Int. J. Nonlinear Anal. Appl. 12 (2021) No. 1, 541-554
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.4836

Grüss type integral inequalities for a new class of k-fractional integrals

Sidra Habib[a], Ghulam Farid[b,*], Shahid Mubeen[c]

[a]*Department of Mathematics, G.C. University Faisalabad, Faisalabad, Pakistan*
[b]*Department of Mathematics, COMSATS University Islamabad, Attock Campus, Pakistan*
[c]*Department of Mathematics, University of Sargodha, Sargodha, Pakistan*

(Communicated by Madjid Eshaghi Gordji)

Abstract

The main aim of this research article is to present the generalized k-fractional conformable integrals and an improved version of Grüss integral inequality via the fractional conformable integral in status of a new parameter $k > 0$. Here for establishing Grüss inequality in fractional calculus the classical method of proof has been adopted also related results with Grüss inequality have been discussed. This work contributes in the current research by providing mathematical results along with their verifications.

Keywords: k-fractional conformable integrals, Fractional integral inequalities, Grüss inequality
2010 MSC: Primary 33E20; Secondary 26A33.

1. Introduction

Inequalities comprising functions of two or more independent variables perform elementary role in the incessant advancement of the notion, approaches and applications of differential and integral equations. Considering the extensive applications, integral inequalities have grasped remarkable attention. Nowadays, several types of such inequalities have been acquired which act as vital tools in the fundamental study of differential and integral equations of different classes. It is well-known that Grüss inequality plays a crucial role in the investigation of qualitative behavior of differential and difference equations in many areas of mathematics in both continuous and discrete cases, Akin et al.

*Corresponding author
Email addresses: sidrahabib1605@gmail.com (Sidra Habib), faridphdsms@hotmail.com (Ghulam Farid), smjhanda@gmail.com (Shahid Mubeen)

Received: 28 May 2018 *Revised:* 11 November 2019

(2015) prove a discrete case of Grüss inequality in fractional calculus.

Fractional integral inequalities and operators are deemed to be profound as they screw many useful applications in science and engineering. Furthermore, the fractional calculus theory is expended in the solutions of differential equations, integral equations and integro-differential equations as well as in many other special function problems. Fractional integral inequalities and operators were came into being due to the most classical fractional integral operator known as the Riemann-Liouville fractional integral operator. Now a days, these inequalities and many generalized fractional integral operators have been studied by Tune et al. (2017), Usta et al. (2017(a), 2017(b), 2017(c)), Usta and Sarikaya (2018).

Over the last decade, the research has been advanced to develop the investigation for fractional integral inequalities. Such types of inequalities and their applications have been extensively studied in many articles for instance by Dragomir (1999), Li (2002), Pachpatte (2002), Elezovic et al. (2007), Liao et al. (2013), Chen (2014), Sarikaya et al. (2016), Farid et al. (2016), Farid et al. (2017), Abbas et al. (2017). Amongst such integral inequalities, Grüss inequality is regarded to be more fascinating, perhaps. Grüss inequality can be characterized as follows.

Theorem 1.1. *Let $f_1, f_2 : [a,b] \to \mathbb{R}$ be integrable functions such that*

$$\varphi < f_1(x) < \Phi \text{ and } \psi < f_2(x) < \Psi \text{ for all } x \in [a,b]$$

Then

$$\left|\frac{1}{b-a}\int_a^b f_1(x)f_2(x)dx - \frac{1}{(b-a)^2}\int_a^b f_1(x)dx \int_a^b f_2(x)dx\right| \leq \frac{1}{4}(\Phi-\varphi)(\Psi-\psi) \tag{1}$$

where the constant $\frac{1}{4}$ is sharp and $\varphi, \Phi, \psi, \Psi \in \mathbb{R}$.

The above described integral inequality (1) (Grüss inequality) actually connects the integral of the product of two functions with the product of their integrals. Researchers focused in investigating such useful inequalities and provided remarkable results for inequalities involving fractional integrals like Riemann-Liouville fractional integral, and many other generalized fractional integrals for example by Dahmani (2012), Wang et al. (2014), Tariboon et al. (2014), Kaçar et al. (2015), Park (2015), Abbas et al. (2016), Ayub et al. (2017), Waheed et al. (2018). We use the following definitions in sequel.

Definition 1.2. *A function f is said to be in $L_{p,s}[a,b]$ if*

$$\left(\int_a^b |f(z)|^p z^s dz\right)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty, s \geq 0.$$

Abdeljawad (2015) gave the following definition of a fractional conformable integral.

Definition 1.3. *Let $f \in L_1[a,b]$. Then the left and right sided fractional conformable integral of order $\alpha \geq 0$ are defined as follows:*

$$I_a^\alpha f(x) = \int_a^x (t-a)^{\alpha-1} f(t)dt, \quad x \in [a,b], 0 < \alpha < 1.$$

and

$$I_b^\alpha f(x) = \int_x^b (b-t)^{\alpha-1} f(t)dt, \quad x \in [a,b], 0 < \alpha < 1.$$

Jarad et al. (2017) defined a generalized fractional conformable integral as follows.

Definition 1.4. *Let $f \in L_{1,s}[a,b]$, then the generalized left and right sided fractional conformable integral $\mathfrak{T}_a^{\alpha,s}$ of order $\alpha \in \mathbb{C}, Re(\alpha) > 0$ and $s > 0$, are defined as follows:*

$$\mathfrak{T}_a^{\alpha,s} f(t) = \frac{s^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \left((t-a)^s - (x-a)^s\right)^{\alpha-1} (x-a)^{s-1} f(x)dx, \quad t \in [a,b],$$

and

$$\mathfrak{T}_b^{\alpha,s} f(t) = \frac{s^{1-\alpha}}{\Gamma(\alpha)} \int_t^b \left((b-x)^s - (b-t)^s\right)^{\alpha-1} (b-x)^{s-1} f(x)dx, \quad t \in [a,b],$$

where Γ is the Euler gamma function.

Kaçar et al. (2015) presented useful results by studying Grüss type inequality. Owing to the significance of such conclusions and presentations, we analyze the current results and present novel results. We generalize the existing fractional conformal integral by involving a new parameter $k > 0$ and apply it to generalize Grüss type inequality and the related results in such a manner that the existing results can be deduced too. Hence, the results presented in this paper are the generalization of the existing results by involving a new class of fractional integrals.

The idea of the generalizations of special functions as well as fractional operators was initiated by Diaz et al. (2007), who defined the generalization of the classical gamma and beta functions in $k-$analogue, called $k-$gamma and $k-$beta functions respectively. Krasniqi (2010) proved the monotonicity and some important inequalities for the ratio of gamma k-function. Later Mubeen et al. (2012) extended the fractional integrals by involving the parameter $k > 0$. Kokologiannaki et al. (2013) presented complete monotonicity characteristics and inequalities involving the k-gamma and k-psi functions. They also proposed the Riemann k-zeta function and obtained some related inequalities of gamma and k-zeta functions. Romero et al. (2013) generalized the Riemann-Liouville fractional derivative by using k-gamma function in the form of k-Riemann-Liouville derivative and also proved some important results of their newly introduced fractional operator and induced its relationship with Riemann-Liouville k-fractional integral.

A large number of inequalities involving fractional integrals and their q-analogues are investigated by many researchers over the past decade. Baleanu et al. (2014) used the two parameters of deformation to establish some inequalities containing Saigo q-fractional integral operator in quantum calculus theory. They also exhibited the analogous inequalities of q-Riemann-Liouville and q-Kober fractional integrals respectively as special cases. Choi et al. (2015) found several new Saigo type fractional integral inequalities and related q-analogue inequalities. They developed the inequalities involving Riemann-Liouville and Erdélyi-Kober type fractional integral operators as their special cases.

Agarwal et al. (2014) established some novel fractional integral inequalities involving generalized q-Erdélyi-Kober fractional integral operator by considering the cases of synchronous functions as well as the functions bounded by integrable functions. Sarikaya at al. (2014) generalized the Riemann-Liouville k-fractional integral including some properties as well as they proved Chebyshev integral inequalities involving this generalized Riemann-Liouville k-fractional integral.

Choi et al. (2016) established some inequalities involving generalized q-Erdélyi-Kober fractional integral of the two parameters of deformation. Liu et al. (2015) verified some new helpful integral inequalities of Gronwall-Bellman-Bihari type with delay for discontinuous functions to discuss the qualitative and quantitative properties for solutions to some nonlinear differential and integral equations which generalize and improve some prior famous results regarding inequalities.

Here, we introduce (k,s)-fractional conformable integral operator as

$$I_{a,k}^{\alpha,s}f(t)=\frac{(s)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t\left((t-a)^s-(x-a)^s\right)^{\frac{\alpha}{k}-1}(x-a)^{s-1}f(x)dx,\quad t\in[a,b], \tag{2}$$

where Γ_k is the Euler gamma k-function.

Motivated by Grüss (1935), our purpose is to prove more general versions of Grüss type inequalities utilizing the (k,s)-fractional conformable integral operator. These fractional integral inequalities are general in the sense that results for Riemann-Liouville k-fractional integrals and Riemann-Liouville fractional integrals are particular results of this paper. We prove a version of the Grüss inequality by following the method of the proof of the classical one via defined fractional integral operator. Several deduced results have been mentioned as applications.

2. Results and Discussion

This section presents some Grüss type inequalities involving the generalized k-fractional conformable integral $I_{a,k}^{\alpha,s}$ defined in (2).

Theorem 2.1. *For $k>0$, let $f\in L_{1,s}[a,b]$ and $s>0$, $\alpha,\beta>0$. Suppose that there exist two integrable functions ζ_1, ζ_2 on $[a,b]$ such that*

$$\zeta_1(t)\le f(t)\le\zeta_2(t),\quad\forall t\in[a,b]. \tag{3}$$

Then the following inequality holds true

$$I_{a,k}^{\beta,s}\zeta_1(t)I_{a,k}^{\alpha,s}f(t)+I_{a,k}^{\alpha,s}\zeta_2(t)I_{a,k}^{\beta,s}f(t)\ge I_{a,k}^{\alpha,s}\zeta_2(t)I_{a,k}^{\beta,s}\zeta_1(t)+I_{a,k}^{\alpha,s}f(t)I_{a,k}^{\beta,s}f(t)$$

Proof . From inequality (3), for all $x,y\in[a,b]$, we have

$$(\zeta_2(x)-f(x))(f(y)-\zeta_1(y))\ge 0.$$

This implies that

$$\zeta_2(x)f(y)+\zeta_1(y)f(x)\ge\zeta_1(y)\zeta_2(x)+f(x)f(y).$$

By multiplying both sides of above inequality with

$$\frac{s^{2-\frac{\alpha}{k}-\frac{\beta}{k}}\left((t-a)^s-(x-a)^s\right)^{\frac{\alpha}{k}-1}\left((t-a)^s-(y-a)^s\right)^{\frac{\beta}{k}-1}(x-a)^{s-1}(y-a)^{s-1}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}$$

and then integrating it with respect to x and y from a to t, we obtain the required inequality. □

Corollary 2.2. *Put $s=1$ in Theorem 2.1, we get following result for Riemann-Liouville k-fractional integral $I_{a,k}^{\alpha}$ of order α defined in [25]*

$$\begin{aligned}&I_{a,k}^{\beta}\zeta_1(t)I_{a,k}^{\alpha}f(t)+I_{a,k}^{\alpha}\zeta_2(t)I_{a,k}^{\beta}f(t)\\ \ge\ &I_{a,k}^{\alpha}\zeta_2(t)I_{a,k}^{\beta}\zeta_1(t)+I_{a,k}^{\alpha}f(t)I_{a,k}^{\beta}f(t).\end{aligned}$$

Theorem 2.3. *For $k > 0$, let $f \in L_1[a,b]$. Suppose that $m_1 \leq f(t) \leq m_2$, for all $t \in [a,b]$ and $m_1, m_2 \in \mathbb{R}$. Then for $s > 0$, $\alpha, \beta > 0$, we have*

$$m_2 \frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)} I_{a,k}^{\beta,s} f(t) + m_1 \frac{(s)^{\frac{-\beta}{k}}(t-a)^{\frac{s\beta}{k}}}{\Gamma_k(\beta+k)} I_{a,k}^{\alpha,s}$$
$$\geq m_1 m_2 \frac{(s)^{-\frac{(\alpha+\beta)}{k}}(t-a)^{(s)\left(\frac{\alpha+\beta}{k}\right)}}{\Gamma_k(\alpha+k)\Gamma_k(\beta+k)} + I_{a,k}^{\alpha,s} f(t) I_{a,k}^{\beta,s} f(t).$$

Proof . Since $m_1 \leq f(t) \leq m_2$ for all $t \in [a,b]$, therefore for all $x, y \in [a,b]$, we have

$$(m_2 - f(x))(f(y) - m_1) \geq 0.$$

The required inequality can be proved by using the above inequality and following the steps of Theorem 2.1. □

Corollary 2.4. *Put $s = 1$ in Theorem 2.3, we get the following result for Riemann-Liouville k-fractional integrals*

$$m_2 \frac{(t-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} I_{a,k}^{\beta} f(t) + m_1 \frac{(t-a)^{\frac{\beta}{k}}}{\Gamma_k(\beta+k)} I_{a,k}^{\alpha} f(t)$$
$$\geq m_1 m_2 \frac{(t-a)^{\frac{\alpha+\beta}{k}}}{\Gamma_k(\alpha+k)\Gamma_k(\beta+k)} + I_{a,k}^{\alpha} f(t) I_{a,k}^{\beta} f(t).$$

Theorem 2.5. *For $k > 0$, let f and g be two integrable functions on $[a,b]$ and $s > 0$, $\alpha, \beta > 0$. Suppose that (3) holds and moreover assume that there exist integrable functions ξ_1 and ξ_2 on $[a,b]$ such that*

$$\xi_1(t) \leq g(t) \leq \xi_2(t), \quad \forall t \in [a,b]. \tag{4}$$

Then the following inequalities hold:
(i) $I_{a,k}^{\beta,s}\xi_1(t) I_{a,k}^{\alpha,s} f(t) + I_{a,k}^{\alpha,s}\zeta_2(t) I_{a,k}^{\beta,s} g(t)$
$\geq I_{a,k}^{\beta,s}\xi_1(t) I_{a,k}^{\alpha,s}\zeta_2(t) + I_{a,k}^{\alpha,s} f(t) I_{a,k}^{\beta,s} g(t)$,
(ii) $I_{a,k}^{\beta,s}\zeta_1(t) I_{a,k}^{\alpha,s} g(t) + I_{a,k}^{\alpha,s}\xi_2(t) I_{a,k}^{\beta,s} f(t)$
$\geq I_{a,k}^{\beta,s}\zeta_1(t) I_{a,k}^{\alpha,s}\xi_2(t) + I_{a,k}^{\alpha,s} g(t) I_{a,k}^{\beta,s} f(t)$,
(iii) $I_{a,k}^{\alpha,s}\zeta_2(t) I_{a,k}^{\beta,s}\xi_2(t) + I_{a,k}^{\alpha,s} f(t) I_{a,k}^{\beta,s} g(t)$
$\geq I_{a,k}^{\alpha,s}\zeta_2(t) I_{a,k}^{\beta,s} g(t) + I_{a,k}^{\beta,s}\xi_2(t) I_{a,k}^{\alpha,s} f(t)$,
(iv) $I_{a,k}^{\alpha,s}\zeta_1(t) I_{a,k}^{\beta,s}\xi_1(t) + I_{a,k}^{\alpha,s} f(t) I_{a,k}^{\beta,s} g(t)$
$\geq I_{a,k}^{\alpha,s}\zeta_1(t) I_{a,k}^{\beta,r} g(t) + I_{a,k}^{\beta,s}\xi_1(t) I_{a,k}^{\alpha,s} f(t)$.

Proof . (i): From (3) and (4) we have

$$(\zeta_2(x) - f(x))(g(y) - \xi_1(y)) \geq 0, \tag{5}$$

then

$$\zeta_2(x) g(y) + \xi_1(y) f(x) \geq \xi_1(y)\zeta_2(x) + f(x) g(y).$$

By multiplying both sides of above inequality with

$$\frac{s^{2-\frac{\alpha}{k}-\frac{\beta}{k}}\left((t-a)^s - (x-a)^s\right)^{\frac{\alpha}{k}-1}\left((t-a)^s - (y-a)^s\right)^{\frac{\beta}{k}-1}(x-a)^s(y-a)^s}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}$$

and then integrating with respect to x and y over (a,t), we obtain (i).
To prove (ii)-(iv), we use the following inequalities instead of (5), then on same lines as done to obtain (i) one can get inequalities (ii)-(iv).
(ii) $(\xi_2(x)-g(x))(f(y)-\zeta_1(y))\geq 0$,
(iii) $(\zeta_2(x)-f(x))(g(y)-\xi_2(y))\leq 0$,
(iv) $(\zeta_1(x)-f(x))(g(y)-\xi_1(y))\leq 0$.
□

Corollary 2.6. *For $k>0$, let f and g be two integrable functions on $[a,b]$ and $s>0$, $\alpha,\beta>0$. Suppose that there exist $m_1,m_2,n_1,n_2\in\mathbb{R}$ such that*

$$m_1\leq f(t)\leq m_2,\quad n_1\leq g(t)\leq n_2,\quad \forall t\in[a,b].$$

Then we have

$$(i^*)\quad n_1\frac{(s)^{\frac{-\beta}{k}}(t-a)^{\frac{s\beta}{k}}}{\Gamma_k(\beta+k)}I_{a,k}^{\alpha,s}f(t)+m_2\frac{s^{\frac{-\alpha}{k}}(t-a)^{(\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}I_{a,k}^{\beta,s}g(t)\\ \geq n_1m_2\frac{s^{-\frac{\alpha+\beta}{k}}(t-a)^{\frac{s(\alpha+\beta)}{k}}}{\Gamma_k(\alpha+k)\Gamma_k(\beta+k)}+I_{a,k}^{\alpha,s}f(t)I_{a,k}^{\beta,s}g(t),$$

$$(ii^*)\quad m_1\frac{(s)^{\frac{-\beta}{k}}(t-a)^{\frac{s\beta}{k}}}{\Gamma_k(\beta+k)}I_{a,k}^{\alpha,s}g(t)+n_2\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}I_{a,k}^{\beta,s}f(t)\\ \geq m_1n_2\frac{s^{-\frac{\alpha+\beta}{k}}(t-a)^{\frac{s(\alpha+\beta)}{k}}}{\Gamma_k(\alpha+k)\Gamma_k(\beta+k)}+I_{a,k}^{\beta,s}f(t)I_{a,k}^{\alpha,s}g(t),$$

$$(iii^*)\quad m_2\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}I_{a,k}^{\beta,s}g(t)+n_2\frac{s^{\frac{-\beta}{k}}(t-a)^{\frac{s\beta}{k}}}{\Gamma_k(\beta+k)}I_{a,k}^{\alpha,s}f(t)\\ \leq m_2n_2\frac{s^{-\frac{\alpha+\beta}{k}}(t-a)^{\frac{s(\alpha+\beta)}{k}}}{\Gamma_k(\alpha+k)\Gamma_k(\beta+k)}+I_{a,k}^{\alpha,s}f(t)I_{a,k}^{\beta,s}g(t),$$

$$(iv^*)\quad m_1\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}I_{a,k}^{\beta,s}g(t)+n_1\frac{s^{\frac{-\beta}{k}}(t-a)^{\frac{s\beta}{k}}}{\Gamma_k(\beta+k)}I_{a,k}^{\alpha,s}f(t)\\ \leq m_1n_1\frac{(s)^{-\frac{\alpha+\beta}{k}}(t-a)^{\frac{s(\alpha+\beta)}{k}}}{\Gamma_k(\alpha+k)\Gamma_k(\beta+k)}+I_{a,k}^{\alpha,s}f(t)I_{a,k}^{\beta,s}g(t).$$

Proof . Proof follows on same lines as the proof Theorem 2.5 just use here $\zeta_1(t)=m_1,\zeta_2(t)=m_2,\xi_1(t)=n_1$ and $\xi_2(t)=n_2$ as constant functions. □

Theorem 2.7. *For $k > 0$, let $f \in L_{1,s}[a,b]$ and let ζ_1, ζ_2 be two integrable functions on $[a,b]$ and $s > 0$, $\alpha > 0$. Suppose that condition (3) holds. Then*

$$\begin{aligned}
&\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)} I_{a,k}^{\alpha,s} f^2(t) - [I_{a,k}^{\alpha,s} f(t)]^2 \\
&= (I_{a,k}^{\alpha,s}\zeta_2(t) - I_{a,k}^{\alpha,s} f(t))(I_{a,k}^{\alpha,s} f(t) - I_{a,k}^{\alpha,s}\zeta_1(t)) \\
&- \frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)} \left(I_{a,k}^{\alpha,s}\zeta_2(t) - I_{a,k}^{\alpha,s} f(t)\right)\left(I_{a,k}^{\alpha,s} f(t) - I_{a,k}^{\alpha,s}\zeta_1(t)\right) \\
&+ \frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)} I_{a,k}^{\alpha,s}(\zeta_1(t)f(t)) - I_{a,k}^{\alpha,s}\zeta_1(t) I_{a,k}^{\alpha,s} f(t) \\
&+ \frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)} I_{a,k}^{\alpha,s}(\zeta_2(t)f(t)) - I_{a,k}^{\alpha,s}\zeta_2(t) I_{a,k}^{\alpha,s} f(t) \\
&- \frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)} I_{a,k}^{\alpha,s}(\zeta_1(t)\zeta_2(t)) + I_{a,k}^{\alpha,s}\zeta_1(t) I_{a,k}^{\alpha,s}\zeta_2(t). \qquad (6)
\end{aligned}$$

Proof . For any $x, y \in [a,b]$ and $s > 0$, we have

$$\begin{aligned}
&(\zeta_2(y) - f(y))(f(x) - \zeta_1(x)) + (\zeta_2(x) - f(x))(f(y) - \zeta_1(y)) \\
&- (\zeta_2(x) - f(x))(f(x) - \zeta_1(x)) - (\zeta_2(y) - f(y))(f(y) - \zeta_1(y)) \\
&= f^2(x) + f^2(y) - 2f(x)f(y) + \zeta_2(y)f(x) + \zeta_1(x)f(y) \\
&- \zeta_1(x)\zeta_2(y) + \zeta_2(x)f(y) + \zeta_1(y)f(x) - \zeta_1(y)\zeta_2(y) \\
&- \zeta_2(x)f(x) + \zeta_1(x)\zeta_2(x) - \zeta_1(x)f(x) - \zeta_2(y)f(y) \\
&+ \zeta_1(y)\zeta_2(y) - \zeta_1(y)f(y).
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
&(\zeta_2(y) - f(y))\left(I_{a,k}^{\alpha,s} f(t) - I_{a,k}^{\alpha,s}\zeta_1(t)\right) + \left(I_{a,k}^{\alpha,s}\zeta_2(t) - I_{a,k}^{\alpha,s} f(t)\right)(f(y) - \zeta_1(y)) \\
&- I_{a,k}^{\alpha,s}\left[(\zeta_2(t) - f(t))(f(t) - \zeta_1(t))\right] - (\zeta_2(y) - f(y))(f(y) - \zeta_1(y)) \frac{(s)^{\frac{-\alpha}{k}}(t-a)^{s\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} \\
&= I_{a,k}^{\alpha,s} f^2(t) + f^2(y)\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{s\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} - 2f(y) I_{a,k}^{\alpha,s} f(t) + \zeta_2(y) I_{a,k}^{\alpha,s} f(t) \\
&+ \zeta_2(y) I_{a,k}^{\alpha,s}\zeta_1(t) + f(y) I_{a,k}^{\alpha,s}\zeta_2(t) + f(y) I_{a,k}^{\alpha,s}\zeta_1(t) + \zeta_1(y) I_{a,k}^{\alpha,s} f(t) - \zeta_1(y) I_{a,k}^{\alpha,s}\zeta_2(t) \\
&- I_{a,k}^{\alpha,s}(\zeta_2(t)f(t)) + I_{a,k}^{\alpha,s}(\zeta_1(t)\zeta_2(t)) - I_{a,k}^{\alpha,s}(\zeta_1(t)f(t)) - \zeta_2(y)f(y)\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)} \\
&+ \zeta_1(y)\zeta_2(y)\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)} - \zeta_1(y)f(y)\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}.
\end{aligned}$$

Multiplying both sides of the above equation by

$$\frac{s^{1-\frac{\alpha}{k}}\left((t-a)^s - (y-a)^s\right)^{\frac{\alpha}{k}-1}(y-a)^{s-1}}{k\Gamma_k(\alpha)}$$

and integrate with respect to y over (a,t), we obtain the required equality (6). □

Corollary 2.8. *For $k>0$, let $s=1$ in Theorem 2.7, we get following result for Riemann-Liouville k-fractional integral $I_{a,k}^{\alpha}$ of order α defined in [25]*

$$\begin{aligned}
&\frac{(t-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} I_{a,k}^{\alpha} f^2(t) - [I_{a,k}^{\alpha} f(t)]^2\\
&= (I_{a,k}^{\alpha}\zeta_2(t) - I_{a,k}^{\alpha} f(t))(I_{a,k}^{\alpha} f(t) - I_{a,k}^{\alpha}\zeta_1(t))\\
&- \frac{(t-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} \left(I_{a,k}^{\alpha}\zeta_2(t) - I_{a,k}^{\alpha} f(t)\right)\left(I_{a,k}^{\alpha} f(t) - I_{a,k}^{\alpha}\zeta_1(t)\right)\\
&+ \frac{(t-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} I_{a,k}^{\alpha}(\zeta_1(t) f(t)) - I_{a,k}^{\alpha}\zeta_1(t) I_{a,k}^{\alpha} f(t)\\
&+ \frac{(t-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} I_{a,k}^{\alpha}(\zeta_2(t) f(t)) - I_{a,k}^{\alpha}\zeta_2(t) I_{a,k}^{\alpha} f(t)\\
&- \frac{(t-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} I_{a,k}^{\alpha}(\zeta_1(t)\zeta_2(t)) + I_{a,k}^{\alpha}\zeta_1(t) I_{a,k}^{\alpha}\zeta_2(t).
\end{aligned}$$

Corollary 2.9. *For $k>0$, let $f \in L_{1,s}[a,b]$. Suppose that $m_1 \le f(t) \le m_2$ for all $t \in [a,b]$ and $m_1, m_2 \in \mathbb{R}$. Then for $s>0$ and $\alpha>0$, we have*

$$\begin{aligned}
&\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)} I_{a,k}^{\alpha,s} f^2(t) - \left[I_{a,k}^{\alpha,s} f(t)\right]^2\\
= -&\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)} I_{a,k}^{\alpha,s}\left[(m_2 - f(t))(f(t)-m_1)\right]\\
&+ \left[\frac{m_2 s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)} - I_{a,k}^{\alpha,s} f(t)\right]\\
&\times \left[I_{a,k}^{\alpha,s} f(t) - \frac{m_1 s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\right].
\end{aligned}$$

Proof . This equality can be proved by setting $\zeta_1(t)=m_1$ and $\zeta_2(t)=m_2$ in the proof of Theorem 2.7. □

Theorem 2.10. *For $k>0$, let f and g be two integrable functions on $[a,b]$ and let ζ_1, ζ_2, ξ_1 and ξ_2 be four integrable functions satisfying the conditions (3) and (4) on $[a,b]$. Then for all $t\in[a,b]$, $s>0$ and $\alpha>0$*

$$\begin{aligned}
&\left|\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)} I_{a,k}^{\alpha,s}\left(f(t)g(t)\right) - I_{a,k}^{\alpha,s} f(t) I_{a,k}^{\alpha,s} g(t)\right|\\
&\le \sqrt{S_k^s(f,\zeta_1,\zeta_2) S_k^s(g,\xi_1,\xi_2)}, \qquad (7)
\end{aligned}$$

where

$$\begin{aligned}
S_k^s(x,y,z) &= \left(I_{a,k}^{\alpha,s} z(t) - I_{a,k}^{\alpha,s} x(t)\right)\left(I_{a,k}^{\alpha,s} x(t) - I_{a,k}^{\alpha,s} y(t)\right) \\
&+ \frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)} I_{a,k}^{\alpha,s}\left(y(t)x(t)\right) - I_{a,k}^{\alpha,s} y(t) I_{a,k}^{\alpha,s} x(t) \\
&+ \frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)} I_{a,k}^{\alpha,s}\left(z(t)x(t)\right) - I_{a,k}^{\alpha,s} z(t) I_{a,k}^{\alpha,s} x(t) \\
&- \frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)} I_{a,k}^{\alpha,s}\left(y(t)z(t)\right) + I_{a,k}^{\alpha,s} y(t) I_{a,k}^{\alpha,s} z(t).
\end{aligned}$$

Proof . Since f and g are integrable functions on $[a,b]$ and satisfying the conditions (3) and (4) so one can have

$$\begin{aligned}
&\frac{(s)^{2-\frac{2\alpha}{k}}}{2k^2\Gamma_k^2(\alpha)} \int_a^t \int_a^t \left((t-a)^s - (x-a)^s\right)^{\frac{\alpha}{k}-1} \left((t-a)^s - (y-a)^s\right)^{\frac{\alpha}{k}-1} \\
&\times (x-a)^{s-1}(y-a)^{s-1}\left[f(x)-f(y)\right]\left[g(x)-g(y)\right] dxdy \\
&= \frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)} I_{a,k}^{\alpha,s}\left(f(t)g(t)\right) - I_{a,k}^{\alpha,s} f(t) I_{a,k}^{\alpha,s} g(t). \qquad (9)
\end{aligned}$$

Now using the Cauchy-Schwarz inequality we have

$$\begin{aligned}
&\Bigg(\frac{(s)^{2-\frac{2\alpha}{k}}}{2k^2\Gamma_k^2(\alpha)} \int_a^t \int_a^t \left((t-a)^s - (x-a)^s\right)^{\frac{\alpha}{k}-1} \left((t-a)^s - (y-a)^s\right)^{\frac{\alpha}{k}-1} \\
&\times (x-a)^{s-1}(y-a)^{s-1}\left[f(x)-f(y)\right]\left[g(x)-g(y)\right] dxdy\Bigg)^2 \\
&\le \frac{(s)^{2-\frac{2\alpha}{k}}}{2k^2\Gamma_k^2(\alpha)} \int_a^t \int_a^t \left((t-a)^s - (x-a)^s\right)^{\frac{\alpha}{k}-1} \left((t-a)^s - (y-a)^s\right)^{\frac{\alpha}{k}-1} \\
&\times (x-a)^{s-1}(y-a)^{s-1}[f(x)-f(y)]^2 dxdy \\
&\times \frac{(s)^{2-\frac{2\alpha}{k}}}{2k^2\Gamma_k^2(\alpha)} \int_a^t \int_a^t \left((t-a)^s - (x-a)^s\right)^{\frac{\alpha}{k}-1} \left((t-a)^s - (y-a)^s\right)^{\frac{\alpha}{k}-1} \\
&\times (x-a)^{s-1}(y-a)^{s-1}[g(x)-g(y)]^2 dxdy. \qquad (10)
\end{aligned}$$

Now since

$$[f(x)-f(y)]^2 = f^2(x) + f^2(y) - 2f(x)f(y),$$

one can easily prove that

$$\begin{aligned}
&\frac{(s)^{2-\frac{2\alpha}{k}}}{2k^2\Gamma_k^2(\alpha)} \int_a^t \int_a^t \left((t-a)^s - (x-a)^s\right)^{\frac{\alpha}{k}-1} \left((t-a)^s - (y-a)^s\right)^{\frac{\alpha}{k}-1} \\
&\times (x-a)^{s-1}(y-a)^{s-1}[f(x)-f(y)]^2 dxdy \\
&= \frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)} I_{a,k}^{\alpha,s} f^2(t) - \left(I_{a,k}^{\alpha,s} f(t)\right)^2. \qquad (11)
\end{aligned}$$

Similarly,

$$\frac{(s)^{2-\frac{2\alpha}{k}}}{2k^2\Gamma_k^2(\alpha)}\int_a^t\int_a^t\left((t-a)^s-(x-a)^s\right)^{\frac{\alpha}{k}-1}\left((t-a)^s-(y-a)^s\right)^{\frac{\alpha}{k}-1}$$
$$\times(x-a)^{s-1}(y-a)^{s-1}[g(x)-g(y)]^2dxdy$$
$$=\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}I_{a,k}^{\alpha,s}g^2(t)-\left(I_{a,k}^{\alpha,s}g(t)\right)^2. \tag{12}$$

Using equations (11) and (12) into (10), we get

$$\left(\frac{(s)^{2-\frac{2\alpha}{k}}}{2k^2\Gamma_k^2(\alpha)}\int_a^t\int_a^t\left((t-a)^s-(x-a)^s\right)^{\frac{\alpha}{k}-1}\left((t-a)^s-(y-a)^s\right)^{\frac{\alpha}{k}-1}\right.$$
$$\left.\times(x-a)^{s-1}(y-a)^{s-1}\left[f(x)-f(y)\right]\left[g(x)-g(y)\right]dxdy\right)^2$$
$$\leq\left[\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}I_{a,k}^{\alpha,s}f^2(t)-\left(I_{a,k}^{\alpha,s}f(t)\right)^2\right]$$
$$\times\left[\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}I_{a,k}^{\alpha,s}g^2(t)-\left(I_{a,k}^{\alpha,s}g(t)\right)^2\right]. \tag{13}$$

Thus the equation (9) together with the inequality (13) implies that

$$\left(\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}I_{a,k}^{\alpha,s}\left(f(t)g(t)\right)-I_{a,k}^{\alpha,s}f(t)I_{a,k}^{\alpha,s}g(t)\right)^2$$
$$\leq\left[\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}I_{a,k}^{\alpha,s}f^2(t)-\left(I_{a,k}^{\alpha,s}f(t)\right)^2\right]$$
$$\times\left[\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}I_{a,k}^{\alpha,s}g^2(t)-\left(I_{a,k}^{\alpha,s}g(t)\right)^2\right]. \tag{13a}$$

Now since

$$\left(\zeta_2(t)-f(t)\right)\left(f(t)-\zeta_1(t)\right)\geq 0$$

and

$$\left(\xi_2(t)-g(t)\right)\left(g(t)-\xi_1(t)\right)\geq 0,$$

therefore,

$$\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}I_{a,k}^{\alpha,s}\left(\zeta_2(t)-f(t)\right)\left(f(t)-\zeta_1(t)\right)\geq 0,\quad t\in[a,b]$$

and

$$\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}I_{a,k}^{\alpha,s}\left(\xi_2(t)-g(t)\right)\left(g(t)-\xi_1(t)\right)\geq 0,\quad t\in[a,b].$$

By Theorem 2.7, we have

$$\begin{aligned}
&\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)} I_{a,k}^{\alpha,s} f^2(t) - [I_{a,k}^{\alpha,s} f(t)]^2 \\
&\leq (I_{a,k}^{\alpha,s}\zeta_2(t) - I_{a,k}^{\alpha,s} f(t))(I_{a,k}^{\alpha,s} f(t) - I_{a,k}^{\alpha,s}\zeta_1(t)) \\
&+ \frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)} I_{a,k}^{\alpha,s}(\zeta_1(t)f(t)) - I_{a,k}^{\alpha,s}\zeta_1(t) I_{a,k}^{\alpha,s} f(t) \\
&+ \frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)} I_{a,k}^{\alpha,s}(\zeta_2(t)f(t)) - I_{a,k}^{\alpha,s}\zeta_2(t) I_{a,k}^{\alpha,s} f(t) \\
&- \frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)} I_{a,k}^{\alpha,s}(\zeta_1(t)\zeta_2(t)) + I_{a,k}^{\alpha,s}\zeta_1(t) I_{a,k}^{\alpha,s}\zeta_2(t) \\
&= S_k^s(f,\zeta_1,\zeta_2).
\end{aligned} \tag{14}$$

Similarly

$$\begin{aligned}
&\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)} I_{a,k}^{\alpha,s} g^2(t) - [I_{a,k}^{\alpha,s} g(t)]^2 \\
&\leq (I_{a,k}^{\alpha,s}\xi_2(t) - I_{a,k}^{\alpha,s} g(t))(I_{a,k}^{\alpha,s} g(t) - I_{a,k}^{\alpha,s}\xi_1(t)) \\
&+ \frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)} I_{a,k}^{\alpha,s}(\xi_1(t)g(t)) - I_{a,k}^{\alpha,s}\xi_1(t) I_{a,k}^{\alpha,s} g(t) \\
&+ \frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)} I_{a,k}^{\alpha,s}(\xi_2(t)g(t)) - I_{a,k}^{\alpha,s}\xi_2(t) I_{a,k}^{\alpha,s} g(t) \\
&- \frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)} I_{a,k}^{\alpha,s}(\xi_1(t)\xi_2(t)) + I_{a,k}^{\alpha,s}\xi_1(t) I_{a,k}^{\alpha,s}\xi_2(t) \\
&= S_k^s(g,\xi_1,\xi_2).
\end{aligned} \tag{15}$$

Equations (14) and (15) together with inequality(13a) yield the inequality (7). □

Corollary 2.11. *Put $s=1$, the inequality (7) reduces to the following result for Riemann-Liouville k-fractional integral $I_{a,k}^{\alpha}$ of order α defined in [25]*

$$\left| \frac{(t-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} I_{a,k}^{\alpha}\left(f(t)g(t)\right) - I_{a,k}^{\alpha} f(t) I_{a,k}^{\alpha} g(t) \right| \leq \sqrt{S_k(f,\zeta_1,\zeta_2) S_k(f,\xi_1,\xi_2)},$$

where

$$\begin{aligned}
S_k(x,y,z) = &\left(I_{a,k}^{\alpha} z(t) - I_{a,k}^{\alpha} x(t)\right)\left(I_{a,k}^{\alpha} x(t) - I_{a,k}^{\alpha} y(t)\right) \\
&+ \frac{(t-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} I_{a,k}^{\alpha}\left(y(t)x(t)\right) - I_{a,k}^{\alpha} y(t) I_{a,k}^{\alpha} x(t) \\
&+ \frac{(t-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} I_{a,k}^{\alpha}\left(z(t)x(t)\right) - I_{a,k}^{\alpha} z(t) I_{a,k}^{\alpha} x(t) \\
&- \frac{(t-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} I_{a,k}^{\alpha}\left(y(t)z(t)\right) + I_{a,k}^{\alpha} y(t) I_{a,k}^{\alpha} z(t).
\end{aligned}$$

Example 2.12. *For $k>0$, let f and g be two functions satisfying $(t-a)^r \leq f(t) \leq (t-a)^r+1$ and $(t-a)^r-1 \leq g(t) \leq (t-a)^r$ for $t \in [a,b]$. Then for $s>0$. $\alpha>0$, we have*

$$\begin{aligned}
&\left|\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)} I_{a,k}^{\alpha,s}\left(f(t)g(t)\right) - I_{a,k}^{\alpha,s} f(t) I_{a,k}^{\alpha,s} g(t)\right| \\
&\leq \sqrt{S_k^s(f,(t-a)^r,(t-a)^r+1) S_k^s(g,(t-a)^r-1,(t-a)^r)}.
\end{aligned}$$

Here,

$$\begin{aligned}
&S_k^s(f,(t-a)^r,(t-a)^r+1) \\
&= \left(\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)} + \frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}+s}\Gamma_k(\frac{rk}{s}+k)}{\Gamma_k(\alpha+\frac{rk}{s}+k)} - I_{a,k}^{\alpha,s} f(t)\right) \\
&\times \left(I_{a,k}^{\alpha,s} f(t) - \frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}+r}\Gamma_k(\frac{rk}{s}+k)}{\Gamma_k(\alpha+\frac{rk}{s}+k)}\right) + \frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)} I_{a,k}^{\alpha,s}(f(t)t^r) \\
&- \frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}+r}\Gamma_k(\frac{rk}{s}+k)}{\Gamma_k(\alpha+\frac{rk}{s}+k)} I_{a,k}^{\alpha,s} f(t) + \frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)} I_{a,k}^{\alpha,s}\left((t-a)^s f(t)\right) \\
&- \left(\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}+r}\Gamma_k(\frac{rk}{s}+k)}{\Gamma_k(\alpha+\frac{rk}{s}+k)} + \frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\right) I_{a,k}^{\alpha,s} f(t) \\
&+ \left(\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}+r}\Gamma_k(\frac{rk}{s}+k)}{\Gamma_k(\alpha+\frac{rk}{s}+k)}\right)\left(\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}+r}\Gamma_k(\frac{rk}{s}+k)}{\Gamma_k(\alpha+\frac{rk}{s}+k)} + \frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\right) \\
&- \frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\left(\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}+r}\Gamma_k(\frac{rk}{s}+k)}{\Gamma_k(\alpha+\frac{rk}{s}+k)} + \frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}+2r}\Gamma_k(\frac{2rk}{s}+k)}{\Gamma_k(\alpha+\frac{2rk}{s}+k)}\right),
\end{aligned}$$

and

$$\begin{aligned}
&S_k^s(f,(t-a)^r-1,(t-a)^r) \\
&= \left(\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)} - \frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}+r}\Gamma_k(\frac{rk}{s}+k)}{\Gamma_k(\alpha+\frac{rk}{s}+k)} + I_{a,k}^{\alpha,s} g(t)\right) \\
&\times \left(\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}+r}\Gamma_k(\frac{rk}{s}+k)}{\Gamma_k(\alpha+\frac{rk}{s}+k)} - I_{a,k}^{\alpha,s} g(t)\right) + \frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)} I_{a,k}^{\alpha,s}\left((t-a)^r - 1g(t)\right) \\
&- \left(\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}+r}\Gamma_k(\frac{rk}{s}+k)}{\Gamma_k(\alpha+\frac{rk}{s}+k)} - \frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\right) I_{a,k}^{\alpha,s} g(t) \\
&+ \frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)} I_{a,k}^{\alpha,s}((t-a)^r g(t)) - \frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}+r}\Gamma_k(\frac{rk}{s}+k)}{\Gamma_k(\alpha+\frac{rk}{s}+k)} I_{a,k}^{\alpha,s} g(t) \\
&+ \left(\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}+r}\Gamma_k(\frac{rk}{s}+k)}{\Gamma_k(\alpha+\frac{rk}{s}+k)} - \frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\right)\left(\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}+r}\Gamma_k(\frac{rk}{s}+k)}{\Gamma_k(\alpha+\frac{rk}{s}+k)}\right) \\
&- \frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\left(\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}+r}\Gamma_k(\frac{rk}{s}+k)}{\Gamma_k(\alpha+\frac{rk}{s}+k)} - \frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}+2r}\Gamma_k(\frac{2rk}{s}+k)}{\Gamma_k(\alpha+\frac{2rk}{s}+k)}\right).
\end{aligned}$$

Conclusions: The presented fractional integral contain the k-generalization of fractional conformable integrals. It produces some known and classical fractional integrals. We have obtained results related to Grüss type inequalities involving k-fractional conformable integrals and a Grüss inequality is presented which is a general version in k-fractional integrals. All results proved here are generalizations of results for well known fractional integrals. If one set $k = 1$ the Grüss type inequalities comprising fractional conformable integrals can be produced while setting $s = 1$ along with $k = 1$ results produced will be for classical Riemann-Liouville fractional integral operators.

Acknowledgment.
This work is supported by the Higher Education Commission, Pakistan. The research work of Ghulam Farid is supported by the Higher Education Commission of Pakistan under NRPU 2016, Project No. 5421.

References

[1] G. Abbas, K.A. Khan, G. Farid and A. Ur Rehman, *Generalizations of some fractional integral inequalities via generalized Mittag-Leffler function*, J. Inequal. Appl. 2017 (2017) 121.

[2] G. Abbas and G. Farid, *Some integral inequalities for m-convex functions via generalized fractional integral operator containing generalized Mittag-Leffler function*, Cogent Math. 3 (2016) 1269589.

[3] Abdeljawad, T., 2015. On conformable fractional calculus. J. Comput. Appl. Math. 279, 57-66.

[4] Agarwal, P., Salahshour, S., Ntouyas, S.K., Tariboon, J., 2014. Certain inequalities involving generalized Erdélyi-Kober fractional q-integral operators. Sci. World J. Article ID 174126, 11 pages.

[5] Ayub, W., Farid, G., Ur Rehman, A., 2017. Generalization of the Fejer-Hadamard type inequalities for p-convex functions via k-fractional integrals. Communication in Mathematical Modeling and Applications. 2(3), 1-15.

[6] Akin, E., Asliyüce, S., Güvenilir, A.F., Kaymakçalan, B., 2015. Discrete Grüss type inequality on fractional calculus. J. Inequal. Appl. 2015:174.

[7] Baleanu, D., Agarwal, P., 2014. Certain inequalities involving the fractional q-integral operators. Abstr. Appl. Anal. 2014, 10 pages.

[8] Choi, J., Agarwal, P., 2014. Some new Saigo type fractional integral inequalities and their q-analogues. Abstr. Appl. Anal. 2014, 11 pages.

[9] Choi, J., Ritelli, D., Agarwal, P., 2015. Some new inequalities involving generalized Erdélyi-Kober fractional q-integral operator. Appl. Math. Sci. 9, 3577-3591.

[10] Chen, F., 2014. A note on Hermite-Hadamard inequalities for products of convex functions via Riemann-Liouville fractional integrals. Ital. J. Pure Appl. Math. 2014, 299-306.

[11] Dragomir, S.S., 1999. A generalization of Grüss inequality in inner product spaces and applications. J. Math. Anal. Appl. 237, 74-82.

[12] Diaz, R., Pariguan, E., 2007. On hypergeometric functions and Pochhammer k-symbol. Divulg. Mat. 15, 179-192.

[13] Dahmani, Z., 2012. About some integral inequalities using Riemann-Liouville integrals. Gen. Math. 20, 63-69.

[14] Elezović, N., Marangunic, L.J., Pećarič, J., 2007. Some improvements of Grüss type inequality. J. Math. Ineq. 1(3), 425-436.

[15] Farid, G., Ur Rehman, A., 2016. On Chebyshev functional and Ostrowski-Grüss type inequalities for two coordinates. Int. J. Anal. App. 12(2), 180-187.

[16] Farid, G., Rafique, S., Ur Rehman, A., 2017. More on Ostrowski and Ostrowski-Gruss type inequalities. Commun. Optim. Theory. 2017, 9 pages.

[17] Grüss, G., 1935. Uber das maximum des absoluten Betrages $\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx \frac{1}{b-a}\int_a^b g(x)dx$. Math. Z. 39, 215-226.

[18] Jarad, F., Ugurlu, E., Abdeljawad, T., Baleanu, D., 2017. On a new class of fractional operators. Adv. Difference Equ. 1, 247.

[19] Kaçar, E., Yildirim, H., 2015. Grüss type integral inequalities for generalized Riemann-Liouville fractional integrals. Int. J. Pure Appl. Math. 101, 55-70.

[20] Krasniqi, V., 2010 Inequalities and monotonicity for the ration of k-gamma functions. Sci. Magna. 6, 40-45.

[21] Kokologiannaki, CG., Krasniqi, V., 2013. Some properties of the k-gamma function. Le Mathematiche. 68, 13-22.

[22] Li, X., Mohapatra, RN., Rodriguez, RS., 2002. Grüss-type inequalities. J. Math. Anal. Appl. 267, 434-443.

[23] Liao, Y., Deng, J., Wang, J., 2013. Riemann-Liouville fractional Hermite-Hadamard inequalities. Part I. for once differentiable geometric-arithmetically s-convex functions. J. Inequal. Appl. 2013:443.

[24] Liu, X., Zhang, L., Agarwal, P., Wang, G., 2016. On some new integral inequalities of Gronwall-Bellman-Bihari type with delay for discontinuous functions and their applications. Indag. Math. 27(1), 1-10.

[25] Mubeen, S., Habibullah, G.M., 2012. k-fractional integrals and application. Int. J. Contemp. Math. Sci. 7, 89-94.

[26] Park, J., 2015. Some integral inequalities for convex functions via Riemann-Liouville integrals. Appl. Math. Sci. 9, 1341-1353.

[27] Pachpatte, B.G., 2002. On multidimensional Grüss type inequalities. J. Inequal. Pure Appl. Math. 3, 1-15.

[28] Romero, L.G., Luque, L.L., Dorrego, G.A., Cerutti, RA., 2013. On the k-Riemann-Liouville Fractional Derivative. Int. J. Contemp. Math. Sciences. 8, 41-51.

[29] Sarikaya, M.Z., Filiz, H., Kiris, M.E., 2015. On some generalized integral inequalities for Riemann-Liouville fractional integrals. Filomat. 29(6), 1307-1314.

[30] Sarikaya, M.Z., Karaca, A., 2014. On the k-Riemann-Liouville fractional integral and applications. Int. J. Math. Stat. 1, 033-043.

[31] Sarikaya, M.Z., Dahmani, Z., Kiris, M.E., Ahmad, F., 2016. (k, s)-Riemann-Liouville fractional integral and applications. Hacet. J. Math. Stat. 45(1), 77–89.

[32] Tariboon, J., Ntouyas, S.K., Sudsutad, W., 2014. Some new Riemann-Liouville fractional integral inequalities. Int. J. Math. Math. Sci. 2014, 6 pages.

[33] Tunc, T., 2013. On new inequalities for h-convex functions via Riemann-Liouville fractional integration. Filomat. 27:4, 559-565.

[34] Tunc, T., Usta, F., Budak, H. and Sarikaya, M.Z., 2017. On Grüss type inequalities utilizing generalized fractional integral operators. AIP Conference Proceedings, 1833:020045.

[35] Usta, F. and Sarkaya, M.Z., 2017. Explicit bounds on certain integral inequalities via conformable fractional calculus. Cogent Mathematics, 4:1277505.

[36] Usta, F., Budak, H., and Sarkaya, M.Z., 2017. On Chebychev type inequalities for fractional integral operators, AIP Conference Proceedings, 1833: 020045.

[37] Usta, F., and Sarkaya, M.Z., 2017. Some Improvements of Conformable Fractional Integral Inequalities. International Journal of Analysis and Applications, 14: 162-166.

[38] Usta, F. and Sarkaya, M.Z., 2018. On generalization conformable fractional integral inequalities, FILOMAT, 32.

[39] G. Wang, P. Agarwal and M. Chand, *Certain Grüss type inequalities involving the generalized fractional integral operator*, J. Inequal. Appl. 2014 (2014) 147.

[40] A. Waheed, G. Farid, A. Ur Rehman and W. Ayub, *k-fractional integral inequalities for harmonically convex functions via Caputo k-fractional derivatives*, Bull. Math. Anal. Appl. 10 (2018) 55–67.