Int. J. Nonlinear Anal. Appl. 14 (2023) 7, 99–106
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.24382.2732

Bounds of the fifth Toeplitz determinant for the classes of functions with bounded turnings

Hormoz Rahmatan

Department of Mathematics, Faculty of Mathematical Sciences, Payame Noor University, P.O.Box 19395-3697, Tehran, Iran

(Communicated by Ali Jabbari)

Abstract

In this paper, we investigate the Toeplitz determinant for a family of functions with bounded turnings, we give estimates of the Toeplitz determinants of fifth order for the set $\mathcal{R}$ of univalent functions with bounded turnings in the unit disc. Also, we obtain bounds of the fifth Toeplitz determinant for the subclasses of the class $\mathcal{R}$.

Keywords: Analytic functions, Univalent functions, Bounded turning functions, Toeplitz determinant
2020 MSC: Primary 30C45; Secondary 30C50

1 Introduction

Assume that $\mathcal{A}$ denotes the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C} : |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$. Also, let, S^* and $\mathcal{C}$ denote the classes of starlike and convex functions respectively and are defined as:

$$S^* = \{f \in \mathcal{S} : Re\ \left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{U}\},$$
$$\mathcal{C} = \{f \in \mathcal{S} : Re\ \left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in \mathbb{U}\}.$$

Suppose that $\mathcal{P}$ denote the class of analytic functions p of the type

$$p(z) = 1 + \sum_{n=2}^{\infty} c_n z^n, \tag{1.2}$$

such that $Re\,(p(z)) > 0$. A function $f \in \mathcal{A}$ is said to be close-to-convex, if there exists a starlike function $g \in S^*$ such that

$$Re\ \left(\frac{zf'(z)}{g(z)}\right) > 0,$$

Email address: h_rahmatan@pnu.ac.ir, h.rahmatan@gmail.com (Hormoz Rahmatan)

Received: August 2021 *Accepted:* November 2022

for $z \in \mathbb{U}$.

Assume that $\mathcal{R}$ denotes the class of functions f in $\mathcal{A}$ satisfying $Re\,(f'(z)) > 0$ in $\mathbb{U}$. It is easy to verify that functions in $\mathcal{R}$ are close-to-convex and hence univalent. Functions in $\mathcal{R}$ are sometimes called functions of bounded turnings.

Also, let $m \in \mathbb{N} = \{1, 2, \dots\}$. An analytic function f is $m-$fold symetric in $\mathbb{U}$, if

$$f(e^{\frac{2\pi i}{m}}z) = e^{\frac{2\pi i}{m}}f(z), \qquad (z \in \mathbb{U}).$$

By $\mathcal{S}^m$, we shall denote the set of $m-$fold univalent functions having the following Taylor series form

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1}z^{mk+1}, \qquad (z \in \mathbb{U}). \tag{1.3}$$

The sub-family $\mathcal{R}^m$ of $\mathcal{S}^m$ is the set of $m-$fold symetric functions with bounded turnings.
An analytic function f of the form(1.3) belongs the family $\mathcal{R}^m$, if and only if

$$f'(z) = p(z),$$

with $p \in \mathcal{P}^m$, where the set $\mathcal{P}^m$ is defined by

$$\mathcal{P}^m = \{p \in \mathcal{P} : p(z) = 1 + \sum_{k=1}^{\infty} c_{mk}z^{mk},\ z \in \mathbb{U}\}. \tag{1.4}$$

Pommerenke [11, 12] introduced the idea of Hankel determinants, and he defined those for univalent functions $f \in \mathcal{S}$ as follows:

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q-2} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_{n+2q-2} \end{vmatrix}$$

In the theory of analytic functions, finding the upper bound of $|H_q(n)|$ is one of the most studied problems. Several researchers found the above-mentioned bound for different subfamilies of univalent functions for fixed values of q and n[14, 15, 19]. For the subfamilies S^*,$\mathcal{C}$ and $\mathcal{R}$ of the set $\mathcal{S}$ the sharp bounds of $|H_2(2)|$ were investigated by Janteng et al.[5, 6]. they proved the bounds as follows:

$$|H_2(2)| \leq \begin{cases} 1 & , \quad f \in \mathcal{S}^*, \\ \frac{1}{8} & , \quad f \in \mathcal{C}, \\ \frac{4}{9} & , \quad f \in \mathcal{R}. \end{cases} \tag{1.5}$$

The accurate estimate of $|H_2(2)|$ was obtained by Krishna et al.[7] for the family of Bazilevic functions. For subfamilies of $\mathcal{S}$, According to Thomas' conjecture [13], if $f \in \mathcal{S}$, then $|H_q(2)| \leq 1$, but it was shown by Li and Srivastava in [9] that this conjecture is not true for $n \geq 4$. Estimation of $|H_3(1)|$ is much more difficult. Babalola [3] published the first paper on $H_3(1)(f)$ in 2010 in which he obtained the upper bound of $|H_3(1)|$ for the subfamilies S^*,$\mathcal{C}$ and $\mathcal{R}$. Zaprawa [18] improved the results of Babalola[3] recently in 2017, by showing

$$|H_3(1)| \leq \begin{cases} 1 & , \quad f \in \mathcal{S}^*, \\ \frac{49}{540} & , \quad f \in \mathcal{C}, \\ \frac{41}{60} & , \quad f \in \mathcal{R}. \end{cases} \tag{1.6}$$

Arif et al. [2] found the upper bounds of $|H_4(1)|$ and $|H_5(1)|$ for the classes of functions with bounded turnings. Toeplitz determinants are closely related to Hankel determinants [10]. Toeplitz matrices have constant entries along the diagonal. Toeplitz matrices have some applications in pure and applied mathematics[17].

Thomas and Halim in [16] introduced the symmetric determinant $T_q(n)$ for analytic functions f of the form (1.1) defined by,

$$T_q(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_n & \dots & a_{n+q-2} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_{n+q-1} & a_{n+q-2} & \dots & a_n \end{vmatrix} \qquad q \in \mathbb{N} \setminus 1, \quad n \in \mathbb{N}.$$

The study of exact upper bound of $|T_q(n)|$ for different subclasses of analytic functions has attracted some authors. The Toeplitz determinant $T_q(n)$ for class $\mathcal{S}$ of univalent functions was studied and improved by Ali et al.[1]. Also Ali et al.[1] have investigated $T_q(n)$ for subclasses of $\mathcal{S}$.
To prove our main results, we need following lemmas and theorems.

Lemma 1.1. If $p \in \mathcal{P}$ and of the form (1.2), then for $n \in \mathbb{N} = \{1, 2, \dots\}$, the following sharp inequality hold

$$|c_n| \leq 2. \tag{1.7}$$

Lemma 1.2. If $p \in \mathcal{P}$ and of the form (1.2), then for $n, k \in \mathbb{N} = \{1, 2, \dots\}$, the following inequalities hold

$$|c_{n+k} - \lambda c_n c_k| \leq 2, \quad for \quad 0 \leq \lambda \leq 1, \tag{1.8}$$

the inequalities (1.7) and (1.8) are proved in [4] and [8] respectively.

Lemma 1.3. [2] If $f \in \mathcal{R}$ and $n + m = k + l$, then

$$|a_n a_m - a_k a_l \leq \frac{4}{\mu} \tag{1.9}$$

where $\mu = \min\{mn, kl\}$.

Theorem 1.4. [1] Let $f \in \mathcal{S}$ be of the form (1.1). Then

$$(i) : |T_2(n)| \leq |a_n^2 - a_{n+1}^2| \leq 2n^2 + 2n + 1 \tag{1.10}$$

$$(ii) : |T_3(1)| \leq 24. \tag{1.11}$$

Both inequalities are sharp.

Theorem 1.5. [1] Let $f \in S^*$ be of the form (1.1). Then

$$T_3(2) \leq 84.$$

The inequality is sharp.

Theorem 1.6. [1] Let $f \in \mathcal{R}$ be of the form (1.1). Then

$$(i) : |T_2(n)| \leq \frac{4}{n^2} + \frac{4}{(n+1)^2}, \quad n \geq 2. \tag{1.12}$$

$$(ii) : |T_3(1)| \leq \frac{35}{9} \tag{1.13}$$

$$(iii) : |T_3(2)| \leq \frac{7}{3}. \tag{1.14}$$

In this paper, we obtain bounds of the fifth Toeplitz determinant for a family of functions with bounded turnings.

2 Bounds of $T_5(1)$ for the set $\mathcal{R}$

In this section, we obtain bounds of $|T_5(1)|$ for the set $\mathcal{R}$. The fifth Toeplitz determinant $|T_5(1)|$ is given by

$$T_5(1) = \begin{vmatrix} 1 & a_2 & a_3 & a_4 & a_5 \\ a_2 & 1 & a_2 & a_3 & a_4 \\ a_3 & a_2 & 1 & a_2 & a_3 \\ a_4 & a_3 & a_2 & 1 & a_2 \\ a_5 & a_4 & a_3 & a_2 & 1 \end{vmatrix} \tag{2.1}$$

we can write $T_5(1)$ in the form

$$T_5(1) = T_4(1) - a_2 A + a_3 B - a_4 C + a_5 D.$$

with

$$T_4(1) = T_3(1) - a_2\Delta_1 + a_3\Delta_2 - a_4\Delta_3, \tag{2.2}$$
$$A = a_2 T_3(1) - a_2\Delta_4 + a_3\Delta_5 - a_4\Delta_6, \tag{2.3}$$
$$B = a_2\Delta_7 - \Delta_8 + a_3\Delta_9 - a_4\Delta_{10}, \tag{2.4}$$
$$C = a_2\Delta_{11} - \Delta_{12} + a_2\Delta_{13} - a_4\Delta_{14}, \tag{2.5}$$
$$D = a_2\Delta_{15} - \Delta_{16} + a_2\Delta_{17} - a_3\Delta_{18}, \tag{2.6}$$

where

$$T_3(1) = (1 - a_2^2) - a_2(a_2 - a_2 a_3) + a_3(a_2^2 - a_3), \tag{2.7}$$
$$\Delta_1 = a_2(1 - a_2^2) - a_2(a_3 - a_2 a_4) + a_3(a_2 a_3 - a_4), \tag{2.8}$$
$$\Delta_2 = a_2(a_2 - a_2 a_3) - (a_3 - a_2 a_4) + a_3(a_3^2 - a_2 a_4), \tag{2.9}$$
$$\Delta_3 = a_2(a_2^2 - a_3) - (a_3 a_2 - a_4) + a_2(a_3^2 - a_2 a_4), \tag{2.10}$$

$$\Delta_4 = a_3(1 - a_2^2) - a_2(a_4 - a_2 a_5) + a_3(a_4 a_2 - a_5), \tag{2.11}$$
$$\Delta_5 = a_3(a_2 - a_2 a_3) - (a_4 - a_2 a_5) + a_3(a_3 a_4 - a_2 a_5), \tag{2.12}$$
$$\Delta_6 = a_3(a_2^2 - a_3) - (a_2 a_4 - a_5) + a_2(a_4 a_3 - a_2 a_5), \tag{2.13}$$

$$\Delta_7 = a_2(1 - a_2^2) - a_2(a_3 - a_2 a_4) + a_3(a_2 a_3 - a_4), \tag{2.14}$$
$$\Delta_8 = a_3(1 - a_2^2) - a_2(a_4 - a_2 a_5) + a_3(a_2 a_4 - a_5), \tag{2.15}$$
$$\Delta_9 = a_3(a_3 - a_2 a_4) - a_2(a_4 - a_2 a_5) + a_2(a_4^2 - a_3 a_5), \tag{2.16}$$
$$\Delta_{10} = a_3(a_2 a_3 - a_4) - a_2(a_2 a_4 - a_5) + a_2(a_4^2 - a_3 a_5), \tag{2.17}$$

$$\Delta_{11} = a_2(a_2 - a_2 a_3) - (a_3 - a_2 a_4) + a_3(a_3^2 - a_2 a_4), \tag{2.18}$$
$$\Delta_{12} = a_3(a_2 - a_2 a_3) - (a_4 - a_2 a_5) + a_3(a_3 a_4 - a_2 a_5), \tag{2.19}$$
$$\Delta_{13} = a_3(a_3 - a_2 a_4) - a_2(a_4 - a_2 a_5) + a_3(a_4^2 - a_3 a_5), \tag{2.20}$$
$$\Delta_{14} = a_3(a_3^2 - a_2 a_4) - a_2(a_3 a_4 - a_2 a_5) + (a_4^2 - a_3 a_5), \tag{2.21}$$

and

$$\Delta_{15} = a_2(a_2^2 - a_3) - (a_2 a_3 - a_4) + a_2(a_3^2 - a_2 a_4), \tag{2.22}$$
$$\Delta_{16} = a_3(a_2^2 - a_3) - (a_2 a_4 - a_5) + a_2(a_3 a_4 - a_2 a_5), \tag{2.23}$$
$$\Delta_{17} = a_3(a_2 a_3 - a_4) - a_2(a_2 a_4 - a_5) + a_2(a_4^2 - a_3 a_5), \tag{2.24}$$
$$\Delta_{18} = a_3(a_3^2 - a_2 a_4) - a_2(a_3 a_4 - a_2 a_5) + (a_4^2 - a_3 a_5), \tag{2.25}$$

From (2.1) we consider that $T_5(1)$ is a polynomial of four successive coefficients a_2, a_3, a_4 and a_5 of a function f in a given class.

Theorem 2.1. If $f \in \mathcal{R}$ and has the form (1.1), then

$$|T_4(1)| \leq \frac{5199}{486} = 10.69 \tag{2.26}$$

Proof . Let $f \in \mathcal{R}$. From Theorem 1.6, using Lemma 1.2 and Lemma 1.3 along with the inequality $|a_n| \leq \frac{2}{n}$ for $n \geq 2$, we have,

$$\begin{aligned}
|T_3(1)| &= \left|(1-a_2^2) - a_2(a_2 - a_2a_3) + a_3(a_2^2 - a_3)\right| \leq \frac{35}{9},\\
|\Delta_1| &= \left|a_2(1-a_2^2) - a_2(a_3 - a_2a_4) + a_3(a_2a_3 - a_4)\right| \leq \frac{20}{6},\\
|\Delta_2| &= \left|a_2(a_2 - a_2a_3) - (a_3 - a_2a_4) + a_3(a_3^2 - a_2a_4)\right| \leq \frac{187}{54},\\
|\Delta_3| &= \left|a_2(a_2^2 - a_3) - (a_3a_2 - a_4) + a_2(a_3^2 - a_2a_4)\right| \leq \frac{14}{6}.
\end{aligned}$$

Since,

$$T_4(1) = T_3(1) - a_2\Delta_1 + a_3\Delta_2 - a_4\Delta_3,$$

by using the triangle inequality, we conclude the proof. □

Theorem 2.2. If $f \in \mathcal{R}$ and has the form (1.1), then

$$|A| \leq \frac{735}{90} = 8.16. \tag{2.27}$$

Proof . Let $f \in \mathcal{R}$. From Theorem 1.6, using Lemma 1.2 and Lemma 1.3 along with the inequality $|a_n| \leq \frac{2}{n}$ for $n \geq 2$, we get,

$$\begin{aligned}
|T_3(1)| &= \left|(1-a_2^2) - a_2(a_2 - a_2a_3) + a_3(a_2^2 - a_3)\right| \leq \frac{35}{9},\\
|\Delta_4| &= \left|a_3(1-a_2^2) - a_2(a_4 - a_2a_5) + a_3(a_4a_2 - a_5)\right| \leq \frac{73}{30},\\
|\Delta_5| &= \left|a_3(a_2 - a_2a_3) - (a_4 - a_2a_5) + a_3(a_3a_4 - a_2a_5)\right| \leq \frac{55}{30},\\
|\Delta_6| &= \left|a_3(a_2^2 - a_3) - (a_2a_4 - a_5) + a_2(a_4a_3 - a_2a_5)\right| \leq \frac{56}{45}.
\end{aligned}$$

Consequently, from (2.3) by using the triangle inequality, we obtain the declared bound. □

Theorem 2.3. If $f \in \mathcal{R}$ and has the form (1.1), then

$$|B| \leq \frac{4054}{540} = 7.50 \tag{2.28}$$

Proof . Let $f \in \mathcal{R}$. Using Lemma 1.2, Lemma 1.3 and $|a_n| \leq \frac{2}{n}$ for $n \geq 2$, it follows that

$$\begin{aligned}
|\Delta_7| &= \left|a_2(1-a_2^2) - a_2(a_3 - a_2a_4) + a_3(a_2a_3 - a_4)\right| \leq \frac{21}{6},\\
|\Delta_8| &= \left|a_3(1-a_2^2) - a_2(a_4 - a_2a_5) + a_3(a_2a_4 - a_5)\right| \leq \frac{73}{30},\\
|\Delta_9| &= \left|a_3(a_3 - a_2a_4) - a_2(a_4 - a_2a_5) + a_2(a_4^2 - a_3a_5)\right| \leq \frac{145}{90},\\
|\Delta_{10}| &= \left|a_3(a_2a_3 - a_4) - a_2(a_2a_4 - a_5) + a_2(a_4^2 - a_3a_5)\right| \leq 1.
\end{aligned}$$

Putting the above values and $a_n \leq \frac{2}{n}$ for $n \geq 2$ in (2.4) gives the desired result. This completed the proof. □

Theorem 2.4. If $f \in \mathcal{R}$ and has the form (1.1), then

$$|C| \leq \frac{1956}{270} = 7.24. \tag{2.29}$$

Proof . Let $f \in \mathcal{R}$. Formulas (1.8), (1.9) and the inequality $|a_n| \leq \dfrac{2}{n}$ for $n \geq 2$, result in

$$\begin{aligned}
|\Delta_{11}| &= \left|a_2(a_2 - a_2a_3) - (a_3 - a_2a_4) + a_3(a_3^2 - a_2a_4)\right| \leq \frac{47}{18},\\
|\Delta_{12}| &= \left|a_3(a_2 - a_2a_3) - (a_4 - a_2a_5) + a_3(a_3a_4 - a_2a_5)\right| \leq \frac{615}{270},\\
|\Delta_{13}| &= \left|a_3(a_3 - a_2a_4) - a_2(a_4 - a_2a_5) + a_3(a_4^2 - a_3a_5)\right| \leq \frac{501}{270},\\
|\Delta_{14}| &= \left|a_3(a_3^2 - a_2a_4) - a_2(a_3a_4 - a_2a_5) + (a_4^2 - a_3a_5)\right| \leq 1.
\end{aligned}$$

From the above values along with the inequality $|a_n| \leq \dfrac{2}{n}$ for $n \geq 2$, in (2.5), we obtain the desired result. □

Theorem 2.5. If $f \in \mathcal{R}$ and has the form (1.1), then

$$|D| \leq \frac{412}{90} = 4.57. \tag{2.30}$$

Proof . Let $f \in \mathcal{R}$. Applying (1.8), (1.9) and $|a_n| \leq \dfrac{2}{n}$ for $n \geq 2$, we get,

$$\begin{aligned}
|\Delta_{15}| &= \left|a_2(a_2^2 - a_3) - (a_2a_3 - a_4) + a_2(a_3^2 - a_2a_4)\right| \leq \frac{5}{3},\\
|\Delta_{16}| &= \left|a_3(a_2^2 - a_3) - (a_2a_4 - a_5) + a_2(a_3a_4 - a_2a_5)\right| \leq \frac{112}{90},\\
|\Delta_{17}| &= \left|a_3(a_2a_3 - a_4) - a_2(a_2a_4 - a_5) + a_2(a_4^2 - a_3a_5)\right| \leq 1,\\
|\Delta_{18}| &= \left|a_3(a_3^2 - a_2a_4) - a_2(a_3a_4 - a_2a_5) + (a_4^2 - a_3a_5)\right| \leq 1.
\end{aligned}$$

Now putting the above estimates and $|a_n| \leq \dfrac{2}{n}$ in (2.6), we obtain the desired result. The proof is complete. □

Theorem 2.6. If $f \in \mathcal{R}$ and has the form (1.1), then

$$|T_5(1)| \leq 32.34. \tag{2.31}$$

Proof . Let $f \in \mathcal{R}$ be of the form 1.1. Clearly,

$$|T_5(1)| \leq |T_4(1)| + |a_2||A| + |a_3||B| + |a_4||C| + |a_5||D|. \tag{2.32}$$

Now putting the bounds found in Theorems 2.1–2.5 and the inequality $|a_n| \leq \dfrac{2}{n}$ for $n \geq 2$ in (2.32), we obtain

$$\begin{aligned}
|T_5(1)| &\leq \frac{5199}{486} + \frac{735}{90} + \left(\frac{2}{3} \times \frac{4054}{540}\right) + \left(\frac{1}{2} \times \frac{1956}{270}\right) + \left(\frac{2}{5} \times \frac{412}{90}\right)\\
&= 10.69 + 8.16 + 8.04 + 3.62 + 1.83\\
&= 32.34
\end{aligned}$$

This concludes the proof. □

3 Bounds of $T_5(1)$ for the sets $\mathcal{R}^2$ and $\mathcal{R}^4$

In this section, we obtained bounds of $|T_5(1)|$ for sub-families $\mathcal{R}^2$ and $\mathcal{R}^4$.

Theorem 3.1. Let $f \in \mathcal{R}^2$ be of the form (1.3). Then

$$|T_5(1)| \leq 3.47$$

Proof . Since $f \in \mathcal{R}^2$, there is a function $p \in \mathcal{P}^2$ such that

$$f'(z) = p(z).$$

Equating coefficients,

$$a_2 = 0, \quad a_3 = \frac{c_2}{3}, \quad a_4 = 0, \quad a_5 = \frac{c_4}{5}. \tag{3.1}$$

By a simple computation, $T_5(1)$ can be written as

$$T_5(1) = (1 - a_3^2)(1 - 2a_3^2 + 2a_5a_3^2 - a_5^2). \tag{3.2}$$

Using (3.1) and triangle inequality, we get

$$\begin{aligned}|T_5(1)| \leq & 1 + \frac{|c_2|^2}{3} + \frac{2}{45}|c_4||c_2|^2 + \frac{1}{25}|c_4|^2 \\ & + \frac{2}{81}|c_2|^4 + \frac{2}{405}|c_4||c_2|^4 + \frac{1}{225}|c_2|^2|c_4|^2.\end{aligned}$$

From Lemma 1.1, it is easily follows that,

$$|T_5(1)| \leq 1 + \frac{4}{3} + \frac{16}{45} + \frac{4}{25} + \frac{32}{81} + \frac{64}{405} + \frac{16}{225}.$$

Therefore, $|T_5(1)| \leq 3.47$. This concludes the proof. □

Theorem 3.2. Let $f \in \mathcal{R}^4$ be of the form (1.3). Then

$$|T_5(1)| \leq 1.16$$

Proof . Since $f \in \mathcal{R}^4$, there is a function $p \in \mathcal{P}^4$ such that

$$f'(z) = p(z).$$

From (1.3) and (1.4), when $m = 4$, we can write

$$a_2 = 0, \quad a_3 = 0, \quad a_4 = 0, \quad a_5 = \frac{c_4}{5}. \tag{3.3}$$

It is easy to see that

$$T_5(1) = 1 - a_5^2.$$

Using (3.3) and triangle inequality, we conclude,

$$|T_5(1)| \leq 1 + \frac{1}{25}|c_4|^2.$$

From Lemma1.1, it easily follows that $|T_5(1)| \leq 1.16$. This concludes the proof. □

4 Conclusion

The bounds of Toeplitz and Hankel determinants have always been the main interest of researchers in univalent and bi-univalent classes. Many studies related to this problem are around analytic normalized functions. Here the fifth Toeplitz determinant is obtained for functions with bounded turnings.

References

[1] M.F. Ali, D.K. Thomas and A. Vasudevarao, *Toeplitz determinants whose elements are the coefficients of analytic and univalent functions*, Bull. Aust. Math. Soc. **97** (2018), no. 2, 253–264.

[2] M. Arif, I. Ullah, M. Raza and P. Zaprawa, *Investigation of the fifth Hankel determinant for a family of functions with bounded turnings*, Math. Slovaca **70** (2020), no. 2,319–328.

[3] K.O. Babalola, *On $H_3(1)$ Hankel determinant for some classes of univalent functions*, Inequl. Theory Appl. **6** (2007), 1–7.

[4] P.L. Duren, *Univalent functions*, Grundlehren der Mathematischen Wissenschaften Band 259, SpringerVerlag, New York, 1983.

[5] A. Janteng, S.A. Halim and M. Darus, *Hankel determinant for starlike and convex functions*, Ient. J. Math. Anal. **13** (2007), no. 1, 619–625.

[6] D.V. Janteng, S.A. Halim and M. Darus, *Coefficient inequality for a function whose derivative has a positive real part*, J. Inequal. Pure Appl. Math. **7** (2006), no. 2, 50.

[7] A. Krishna and T. RamReddy, *Second Hankel determinant for the class of Bazilevic functions*, Stud. Univ. Babeş¸Bolyai Math. **60** (2015), no. 3, 413–420.

[8] A.E. Livingston, *The coefficients of multivalent close-to-convex functions*, Proc. Amer. Math. Soc. **21** (1969), 545–552.

[9] J.L. Li and H.M. Srivastava, *Some questions and conjectures in the theory of univalent functions*, Rocky Mount. J. Math. **28** (1998), no. 3, 1035–552.

[10] Sh. Najafzadeh, H. Rahmatan and H. Haji, *Application of subordination for estimating the Hankel determinant for subclass of univalent functions*, Creat. Math. Inf. **30** (2021), 69–74.

[11] C. Pommerenke, *On the coefficients and Hankel determinants of univalent functions*, J. London Math. Soc. **41** (1966), 111–122.

[12] C. Pommerenke, *On the Hankel determinants of univalent functions*, Mathematika **14** (1967), 108–112.

[13] R. Parvatham and S. Ponnusamy, *New Trends in Geometric Function 4eory and Application*, World Scientific Publishing Company, London, UK, 1981.

[14] H. Rahmatan, A. Shokri, H. Ahmad and T. Botmart, *Subordination method for the estimation of certain subclass of analytic functions defined by the q-derivative operator*, J. Funct. Spaces **2022** (2022), Article ID 5078060, 9 pages.

[15] H. Rahmatan, H. Haji and Sh. Najafzadeh, *Coefficient estimates and Fekete-Szegö coefficient inequality for new subclasses of Bi-univalent functions*, Caspian J. Math. Sci. **10** (2021), no. 1, 39–50.

[16] D.K. Thomas and A.A. Halim, *Toeplitz matrices whose elements are the coefficients of starlike and close-to-convex functions*, Bull. Malays. Math. Sci. Soc. **40** (2017), no. 4, 1781–1790.

[17] K. Ye and L.H. Lim, *Every Matrix is product of Toeplitz matrices*, Found. Comput. **16** (2016), 577–598.

[18] P. Zaprawa, *Third Hankel determinants for subclasses of univalent functions*, Mediter. J. Math. **14** (2017), no. 1, 19.

[19] P. Zaprawa, *Fourth-order Hankel determinants and Toeplitz determinants for convex functions connected with sine functions*, J. Math. **2022** (2022), Article ID 2871511, 12 pages.