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Existence of solutions of system of functional-integral equations using measure of noncompactness

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Abstract

We propose to investigate the solutions of system of functional-integral equations in the setting of measure of noncompactness on real-valued bounded and continuous Banach space. To achieve this, we first establish some new Darbo type fixed and coupled fixed point results for μ-set (ω, ϑ)-contraction operator using arbitrary measure of noncompactness in Banach spaces. An example is given in support for the solutions of a pair of system of functional-integral equations.

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1. Introduction and preliminaries

The measure of noncompactness (MNC, in short) is coined by Kuratowski [11] and combined with some algebraic arguments are useful for studying the mathematical formulations, particularly for solving the existence of solutions of some nonlinear problems under certain conditions.
Denote by $\mathbb{R}$ the set of real numbers and put $\mathbb{R}_+ = [0, +\infty)$. Let $(E, \|\cdot\|)$ be a real Banach space with zero element 0. Let $\overline{B}(x, r)$ denote the closed ball centered at x with radius r. The symbol

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$\overline{B}_r$ stands for the ball $\overline{B}(0,r)$. For X, a nonempty subset of E, we denote by $\overline{X}$ and $ConvX$ the closure and the closed convex hull of X, respectively. Moreover, let us denote by $\mathfrak{M}_E$ the family of nonempty bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact subsets of E.

Definition 1.1. *[7] A mapping $\mu : \mathfrak{M}_E \longrightarrow \mathbb{R}_+$ is said to be a MNC in E if it satisfies the following conditions:*

(1°) The family $ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $ker\mu \subset \mathfrak{N}_E$;

(2°) $X \subset Y \Longrightarrow \mu(X) \leq \mu(Y)$;

(3°) $\mu(\overline{X}) = \mu(X)$;

(4°) $\mu(ConvX) = \mu(X)$;

(5°) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$;

(6°) If $\{X_n\}$ is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1,2,\cdots$, and if $\lim_{n\to\infty} \mu(X_n) = 0$, then the intersection set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

The subfamily $ker\mu$ defined (1°) represents the kernel of the measure μ of noncompactness and since

$$\mu(X_\infty) = \mu(\bigcap_{n=1}^{\infty} X_n) \leq \mu(X_n),$$

we see that

$$\mu(\bigcap_{n=1}^{\infty} X_n) = 0.$$

Therefore, $X_\infty \in ker\mu$.

For a bounded subset A of a metric space (X,d) the Kuratowski MNC is defined as

$$\alpha(A) = \inf\left\{\delta > 0 : A = \bigcup_{i=1}^{n} A_i,\ diam\,(A_i) \leq \delta \text{ for } 1 \leq i \leq n \leq \infty\right\},$$

where $diam\,(A_i) = \sup\{d(x,y) : x,y \in A_i\}$.
The Hausdorff MNC for a bounded set A is defined by

$$\chi(A) = \inf\{\epsilon > 0 : A \text{ has finite } \epsilon - \text{net in } X\}.$$

From now onwards unless otherwise specified, we take $\mu(\cdot)$ as an arbitrary MNC in Banach space $\mathcal{X}$.

In [8], the notion of MNC is very well utilized by Darbo to generalized Schauder's and Banach's fixed point theorems.
We denote $\Gamma = \{C : \emptyset \neq C$, closed, bounded and convex subset of a Banach space $E\}$.

Theorem 1.2. *(Schauder's fixed point theorem)[6] Let $C \in \Gamma$ without boundedness. Then every compact, continuous map $T : C \to C$ has at least one fixed point.*

Theorem 1.3. *(Darbo's fixed point theorem) [5]. Let $C \in \Gamma$ and let $T : C \longrightarrow C$ be a continuous mapping such that $\exists$ a constant $k \in [0,1)$ with the property*

$$\mu(TX) \leq k\mu(X),$$

for any $\emptyset \neq X \subset C$. Then T has a fixed point in the set C.

2. Fixed point theorems for μ-set (ω, ϑ)-contraction condition

In this section, we propose some new fixed point results for new notion of μ-set (ω, ϑ)-contraction condition in the frame of Banach space. Before introducing μ-set (ω, ϑ)-contraction, we recall following definitions:

Definition 2.1. *(Altun and Turkoglu [1]) Let $F([0,\infty))$ be class of all function $f : [0,\infty) \to [0,\infty]$ and let Θ be class of all operators*

$$\mathcal{O}(\circ;\cdot) : F([0,\infty)) \to F([0,\infty)), f \to \mathcal{O}(f;\cdot)$$

satisfying the following conditions:

(i) $\mathcal{O}(f;t) > 0$ *for* $t > 0$ *and* $\mathcal{O}(f;0) = 0$,
(ii) $\mathcal{O}(f;t) \leq \mathcal{O}(f,s)$ *for* $t \leq s$,
(iii) $\lim_{n\to\infty} \mathcal{O}(f;t_n) = \mathcal{O}(f;\lim_{n\to\infty} t_n)$,
(iv) $\mathcal{O}(f;\max\{t,s\}) = \max\{\mathcal{O}(f;t), \mathcal{O}(f;s)\}$ *for some* $f \in F([0,\infty))$.

Definition 2.2. *[12] A function $\varphi : [0,\infty) \to [0,1)$ is said to be an MT-function if*

$$\limsup_{s\to t^+} \varphi(s) < 1 \ \text{ for all } t \in [0,\infty).$$

Definition 2.3. *[10] A function $\vartheta : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is called a GMT function if the following conditions hold:*

(ϑ_1) $0 < \vartheta(t,s) < 1$ *for all* $t,s > 0$;

(ϑ_2) for any bounded sequence $\{t_n\} \subset (0,+\infty)$ and any non-increasing sequence $\{s_n\} \subset (0,+\infty)$, we have

$$\limsup_{n\to\infty} \vartheta(t_n, s_n) < 1.$$

We denote the set of all GMT functions by $\widehat{GMT(R)}$.

Definition 2.4. *Let Ω denote the set of all functions $\omega : [0;+\infty) \to [0,+\infty)$ satisfying:*

(I) ω *is non-decreasing,*
(II) $\omega(t) = 0 \Leftrightarrow t = 0$.

Now we are in position to establish generalized form of Darbo fixed point theorem.

Theorem 2.5. *Let $C \in \Gamma$ and $T : C \to C$ is continuous function and satisfying*

$$\omega(\mathcal{O}(f;\mu(TX))) \leq \vartheta(\mathcal{O}(f;\mu(TX)), \omega(\mathcal{O}(f;\mu(X))))\omega(\mathcal{O}(f;\mu(X))), \tag{2.1}$$

for any $\emptyset \neq X \subset C$, where $\mathcal{O}(\circ;\cdot) \in \Theta$, $\vartheta \in \widehat{GMT(R)}$ and $\omega \in \Omega$. Then T has at least one fixed point in C.

Proof . We start with constructing a sequence $\{C_n\}$ such that $C_0 = C$, $C_{n+1} = Conv(TC_n)$, for $n \geq 0$. If $\mu(C_N) = 0$ for some natural number $n_0 \in \mathbb{N}$, then $\mu(C_{n_0}) = 0$, then C_{n_0} is compact and since $T(C_{n_0}) \subseteq Conv(TC_{n_0}) = C_{n_0+1} \subseteq C_{n_0}$. Thus we conclude the result from Theorem 1.2, hence we assume that

$$0 < \mu(C_n), \forall\, n \geq 1.$$

From (2.1), we have

$$\begin{aligned} \omega(\mathcal{O}(f;\mu(C_{n+1}))) &= \omega(\mathcal{O}(f;\mu(Conv(TC_n)))) = \omega(\mathcal{O}(f;\mu(TC_n))) \\ &\leq \vartheta(\mathcal{O}(f;\mu(TC_n)), \omega(\mathcal{O}(f;\mu(C_n))))\omega(\mathcal{O}(f;\mu(C_n))), \end{aligned} \tag{2.2}$$

which, by the fact that $\vartheta < 1$ implies

$$\omega(\mathcal{O}(f;\mu(C_{n+1}))) \leq \omega(\mathcal{O}(f;\mu(C_n))).$$

Therefore the sequence $\{\omega(\mathcal{O}(f;\mu(C_n)))\}$ is nonincreasing and nonnegative, we suppose that

$$\delta_1 = \lim_{n\to\infty} \omega(\mathcal{O}(f;\mu(C_n))), \quad \delta_2 = \lim_{n\to\infty} \mathcal{O}(f;\mu(C_n)), \tag{2.3}$$

where $\delta_1, \delta_2 \geq 0$ are nonnegative real numbers.
We show that $\delta_1 = \delta_2 = 0$. Suppose, to the contrary, that $\delta_1, \delta_2 > 0$.
Since $\{\mathcal{O}(f;\mu(C_n))\}$ is a non-increasing sequence and $\{\mathcal{O}(f;\mu(TC_n)))\}$ is a bounded sequence. By (ϑ_2), we have

$$\limsup_{n\to\infty} \vartheta(\mathcal{O}(f;\mu(TC_n)), \omega(\mathcal{O}(f;\mu(C_n)))) < 1.$$

Passing to the limit as $n \to \infty$ in (2.2) and using (2.3) with (iii) property of $\mathcal{O}(\circ;\cdot)$, we obtain that

$$\delta_1 \leq \limsup_{n\to\infty} \vartheta(\mathcal{O}(f;\mu(TC_n)), \omega(\mathcal{O}(f;\mu(C_n))))\delta_1 < \delta_1.$$

Therefore $\delta_1 = 0$, that is,

$$\lim_{n\to\infty} \omega(\mathcal{O}(f;\mu(C_n))) = 0. \tag{2.4}$$

Since $\{\mu(C_n)\}$ is a non-increasing sequence of positive numbers. This implies that there exists $\delta \geq 0$ such that

$$\lim_{n\to\infty} \mu(C_n) = \delta. \tag{2.5}$$

Since ω is non-decreasing and by the (ii)-(iii) properties of $\mathcal{O}(\circ;\cdot)$, we have

$$\omega(\mathcal{O}(f;\mu(C_n))) \geq \omega(\mathcal{O}(f;\delta)). \tag{2.6}$$

Passing to the limit as $n \to \infty$ in (2.6) and using (2.4) with (i) and (iii) properties of $\mathcal{O}(\circ;\cdot)$, we get $0 \geq \omega(\delta)$ which, by (II) implies that $\delta = 0$. Therefore

$$\lim_{n\to\infty} \mu(C_n) = 0.$$

Since $C_n \supseteq C_{n+1}$ and $TC_n \subseteq C_n$ for all $n = 1, 2, \ldots$, then from (6^0), $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty convex closed set, invariant under T and belongs to $Ker\mu$. Therefore, by Theorem 1.2, we conclude the result. □
Following are the some special cases of Theorem 2.5.

Corollary 2.6. *Let $C \in \Gamma$ and $T : C \to C$ is continuous function and satisfying*

$$\omega(\mathcal{O}(f;\mu(TX))) \leq \lambda\omega(\mathcal{O}(f;\mu(X))),$$

for any $\emptyset \neq X \subset C$, where $0 \leq \lambda < 1$, $\mathcal{O}(\circ;\cdot) \in \Theta$ and $\omega \in \Omega$. Then T has at least one fixed point in C.

Proof . It suffices to take $\vartheta(t,s) = \lambda$ and apply Theorem 2.5. □

Corollary 2.7. *Let $C \in \Gamma$ and $T : C \to C$ is continuous function and satisfying*

$$\omega(\mathcal{O}(f;\mu(TX))) \leq \varphi(\omega(\mathcal{O}(f;\mu(X))))\omega(\mathcal{O}(f;\mu(X))),$$

for any $\emptyset \neq X \subset C$, where $\varphi : [0,\infty) \to [0,1)$ be an MT-function, $\mathcal{O}(\circ;\cdot) \in \Theta$ and $\omega \in \Omega$. Then T has at least one fixed point in C.

Proof . It suffices to take $\vartheta(t,s) = \varphi(s)$ and apply Theorem 2.5. □

Corollary 2.8. *Let $C \in \Gamma$ and $T : C \to C$ is continuous function and satisfying*

$$\omega(\mathcal{O}(f;\mu(TX))) \leq \varphi(\omega(\mathcal{O}(f;\mu(X)))),$$

for any $\emptyset \neq X \subset C$, where $\mathcal{O}(\circ;\cdot) \in \Theta$, $\omega \in \Omega$ and $\varphi : [0,\infty) \to [0,1)$ be a function such that $\varphi(s) < s$ and $\limsup_{s\to t^+} \dfrac{\varphi(s)}{s} < 1$. Then T has at least one fixed point in C.

Proof . It suffices to take $\vartheta(t,s) = \dfrac{\varphi(s)}{s}$ and apply Theorem 2.5. □

3. Darbo type coupled fixed point

Definition 3.1. *[9] An element $(u^*,v^*) \in E^2$ is called a coupled fixed point of a mapping $G : E^2 \to E$ if $G(u^*,v^*) = u^*$ and $G(v^*,u^*) = v^*$.*

Theorem 3.2. *[3] Suppose μ_i $(i = 1,2,3,\dots,n)$ are MNCs in Banach spaces E_i respectively. Moreover assume that the function $F : [0,\infty)^n \to [0,\infty)$ is convex and $F(x_1,x_2,\dots,x_n) = 0$ if and only if $x_i = 0$ $(i = 1,2,3,\dots,n)$. Then*

$$\mu(C) = F(\mu_1(C_1),\mu_2(C_2),\dots,\mu_n(C_n)),$$

defines a MNC in $\prod_{i=1}^n E_1$ where C_i denotes the natural projection of C into E_i, for $i = 1,2,3,\dots,n$.

Theorem 3.3. *Let $C \in \Gamma$ and $F : C^2 \to C$ be a continuous function such that*

$$\begin{aligned}\omega(\mathcal{O}(f;\mu(F(X_1\times X_2)))) \leq{}& \frac{1}{2}\vartheta(\mathcal{O}(f;\mu(F(X_1))\\ &+\mu(F(X_2))),\omega(\mathcal{O}(f;\mu(X_1)+\mu(X_2))))\omega(\mathcal{O}(f;\mu(X_1)+\mu(X_2))),\end{aligned}$$

for any $\emptyset \neq X_1,X_2 \subset C$, where $\vartheta \in \widehat{GMT(R)}$ and ω is sub-additive and $\omega \in \Omega$. Also $\mathcal{O}(\circ;.) \in \Theta$ and $\mathcal{O}(f;t+s) \leq \mathcal{O}(f;t) + \mathcal{O}(f;s)$ for all $t,s \geq 0$. Then F has at least a coupled fixed point.

Proof . Consider the map $\hat{F}: C^2 \to C^2$ defined by the formula

$$\hat{F}(u,v) = (F(u,v), F(v,u)).$$

Since F is continuous, $\hat{F}$ is also continuous. We define a new MNC in the space C^2 as

$$\hat{\mu}(X) = \mu(X_1) + \mu(X_2)$$

where X_i, $i = 1,2$ denote the natural projections of C. Now let $X \subset C^2$ be a nonempty subset. Hence, due to (3.1) and the condition (2^0) of Definition 1.1 we conclude that

$$\begin{aligned}
&\omega(\mathcal{O}(f;\hat{\mu}(\hat{F}(X))))\\
&\le \omega(\mathcal{O}(f;\hat{\mu}(F(X_1\times X_2)\times F(X_2\times X_1))))\\
&\le \omega(\mathcal{O}(f;\mu(F(X_1\times X_2)))) + \omega(\mathcal{O}(f;\mu(F(X_2\times X_1))))\\
&\le \frac{1}{2}\vartheta(\mathcal{O}(f;\mu(F(X_1))+\mu(F(X_2))),\omega(\mathcal{O}(f;\mu(X_1)+\mu(X_2))))\omega(\mathcal{O}(f;\mu(X_1)+\mu(X_2)))\\
&+\frac{1}{2}\vartheta(\mathcal{O}(f;\mu(F(X_2))+\mu(F(X_1))),\omega(\mathcal{O}(f;\mu(X_2)+\mu(X_1))))\omega(\mathcal{O}(f;\mu(X_2)+\mu(X_1)))\\
&= \vartheta(\mathcal{O}(f;\mu(F(X_1))+\mu(F(X_2))),\omega(\mathcal{O}(f;\mu(X_1)+\mu(X_2))))\omega(\mathcal{O}(f;\mu(X_1)+\mu(X_2)))\\
&= \vartheta(\mathcal{O}(f;\hat{\mu}(\hat{F}(X))),\omega(\mathcal{O}(f;\hat{\mu}(X)))\omega(\mathcal{O}(f;\hat{\mu}(X))),
\end{aligned}$$

that is,

$$\omega(\mathcal{O}(f;\hat{\mu}(\hat{F}(X)))) \le \vartheta(\mathcal{O}(f;\hat{\mu}(\hat{F}(X)))),\omega(\mathcal{O}(f;\hat{\mu}(X)))\omega(\mathcal{O}(f;\hat{\mu}(X))).$$

Theorem 2.5 suggest that $\hat{F}$ has a fixed point, and hence F has a coupled fixed point. □

Corollary 3.4. *Let $C \in \Gamma$ and $F: C^2 \to C$ be a continuous function such that*

$$\omega(\mathcal{O}(f;\mu(F(X_1\times X_2))))) \le \frac{\lambda}{2}\omega(\mathcal{O}(f;\mu(X_1)+\mu(X_2))), \quad 0\le\lambda<1$$

for any $\emptyset \ne X_1, X_2 \subset C$, where ω is sub-additive and $\omega \in \Omega$. Also $\mathcal{O}(\circ;.) \in \Theta$ and $\mathcal{O}(f;t+s) \le \mathcal{O}(f;t)+\mathcal{O}(f;s)$ for all $t,s\ge 0$. Then F has at least a coupled fixed point.

Corollary 3.5. *Let $C \in \Gamma$ and $F: C^2 \to C$ be a continuous function such that*

$$\begin{aligned}
&\mathcal{O}(f;\mu(F(X_1\times X_2)))\\
&\le \frac{1}{2}\vartheta(\mathcal{O}(f;\mu(F(X_1))+\mu(F(X_2))),\omega(\mathcal{O}(f;\mu(X_1)+\mu(X_2))))\mathcal{O}(f;\mu(X_1)+\mu(X_2)),
\end{aligned}$$

for any $\emptyset \ne X_1, X_2 \subset C$, where $\vartheta \in \widehat{GMT(R)}$, $\mathcal{O}(\circ;.) \in \Theta$ and $\mathcal{O}(f;t+s) \le \mathcal{O}(f;t)+\mathcal{O}(f;s)$ for all $t,s\ge 0$. Then F has at least a coupled fixed point.

Corollary 3.6. *Let $C \in \Gamma$ and $F: C^2 \to C$ be a continuous function such that*

$$\omega(\mathcal{O}(f;\mu(F(X_1\times X_2)))) \le \frac{1}{2}\varphi(\omega(\mathcal{O}(f;\mu(X_1)+\mu(X_2))),$$

for any $\emptyset \ne X_1, X_2 \subset C$, where $\mathcal{O}(\circ;\cdot) \in \Theta$, $\omega \in \Omega$ and $\varphi: [0,\infty) \to [0,1)$ be a function such that $\varphi(s) < s$ and $\limsup_{s\to t^+} \dfrac{\varphi(s)}{s} < 1$. Then T has at least one fixed point in C.

Proof . It suffices to take $\vartheta(t,s) = \dfrac{\varphi(s)}{s}$ and apply Theorem 3.3. □

4. Applications

Writing classical Banach space $E = BC(\mathbb{R}^+)$ consisting of all real functions defined, bounded and continuous on $\mathbb{R}_+$ equipped with the standard norm

$$||x|| = sup\{|x(t)| : t \geq 0\}.$$

Following [5], the MNC in $BC(\mathbb{R}^+)$ is defined in the below.
Let us fix X as a nonempty and bounded subset of $BC(\mathbb{R}^+)$ and T as a positive number. For $x \in X$ and $\epsilon > 0$, denote by $\omega^T(x, \epsilon)$ the modulus of the continuity of function x on the interval $[0, T]$, i.e.,

$$\omega^T(x, \epsilon) = sup\{|x(t) - x(u)| : t, u \in [0, T], |t - u| \leq \epsilon\}.$$

Further, let us put

$$\omega^T(X, \epsilon) = sup\{\omega^T(x, \epsilon) : x \in X\},$$
$$\omega_0^T(X) = \lim_{\epsilon \to 0} \omega^T(X, \epsilon)$$

and

$$\omega_0(X) = \lim_{T \to \infty} \omega_0^T(X).$$

Moreover, for two fixed numbers $t \in \mathbb{R}^+$ let us the define the function μ on the family $\mathfrak{M}_{BC(\mathbb{R}^+)}$ by the following formula

$$\mu(X) = \omega_0(X) + \alpha(X),$$

where $\alpha(X) = \limsup_{t \to \infty} diamX(t)$, $X(t) = \{x(t) : x \in X\}$ and $diamX(t) = sup\{|x(t) - y(t)| : x, y \in X\}$. Following [5] (cf. also [4]), it easy to see the function μ is the MNC in the space $E = BC(\mathbb{R}^+)$.
Now we consider the system of integral equations

$$\begin{cases} x(t) = h\left(t, x(t), y(t), \int\limits_0^t g\left(t, s, x(s), y(s)\right) ds\right) \\ y(t) = h\left(t, y(t), x(t), \int\limits_0^t g\left(t, s, y(s), x(s)\right) ds\right). \end{cases} \tag{4.1}$$

Our aim here is to find the existence of solutions of (4.1).

Consider the following assumptions:

(a_1) The function $h : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists an upper semicontinuous, nondecreasing and concave function $\varphi : \mathbb{R}^+ \longrightarrow [0, 1)$ such that $\varphi(t) < t$ for all $t > 0$, $\limsup_{s \to t^+} \frac{\varphi(s)}{s} < 1$ and a non-decreasing continuous function $\psi : \mathbb{R}^+ \to \mathbb{R}$ with $\psi(0) = 0$, for any $t \geq 0$ and for all $x, y, u, v \in \mathbb{R}$

$$\mathcal{O}(f; |h(t, x, y, z) - h(t, u, v, w)|) \leq \frac{1}{4}\varphi(\mathcal{O}(f; |x - u| + |y - v|) + \psi(|z - w|),$$

where $\mathcal{O}(\circ; .) \in \Theta$ and $\mathcal{O}(f; t + s) \leq \mathcal{O}(f; t) + \mathcal{O}(f; s)$ for all $t, s \geq 0$.

(a_2) The function defined by $t \to |h(t, 0, 0, 0)|$ is bounded on $\mathbb{R}^+$, i.e.

$$M_1 = \sup\{\mathcal{O}(f; |h(t, 0, 0, 0)|) : t \in \mathbb{R}^+\} < \infty,$$

and $\mathcal{O}(f; \epsilon) < \epsilon$.

(a_3) $g : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a positive constant M_2 such that

$$M_2 = sup\{\mathcal{O}(f; \int_0^t |g(t,s,x(s),y(s))|ds) : t \in \mathbb{R}^+, x, y \in E)\}.$$

Morever,

$$\lim_{t\to\infty} | \int_0^t [g(t,s,x(s),y(s)) - g(t,s,u(s),v(s))]ds| = 0$$

uniformly respect to $x, y \in E$.

(a_4) There exists a positive solution r_0 of the inequality $\frac{1}{4}\varphi(\mathcal{O}(f; 2r_0)) + M_1 + \psi(M_2) \leq r_0$.

Theorem 4.1. *If the assumptions $(a_1) - (a_4)$ are satisfied, then the equation (4.1) has at least one solution $x \in E$.*

Proof . Let $F : E \times E \to E$ be defined by,

$$F(x,y)(t) = h\left(t, x(t), y(t), \int_0^t g\left(t,s,x(s),y(s)\right) ds\right).$$

We know that $E \times E$ is a Banach space equipped with the norm,

$$\| (x,y) \| = \| x \|_{BC(\mathbb{R}_+)} + \| y \|_{BC(\mathbb{R}_+)}$$

where $\| u \|_{BC(\mathbb{R}_+)} = \sup\{|u(t)| : t \geq 0\}$ and $u \in BC(\mathbb{R}_+)$. It is obvious that $F(x,y)(t)$ is continuous for any $x, y \in BC(\mathbb{R}_+)$.

Let $\bar{B}_r = \left\{x \in BC(\mathbb{R}_+) : \| x \|_{BC(\mathbb{R}_+)} \leq r\right\}$. By considering conditions of theorem we infer that $F(x,y)$ is continuous on $\mathbb{R}^+$. Now we prove that $F(x,y) \in E$ for any $x, y \in E$. For arbitrarily fixed $t \in \mathbb{R}^+$ and $f \in F([0,\infty))$ we have

$$\begin{aligned}
&\mathcal{O}(f; |F(x,y)(t)|) \\
&\leq \mathcal{O}\left(f; \left|h(t, x(t), y(t), \int_0^{\alpha(t)} g\left(t,s,x(s),y(s)\right) ds) - h(t,0,0,0)\right|\right) \\
&\quad + \mathcal{O}(f; |h(t,0,0,0)|) \\
&\leq \frac{1}{4}\varphi(\mathcal{O}(f; |x(t)| + |y(t)|)) + \psi\left(\left|\int_0^t g\left(t,s,x(s),y(s)\right) ds\right|\right) + M_1 \\
&\leq \frac{1}{4}\varphi(\mathcal{O}(f; ||x|| + ||y||)) + \psi(M_2) + M_1.
\end{aligned}$$

Thus F is well defined and condition (a_4) implies that $F\left(\bar{B}_r \times \bar{B}_r\right) \subseteq \bar{B}_r$.
Now, we have to prove that F is continuous on $\bar{B}_r \times \bar{B}_r$. For this, take $(x,y) \in \bar{B}_{r_0} \times \bar{B}_{r_0}$ and $\varepsilon > 0$

arbitrarily. Moreover consider $(p,q) \in \bar{B}_{r_0} \times \bar{B}_{r_0}$ with $\|(x,y)-(p,q)\|_{BC(\mathbb{R}_+)\times BC(\mathbb{R}_+)} < \frac{\epsilon}{2}$. Then we have

$$\begin{aligned}
&\mathcal{O}(f;|F(x,y)(t)-F(p,q)(t)|)\\
&= \mathcal{O}\left(f; \left| \begin{array}{c} h\left(t, x(t), y(t), \int_0^t g\left(t,s,x(s),y(s)\right) ds\right) \\ -h\left(t, p(t), q(t), \int_0^t g\left(t,s,p(s),q(s)\right) ds\right) \end{array} \right| \right)\\
&\le \frac{1}{4}\varphi\left(\mathcal{O}(f;|x-p|+|y-q|)\right)\\
&\quad + \psi\left(\left|\int_0^t \left\{g\left(t,s,x(s),y(s)\right)-g\left(t,s,p(s),q(s)\right)\right\} ds\right|\right)\\
&\le \frac{1}{4}\varphi\left(\mathcal{O}(f;\| x-p \| + \| y-q \|)\right)\\
&\quad + \psi\left(\int_0^t \left|g\left(t,s,x(s),y(s)\right)-g\left(t,s,p(s),q(s)\right)\right| ds\right).
\end{aligned}$$

By applying assumption (a_1) and (a_3) we get for $\epsilon > 0$ there exists $T > 0$ such that if $t > T$ then $\psi\left(\int_0^t |g(t,s,x(s),y(s)) - g(t,s,p(s),q(s))| ds\right) \le \frac{\epsilon}{2}$, for any $x,y,p,q \in BC(\mathbb{R}_+)$.

Case 1:

If $t > T$, then we get

$$\mathcal{O}(f;|F(x,y)(t)-F(p,q)(t)|) \le \frac{1}{4}\varphi(\mathcal{O}(f;\frac{\epsilon}{2}+\frac{\epsilon}{2})) + \frac{\epsilon}{2} < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon.$$

Case 2:

If $t \in [0,T]$ then

$$\mathcal{O}(f;|F(x,y)(t)-F(p,q)(t)|) \le \frac{1}{4}\varphi(\mathcal{O}(f;\frac{\epsilon}{2}+\frac{\epsilon}{2})) + \psi(T\hat{\omega}) < \frac{\epsilon}{4} + \psi(T\hat{\omega})$$

where

$$\hat{\omega} = \sup\left\{ \begin{array}{c} |g(t,s,x,y) - g(t,s,p,q)| : t,s \in [0,T], x,y,p,q \in [-r,r], \\ \| (x,y)-(p,q) \| < \frac{\epsilon}{2} \end{array} \right\}.$$

Since g is continuous on $[0,T]\times[0,T]\times[-r,r]\times[-r,r]$ therefore $\hat{\omega} \to 0$ as $\epsilon \to 0$ i.e. since $\epsilon \to 0$ gives $T\hat{\omega} \to 0$ therefore $\psi_2(T\hat{\omega}) \to 0$.

Thus F is a continuous function from $\bar{B}_r \times \bar{B}_r$ into $\bar{B}_r$.

We have, $T, \epsilon \in \mathbb{R}_+$ and X_1, X_2 are arbitrary non-empty subset of $\bar{B}_r$ and let $t, s \in [0,T]$ such that $|t-s| \le \epsilon$.

Without loss of generality we can assume that $t < s$. Also let $x \in X_1, y \in X_2$ and

$$
\begin{aligned}
\hat{K} &= T\sup\left\{|g(t,s,x(s),y(s)|:t,s\in[0,T],x,y\in[-r,r]\right\},\\
\omega^T(x,\epsilon) &= \sup\left\{|x(t)-x(s)|:t,s\in[0,T],|t-s|\le\epsilon,\right\},\\
\omega^T(y,\epsilon) &= \sup\left\{|y(t)-y(s)|:t,s\in[0,T],|t-s|\le\epsilon,\right\},\\
\omega_r^T(h,\epsilon) &= \sup\left\{\begin{array}{c}|h\left(t,x,y,z\right)-h\left(s,x,y,z\right)|:t,s\in[0,T],\\ |t-s|\le\epsilon,x,y\in[-r,r],z\in[-\hat{K},\hat{K}]\end{array}\right\},\\
\omega_r^T(g,\epsilon) &= \sup\left\{\begin{array}{c}|g\left(t,u,x(u),y(u)\right)-g\left(s,u,x(u),y(u)\right)|:t,s,u\in[0,T],\\ |t_2-t_1|\le\epsilon,x,y\in[-r,r],z\in[-\hat{K},\hat{K}],\end{array}\right\}.
\end{aligned}
$$

Then we get

$$
\begin{aligned}
&\mathcal{O}(f;|F(x,y)(t)-F(x,y)(s)|)\\
&=\mathcal{O}\left(f;\left|\begin{array}{c}h\left(t,x(t),y(t),\int_0^t g\left(t,u,x(u),y(u)\right)du\right)\\ -h\left(s,x(s),y(s),\int_0^s g\left(s,u,x(u),y(u)\right)du\right)\end{array}\right|\right)\\
&\le\mathcal{O}\left(f;\left|\begin{array}{c}h\left(t,x(t),y(t),\int_0^t g\left(t,u,x(u),y(u)\right)du\right)\\ -h\left(t,x(s),y(s),\int_0^t g\left(t,u,x(u),y(u)\right)du\right)\end{array}\right|\right)\\
&+\mathcal{O}\left(f;\left|\begin{array}{c}h\left(t,x(s),y(s),\int_0^t g\left(t,u,x(u),y(u)\right)du\right)\\ -h\left(s,x(s),y(s),\int_0^t g\left(t,u,x(u),y(u)\right)du\right)\end{array}\right|\right)\\
&+\mathcal{O}\left(f;\left|\begin{array}{c}h\left(s,x(s),y(s),\int_0^t g\left(t,u,x(u),y(u)\right)du\right)\\ -h\left(s,x(s),y(s),\int_0^t g\left(s,u,x(u),y(u)\right)du\right)\end{array}\right|\right)\\
&+\mathcal{O}\left(f;\left|\begin{array}{c}h\left(s,x(s),y(s),\int_0^t g\left(s,u,x(u),y(u)\right)du\right)\\ -h\left(s,x(s),y(s),\int_0^s g\left(s,u,x(u),y(u)\right)du\right)\end{array}\right|\right)\\
&\le\frac{1}{4}\varphi(\mathcal{O}(f;|x(t)-x(s)|+|y(t)-y(s)|))+\mathcal{O}(f;\omega_r^T(h,\epsilon))\\
&+\psi\left(\left|\int_0^t\left(g\left(t,u,x(u),y(u)\right)-g\left(s,u,x(u),y(u)\right)\right)du\right|\right)\\
&+\psi\left(\left|\int_0^t g\left(s,u,x(u),y(u)\right)du-\int_0^s g\left(s,u,x(u),y(u)\right)du\right|\right)
\end{aligned}
$$

$$\leq \frac{1}{4}\varphi\left(\mathcal{O}(f;\omega^T(x,\epsilon)+\omega^T(y,\epsilon))\right)+\mathcal{O}(f;\omega_r^T(h,\epsilon))+\psi\left(\hat{\alpha}\omega_r^T(g,\epsilon)\right)$$
$$+\psi\left(\left|\int_0^t g\left(s,u,x(u),y(u)\right)du-\int_0^s g\left(s,u,x(u),y(u)\right)du\right|\right).$$

By the uniform continuity of h on $[0,T]\times[-r,r]\times[-r,r]\times[-\hat{K},\hat{K}]$ and g on $[0,T]\times[0,T]\times[-r,r]\times[-r,r]$ we have as $\epsilon\to 0$, gives $\omega_r^T(g,\epsilon)\to 0, \omega_r^T(h,\epsilon)\to 0$. Thus as $\epsilon\to 0$ we have

$$\left|\int_0^t g\left(s,u,x(u),y(u)\right)du-\int_0^s g\left(s,u,x(u),y(u)\right)du\right|\to 0$$

which gives

$$\psi\left(\left|\int_0^t g\left(s,u,x(u),y(u)\right)du-\int_0^s g\left(s,u,x(u),y(u)\right)du\right|\right)\to 0.$$

Now taking the limit as $\epsilon\to 0$ we have

$$\mathcal{O}(f;\omega_0^T(F(X_1\times X_2)))\leq\frac{1}{4}\varphi(\mathcal{O}(f;\omega_0^T(X_1)+\omega_0^T(X_2))).$$

As $T\to\infty$ we get

$$\mathcal{O}(f;\omega_0(F(X_1\times X_2)))\leq\frac{1}{4}\varphi(\mathcal{O}(f;\omega_0(X_1)+\omega_0(X_2))). \tag{4.2}$$

For arbitrary $(x,y),(p,q)\in X_1\times X_2$ and $t\in\mathbb{R}_+$ we have,

$$\begin{aligned}&\mathcal{O}(f;|F(x,y)(t)-F(p,q)(t)|)\\&\leq\frac{1}{4}\varphi(\mathcal{O}(f;|x(t)-p(t)|+|y(t)-q(t)|)\\&\quad+\psi\left(\left|\int_0^t\{g(t,u,x,y)-g(t,u,p.q)\}\,du\right|\right)\\&\leq\frac{1}{4}\varphi(\mathcal{O}(f;diam\,(X_1(t))+diam\,(X_2(t))))\\&\quad+\psi\left(\left|\int_0^t\{g(t,u,x,y)-g(t,u,p.q)\}\,du\right|\right).\end{aligned}$$

Since $(x,y),(p,q)$ and t are arbitrary, therefore we have,

$$\begin{aligned}\mathcal{O}(f;diamF(X_1\times X_2)(t))\leq&\frac{1}{4}\varphi(\mathcal{O}(f;diam(X_1(t))+diam(X_2(t))))\\&+\psi\left(\left|\int_0^t\{g(t,u,x,y)-g(t,u,p.q)\}\,du\right|\right).\end{aligned}$$

As $t \to \infty$, by applying (a_3) we get

$$\begin{aligned} &\mathcal{O}(f; \limsup_{t\to\infty} diam F(X_1 \times X_2)(t)) \\ &\leq \frac{1}{4}\varphi(\mathcal{O}(f; \limsup_{t\to\infty} diam\,(X_1(t)) + \limsup_{t\to\infty} diam(X_2(t)))). \end{aligned} \tag{4.3}$$

From (4.2)and (4.3) we have

$$\begin{aligned} &\mathcal{O}(f; \mu(F(X_1 \times X_2) \\ &= \mathcal{O}(f; \omega_0\,(F(X_1 \times X_2)) + \limsup_{t\to\infty} diam F(X_1 \times X_2)(t)) \\ &\leq \mathcal{O}(f; \omega_0\,(F(X_1 \times X_2)) + \mathcal{O}(f; \limsup_{t\to\infty} diam F(X_1 \times X_2)(t))) \\ &\leq \frac{1}{4}\varphi(\mathcal{O}(f; \omega_0(X_1) + \omega_0(X_2))) \\ &\quad + \frac{1}{2}\phi(\mathcal{O}(f; \limsup_{t\to\infty} diam\,(X_1(t)) + \limsup_{t\to\infty} diam(X_2(t))) \\ &\leq \frac{1}{4}\varphi(\mathcal{O}(f; \omega_0(X_1) + \omega_0(X_2) + \limsup_{t\to\infty} diam\,(X_1(t)) + \limsup_{t\to\infty} diam(X_2(t)))) \\ &+ \frac{1}{4}\varphi(\mathcal{O}(f; \limsup_{t\to\infty} diam\,(X_1(t)) + \limsup_{t\to\infty} diam(X_2(t))) + \omega_0(X_1) + \omega_0(X_2))) \\ &= \frac{1}{2}\varphi(\mathcal{O}(f; \mu(X_1) + \mu(X_2)). \end{aligned}$$

Therefore

$$\mathcal{O}(f; \mu(F(X_1 \times X_2) \leq \frac{1}{2}\varphi(\mathcal{O}(f; \mu(X_1) + \mu(X_2))$$

Therefore by Corollary 3.6, F has at least a coupled fixed point in the space $E \times E$. Thus, the system of equation (4.1) has at least a solution in $E \times E$. □

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