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On sum of range sets of sum of two maximal monotone operators

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Abstract

In the setting of non-reflexive spaces (Grothendieck Banach spaces), we establish

1. $\overline{\operatorname{ran}(A+B)} = \overline{\operatorname{ran} A + \operatorname{ran} B}$,
2. $\operatorname{int}(\operatorname{ran}(A+B)) = \operatorname{int}(\operatorname{ran} A + \operatorname{ran} B)$

with the assumption that A is a maximal monotone operator and B is a single-valued maximal monotone operator such that $A+B$ is ultramaximally monotone. Conditions (1) and (2) are known as Brézis-Haraux conditions.

Keywords: Monotone Operator, Maximal Monotone Operator, Ultramaximal Monotone Operator, Brézis-Haraux conditions
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1. Introduction

Monotone operators are important class of operators used in study of modern nonlinear analysis and various classes of optimization problems. The theory of monotone operators (multifunctions) were first introduced by George Minty [18] and later it was used substantially in proving existence results in partial differential equations by Felix Browder and his school [1, 2, 9, 11, 12, 14, 16, 17, 29]. In particular, maximal monotone operators have found their plethora of applications in partial differential equations, optimization problems, variational inequalities and mathematical economics.

Among all the problems related to the monotone operators and maximal monotone operators, most studied and celebrated problem concerns the maximality of sum of two maximal monotone

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operators (sum problem) [3, 4, 6, 7, 8, 12, 21, 23, 24, 25, 27, 28]. It is observed that certain qualification constraints are required to prove the maximality of sum of two maximal monotone operators.

In 1970, Rockafellar [21] established the maximal monotonicity of sum of two maximal monotone operators in reflexive spaces with the constraint that one domain must intersect the interior of the other domain (Rockafellar's constraint qualification). If the maximal monotone operators have domain with empty interior, then the preceding results cannot be applied. One may note that there are many maximal monotone operators those having domain with empty interior [12].

Apart from sum problem the other problem that draws attention in monotone operator theory is the algebraic sum of range sets of two maximal monotone operators [11, 16, 20]. Sum of the range sets of two maximal monotone operators is applied in solving equations of Hammerstein type [10]. In [9], Brézis established a relation between range of sum of two maximal monotone operators. Namely, If A is a maximal monotone operator and B is the subdifferential of a lower semi-continuous convex function, then

1. $\overline{\operatorname{ran}(A+B)} = \overline{\operatorname{ran} A + \operatorname{ran} B}$,
2. $\operatorname{int}(\operatorname{ran}(A+B)) = \operatorname{int}(\operatorname{ran} A + \operatorname{ran} B)$.

Later, Brézis and Haraux [11] extended the above results to any monotone operators in Hilbert spaces. Conditions (1) and (2) are called as Brézis-Haraux conditions. Further, Simeon Reich generalized the Brézis-Haraux conditions to reflexive Banach spaces [20]. We remind that condition (2) is quite frequently applied to the theory of partial differential equations [17]. Also, F. E. Browder generalized (2), by giving various conditions on A and B in reflexive spaces as well as in Banach spaces [13]. For more results and extension related to condition (2) one may refer [14, 17].

Here, we prove the Brézis-Haraux conditions for certain classes of monotone operators in Grothendieck Banach spaces. The remainder of this paper is organized as follows. In Section 2, we provide some basic notions and auxiliary results which are frequently used in our main results. Section 3 contains our main result followed by one proposition. Finally, the conclusion is presented in Section 4.

2. Basic Notations and Auxiliary Results

In this note, we assume that X is a Grothendieck real Banach space. A real Banach space X is said to be Grothendieck [19] if every weak star convergence sequence is weakly convergent in X^*. The weak and weak star convergence are denoted by the notation $\xrightarrow{w}$ and $\xrightarrow{w^*}$ respectively. Every reflexive Banach space is a Grothendieck space. But the converse is not true, e.g., the space of bounded nets on some directed set Γ, $l_\infty(\Gamma)$ is a Grothendieck Banach space but not reflexive [19]. The dual of X is denoted as X^*; X and X^* are paired by $\langle x, x^*\rangle = x^*(x)$ for $x \in X$ and $x^* \in X^*$. If necessary, we identify $X \subset X^{**}$ with its image under the canonical embedding of X into X^{**}. For a given subset C of X, we denote interior of C as int$C$, closure of C as $\overline{C}$, boundary of C as bdry C and $|C| = \inf_{c\in C} \|c\|$. For any $C, D \subseteq X$, $C + D = \{x + y : \ x \in C, \ y \in D\}$. For any $\alpha > 0$, $\alpha C := \{\alpha x | \ \ x \in C\}$. Let $A : X \rightrightarrows X^*$ be a set-valued operator (also known as multifunction or point-to-set mapping) from X to X^*, i.e., for every $x \in X$, $Ax \subseteq X^*$. Domain of A is denoted as $\operatorname{dom}A := \{x \in X | \ Ax \neq \phi\}$ and range of A is $\operatorname{ran}A = \{x^* \in Ax | \ x \in \operatorname{dom}A\}$. Graph of A is denoted as $\operatorname{gra}A = \{(x, x^*) \in X \times X^* | \ x^* \in Ax\}$. The set-valued mapping $A : X \rightrightarrows X^*$ is said to be monotone if

$$\langle x - y, x^* - y^*\rangle \geq 0, \quad \forall (x, x^*), (y, y^*) \in \operatorname{gra}A.$$

Let $A : X \rightrightarrows X^*$ be monotone and $(x, x^*) \in X \times X^*$. We say that (x, x^*) is monotonically related to gra$A$ if

$$\langle x - y, x^* - y^* \rangle \geq 0, \quad \forall (y, y^*) \in \mathrm{gra}A.$$

A set valued mapping A is said to be maximal monotone if A is monotone and A has no proper monotone extension (in the sense of graph inclusion). In other words, A is maximal monotone if for any $(x, x^*) \in X \times X^*$ is monotonically related to gra$A$ then $(x, x^*) \in$ gra$A$. A monotone operator $A : X \rightrightarrows X^*$ is said to be ultramaximal monotone [5, 26] if A is maximally monotone with respect to $X^{**} \times X^*$.

Let $f : X \rightarrow]-\infty, +\infty]$ be a function and its domain is defined as dom$f := f^{-1}(\mathbb{R})$. f is said to be proper if dom$f \neq \phi$. Let f be any proper convex function then the subdifferential operator of f is defined as $\partial f : X \rightrightarrows X^* : x \mapsto \{x^* \in X^* | \ \langle y - x, x^* \rangle + f(x) \leq f(y), \forall y \in X\}$. The duality map $J : X \rightarrow X^*$ is defined as $J = \partial(\frac{1}{2}\|.\|^2)$. Using $f(x) = \frac{1}{2}\|x\|^2$ in the above definitions, we get

$$x^* \in J(x) \Leftrightarrow \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 = \langle x, x^* \rangle$$

or equivalently,

$$J(x) = \{x^* \in X^* | \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}.$$

For any two monotone operators A and B, the sum operator is defined as $A + B : X \rightrightarrows X^* : x \mapsto Ax + Bx = \{a^* + b^* | \ a^* \in Ax \ \text{ and } \ b^* \in Bx\}$.

For our convenience, we recall the following fundamental fact for our main result. The fact establishes the surjectivity properties of maximal monotone operators in general Banach spaces.

Fact 2.1. *[26, Corollary 3.6] Let $A : X \rightrightarrows X^*$ be ultramaximally monotone. Then $A + J$ is ultramaximally monotone and* $\mathrm{ran}(A + J) = X^*$.

3. Sum of range sets of monotone operators

We remind that we assume X as a real Grothendieck space. Also we assume it satisfies weakly compactly generated property. A Banach space X is called as weakly compactly generated if there is a weakly compact set K in X such that $X = \overline{\mathrm{span}}(K)$. Here, we prove the Brézis-Haraux conditions in the setting of non-reflexive Banach spaces for certain classes of maximal monotone operators. The following technical results are important for our next main result.

The proof of the following lemma closely follows the lines of the proof of [12, Lemma 1.2].

Lemma 3.1. *Let $A : X \rightrightarrows X^*$ be a monotone operator. If $(x_n, x_n^*) \in \mathrm{gra}\, A$, $x_n \xrightarrow{w^*} x^{**}$ in X^{**}, $x_n^* \xrightarrow{w^*} x^*$ in X^* and*

$$\limsup_{m,n \to \infty} \langle x_n - x_m, x_n^* - x_m^* \rangle \leq 0. \tag{3.1}$$

Then $\langle x_n, x_n^* \rangle \longrightarrow \langle x^{**}, x^* \rangle$.

Proof . Since $(x_n, x_n^*), (x_m, x_m^*) \in \mathrm{gra}\, A$. By monotonicity of A and (3.1),

$$\lim_{m,n \to \infty} \langle x_n - x_m, x_n^* - x_m^* \rangle = 0. \tag{3.2}$$

Let $\{n_i\}$ be a subsequence of $\{n\}$ such that $\langle x_{n_i}, {x_{n_i}}^* \rangle \to L$ (say). Thus, from (3.2), we have

$$\begin{aligned} 0 &= \lim_{n_i \to \infty} \left[\lim_{n_k \to \infty} \langle x_{n_i} - x_{n_k}, x_{n_i}^* - x_{n_k}^* \rangle \right] \\ &= \lim_{n_i \to \infty} \left[\langle x_{n_i}, x_{n_i}^* \rangle - \langle x_{n_i}, x^* \rangle - \langle x^{**}, x_{n_i}^* \rangle + L \right]. \end{aligned}$$

Since X is a Gronthendieck space, now we treat $x^*_{n_i} \overset{w^*}{\to} x^*$ as $x^*_{n_i} \overset{w}{\to} x^*$ in X^*. Thus,

$$\begin{aligned} 0 &= L - \langle x^{**}, x^* \rangle - \langle x^{**}, x^* \rangle + L \\ &= 2L - 2\langle x^{**}, x^* \rangle. \end{aligned}$$

Hence, $L = \langle x^{**}, x^* \rangle$. Therefore, $\langle x_n, x^*_n \rangle \longrightarrow \langle x^{**}, x^* \rangle$. □

There is one more block to fit in our main result.

Proposition 3.2. *Let $A : X \rightrightarrows X^*$ be a maximal monotone operator and $B : X \to X^*$ be a single-valued maximal monotone operator. Assume that $A + B$ is an ultramaximal monotone operator. For any positive r, let x_r satisfy $x^* \in x^*_r + B(x_r) + rJ(x_r)$ where $x^* \in X^*$ and $x^*_r \in A(x_r)$. If for each $x^* \in \operatorname{ran} A + \operatorname{ran} B$, $\{x^*_r\}$ is bounded as $r \to 0$. Then*

$$\overline{\operatorname{ran}(A + B)} = \overline{\operatorname{ran} A + \operatorname{ran} B}$$

and

$$\operatorname{int} \operatorname{ran}(A + B) = \operatorname{int}(\operatorname{ran} A + \operatorname{ran} B).$$

Proof . Since $\operatorname{ran}(A + B) \subseteq \operatorname{ran} A + \operatorname{ran} B$. Then it is enough to show that

$$\operatorname{ran} A + \operatorname{ran} B \subseteq \overline{\operatorname{ran}(A + B)} \tag{3.3}$$

and

$$\operatorname{int}(\operatorname{ran} A + \operatorname{ran} B) \subseteq \operatorname{ran}(A + B). \tag{3.4}$$

Let $x^* = a^* + b^*$, $a^* \in A(a)$, $b^* \in B(b)$, and let x_r satisfy $x^* \in x^*_r + B(x_r) + rJ(x_r)$, where $x^*_r \in A(x_r)$. By monotonicity of A and B,

$$\langle x_r - a, x^*_r - a^* \rangle \geq 0 \tag{3.5}$$

and

$$\langle x_r - b, Bx_r - Bb \rangle \geq 0. \tag{3.6}$$

By assumption, there exists $w^*_r \in J(x_r)$ such that

$$x^* = x^*_r + B(x_r) + rw^*_r. \tag{3.7}$$

By (3.6),

$$\begin{aligned} &\langle b - x_r, Bb - B(x_r) \rangle \geq 0 \\ \Rightarrow &\langle b - x_r, Bb - x^* + x^*_r + rw^*_r \rangle \geq 0 \quad (\text{by } \ (3.7)) \\ \Rightarrow &\langle b - x_r, Bb - x^* \rangle \geq \langle x_r - b, x^*_r \rangle + \langle x_r, rw^*_r \rangle - \langle b, rw^*_r \rangle. \end{aligned} \tag{3.8}$$

Note that,

$$\langle b, w^*_r \rangle \leq \frac{1}{2}\|b\|^2 + \frac{1}{2}\|w^*_r\|^2$$

and

$$\langle x_r, w^*_r \rangle = \frac{1}{2}\|x_r\|^2 + \frac{1}{2}\|w^*_r\|^2.$$

Thus, (3.8) implies

$$\begin{aligned}
&\langle b - x_r, Bb - x^*\rangle \geq \langle x_r - b, x_r^*\rangle + \frac{1}{2}r\|x_r\|^2 - \frac{1}{2}r\|b\|^2 \\
\Rightarrow &\frac{1}{2}r\|x_r\|^2 \leq \langle b - x_r, Bb - x^*\rangle + \langle b - x_r, x_r^*\rangle + \frac{1}{2}r\|b\|^2 \\
\Rightarrow &\|rx_r\|^2 \leq 2r\langle b - x_r, Bb - x^*\rangle + 2r\langle b - x_r, x_r^*\rangle + \|rb\|^2.
\end{aligned}$$

As $\{x_r^*\}$ is bounded as $r \to 0$, the above inequality shows that $\{rx_r\}$ is bounded. Also by definition J, $\{rw_r^*\}$ is bounded. Therefore, by (3.7), $\|B(x_r)\|$ is bounded as $r \to 0$. Note that,

$$\begin{aligned}
r\|x_r\|^2 &= \langle x_r, rw_r^*\rangle \\
&= \langle x_r, x^* - x_r^* - B(x_r)\rangle \quad \text{(by(3.7))} \\
&= \langle x_r, a^* + b^* - x_r^* - Bx_r\rangle \\
&= \langle x_r, a^* - x_r^*\rangle + \langle x_r, b^* - Bx_r\rangle \\
&\leq \langle a, a^* - x_r^*\rangle + \langle b, Bb - Bx_r\rangle \quad \text{(by (3.5) and (3.6))}.
\end{aligned}$$

This shows that $r^{\frac{1}{2}}\|x_r\|$ is bounded as $r \to 0$. Thus, $\|x_r^* + Bx_r - x^*\| = \|rw_r^*\| = r^{\frac{1}{2}}\|r^{\frac{1}{2}}w_r^*\| \to 0$ as $r \to 0$ and therefore, $x^* \in \overline{\operatorname{ran}(A+B)}$. Hence, (3.3) holds. Next, let $x^* \in \operatorname{int}(\operatorname{ran} A + \operatorname{ran} B)$. Then, there exist $h^* \in X^*$ for sufficiently small $\|h^*\|$ such that $x^* + h^* = c_{h^*}^* + d_{h^*}^*$, where $c_{h^*}^* \in Aa_{h^*}$ and $d_{h^*}^* \in Bb_{h^*}$. By monotonicity of A and B,

$$\langle x_r - a_{h^*}, x_r^* - c_{h^*}^*\rangle \geq 0 \tag{3.9}$$

and

$$\langle x_r - b_{h^*}, Bx_r - d_{h^*}^*\rangle. \tag{3.10}$$

Thus,

$$\begin{aligned}
\langle x_r, h^*\rangle &= \langle x_r, c_{h^*}^* + d_{h^*}^* - x^*\rangle \\
&= \langle x_r, c_{h^*}^* + d_{h^*}^* - x_r^* - Bx_r - rw_r^*\rangle \quad \text{by (3.7)} \\
&= \langle x_r, c_{h^*}^* - x_r^*\rangle + \langle x_r, d_{h^*}^* - Bx_r\rangle - \langle x_r, rw_r^*\rangle \\
&= \langle x_r, c_{h^*}^* - x_r^*\rangle - \langle x_r, Bx_r - d_{h^*}^*\rangle - \langle x_r, rw_r^*\rangle \\
&\leq \langle a_{h^*}, c_{h^*}^* - x_r^*\rangle + \langle b_{h^*}, d_{h^*}^* - Bx_r\rangle - r\langle x_r, w_r^*\rangle \quad \text{(by (3.9) and (3.10)} \\
&= \langle a_{h^*}, c_{h^*}^* - x_r^*\rangle + \langle b_{h^*}, d_{h^*}^* - Bx_r\rangle + r\|x_r\|^2.
\end{aligned}$$

Since $\{x_r^*\}$, $\{Bx_r\}$ and $r^{\frac{1}{2}}\|x_r\|$ is bounded as $r \to 0$, then $\langle x_r, h^*\rangle \leq C(h^*)$. This shows that (x_r) is point-wise bounded. By uniform bounded principle, we obtain $\|\hat{x_r}\| = \|x_r\|$ is bounded. Since X satifies both Grothendieck and weakly compactly generated property. Thence, by [19, Theorem 4.9 (iii)] and Amir-Lindenstrauss Theorem [19, Theorem 4.8], let (x_{r_n}), $(x_{r_n}^*)$ and (Bx_{r_n}) subsequence of $(x_r), (x_r^*)$ and (Bx_r) such that $x_{r_n} \to x_0$ in the weak topology of X, $x_{r_n}^* \to x_0^*$ and $Bx_{r_n} \to b_0^*$ in the weak* topology of X^*. Thus, $x_{r_n}^* + Bx_{r_n} \to x_0^* + b_0^*$ in the weak star topology of X^*. If we replace $x_{\lambda_n} \to x_0$ in the weak topology of X instead of weak star topology. Then Lemma 3.1 holds. Thus, by Lemma 3.1, $\langle x_{r_n}, x_{r_n}^* + Bx_{r_n}\rangle \to \langle x_0, x_0^* + b_0^*\rangle$. For $(y, y^*) \in \operatorname{gra}(A+B)$, $\langle x_n - y, x_n^* + Bx_n - y^*\rangle \geq 0$. By passing limit along the subsequence and maximal monotonicity of $A+B$ implies that $x_0^* + b_0^* \in (A+B)(x_0)$. Again, by passing limit in (3.7) along the subsequences, we obtain $x^* = x_0^* + b_0^*$. Hence, $x^* \in \operatorname{ran}(A+B)$ and (3.4) holds. $\square$

Now we are ready for our next main result.

Theorem 3.3. *Let $A : X \rightrightarrows X^*$ be a maximal monotone operator and, let $B : X \to X^*$ be a single-valued maximal monotone operator. Suppose that $A + B$ is ultramaximally monotone and*

1. $\operatorname{dom} A \subseteq \operatorname{dom} B$,
2. $\|Bx\| \leq a|A(x)| + c,$ *for any* $x \in \operatorname{dom} A$

where $0 \leq a < 1$ *and* $c > 0$. *Then*

$$\overline{\operatorname{ran}(A + B)} = \overline{\operatorname{ran} A + \operatorname{ran} B}$$

and

$$\operatorname{int} \operatorname{ran}(A + B) = \operatorname{int}(\operatorname{ran} A + \operatorname{ran} B).$$

Proof . We can suppose that $(0, 0) \in \operatorname{gra} A$ and $(0, 0) \in \operatorname{gra} B$. Since ultramaximal monotone operators are invariant under translation. Then $A + B$ can be replaced by $\frac{1}{r}(A + B)$, for any $r > 0$. By assumption and Fact 2.1, we obtain $\operatorname{ran}(\frac{1}{r}(A + B) + J) = X^*$ and it is elementary to verify that $\operatorname{ran}(A + B + rJ) = X^*$. Thus, for any $x^* \in X^*$ and $r > 0$, there exists $x_r \in X$, satisfy $x^* \in x_r^* + B(x_r) + rJ(x_r)$, where $x_r^* \in A(x_r)$. By Proposition 3.2, it is enough to show that for $x^* \in \operatorname{ran} A + \operatorname{ran} B$, $\{x_r^*\}$ is bounded as $r \to 0$. Since (x_r) satisfy $x^* \in x_r^* + B(x_r) + rJ(x_r)$, then there exists $w_r^* \in J(x_r)$ such that

$$x^* = x_r^* + Bx_r + rw_r^*. \tag{3.11}$$

Now

$$\begin{aligned} r\|x_r\|^2 &= \langle x_r, rw_r^* \rangle \\ &= \langle x_r, x^* - x_r^* - Bx_r \rangle \quad \text{(by (3.11))} \\ &= \langle x_r, x^* \rangle - \langle x_r, x_r^* \rangle - \langle x_r, Bx_r \rangle \quad \text{(by monotonicity of A and B)} \\ &\leq \langle x_r, x^* \rangle \leq \|x_r\| \|x^*\|. \end{aligned}$$

By multiplying r in the above inequality, we obtain $\{rx_r\}$ is bounded as $r \to 0$ and so $\{rw_r^*\}$. By hypothesis,

$$\begin{aligned} \|Bx_r\| &\leq a|Ax_r| + c \\ &\leq a\|x_r^*\| + c \\ &\leq a\|x^*\| + a\|Bx_r\| + a\|rx_r\| + c. \end{aligned}$$

Since $a < 1$, then $\|Bx_r\| \leq \frac{a}{1-a}\|x^*\| + \frac{a}{1-a}\|rx_r\| + \frac{c}{1-a}$. This shows that $\|Bx_r\|$ is bounded as $r \to 0$. Therefore, by (3.11), $\{x_r^*\}$ is bounded as $r \to 0$. This completes the proof. □

4. Conclusion

This approach shows a way of proving Brézis-Haraux conditions in non-reflexive Banach spaces. For extending these results to Banach spaces one need to relax Grothendieck property as well as weakly compactly generated property. We mention here that both the properties are automatically satisfied in reflexive spaces.

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