16] studied the stability of the quartic functional equation in the Lipschitz norms.

A mapping $F : G^n \to E$ is said to be multi $\{m,k\}$-quartic if it satisfies the $\{m,k\}$-quartic equation in each of its n arguments, that is

$$\begin{aligned} &F(x_1,...,x_{i-1},mx_i+ky_i,x_{i+1},...,x_n) + F(x_1,...,x_{i-1},mx_i-ky_i,x_{i+1},...,x_n)\\ &= (mk)^2[F(x_1,...,x_{i-1},x_i+y_i,x_{i+1},...,x_n) + F(x_1,...,x_{i-1},x_i-y_i,x_{i+1},...,x_n)]\\ &\quad + 2m^2(m^2-k^2)F(x_1,...,x_{i-1},x_i,x_{i+1},...,x_n) - 2k^2(m^2-k^2)F(x_1,...,x_{i-1},y_i,x_{i+1},...,x_n), \end{aligned}$$

where $i \in \{1,...,n\},\ x_1,...,x_{i-1},x_i,y_i,x_{i+1},...,x_n \in G$.

(i) Is the Nikoufar's result true for $\{m,k\}$-quartic functional equation?

(ii) Is the Nikoufar's result true for multi $\{m,k\}$-quartic mappings?

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