Int. J. Nonlinear Anal. Appl. 14 (2023) 5, 95–101
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2023.33574.4187

The Daniell's functional on a Banach lattice

Noori F. Al-Mayahi, Haitham Irhayyim Mosa*

Mathematics Department, College of Science, Al-Qadisiyah University, Iraq

(Communicated by Farshid Khojasteh)

Abstract

In this paper, we presented both the concept of Daniell space and the extension of Daniell space with some basic results related to these spaces when the Daniell functional on a Banach lattice space. The extension of Daniell's space has been proven a complete space.

Keywords: Daniell space, Riesz space, Banach lattice, normed space, Banach space, complete
2020 MSC: Primary 90C33; Secondary 26B25

1 Introduction

The area of study of computing areas of geometrical shapes was originated more than 2 millenniums ago with the introduction by Greek mathematicians of the celebrated "method of exhaustion". The process of computing areas and volumes of geometrical figures is called integration. The method of exhaustion assumes that a convex figure is approximated by inscribed polygons whose areas can be calculated and the number of the vertexes of the inscribed polygons is increased until the convex region becomes "exhausted". So, the area of the convex region can be computed as the limit of the areas of the inscribed polygons. The method of exhaustion was used by Archimedes (287-212 B.C.) to calculate the area of circles and the volume of spheres as well as other geometrical figures. The method of exhaustion is, in fact, considered the core of all modern integration techniques. Cauchy (1789-1857) and B. Riemann (1826–1866) were among the first to present axiomatic abstract foundations of integration. The work of Riesz (1909) and Daniell (1889) established fundamental connections between integration and functional analysis and a connection between linear continuous functional and measure. In his research paper " A General Form of Integral " in 1918, Daniell defined an integral as a function defined on a certain class of functions as a continuous function or step functions this functional is linear, nonnegative and to have a monotone convergence property. He then devised a procedure for extending this functional to a larger class of functions in such a way that it still satisfies the given condition.

In [14] He introduced the definition of Daniell space and extended this space and showed that the extended space contains the first, and he defined the lower and upper Daniell integral, in [4] Banasiak briefly present basic concepts of the theory of Banach lattices. In [6] Jeurnink study the integration theories for functions which are defined on a finite measure space and which take on values in a Banach lattice.

In our paper, we presented the definition of Daniell space when the Daniell functional on a Banach lattice instead of a Riesz space and the definition of the extension space and the complete space as it was known in [5] and we showed

*Corresponding author
Email addresses: nafm60@yahoo.com (Noori F. Al-Mayahi), hr.musa82@gmail.com (Haitham Irhayyim Mosa)

Received: December 2022 *Accepted:* March 2023

that this space is complete We have presented these concepts based on the detailed information provided by both [4, 6]. We also presented the concept of the lower and the upper Daniell integral.

2 Vector lattices

In this section we presented a general introduction of the vector lattice or Riesz space and Daniell space.

Definition 2.1. [14] A linear space over F is a set L, whose elements are called vectors, and in which two operations, addition $(+ : L \times L \to L)$ and scalar multiplication $(\because F \times L \to L)$ such that f and g in S implies that $f + g$ is in $S; f$ in S, a in F implies af in S and further:

(1) L is an abelian group under addition,

(2) for all f and g in L and for all a and b in F,

$$a(f+g) = af + bg, (a+b)f = af + bf, (ab)f = a(bf), 1f = f.$$

Remark 2.2. [10] A real linear space is one for which $F = R$, a complex is linear space is one for which $F = C$

Definition 2.3. [7] If L be a nonempty set , the set of all ordered pairs (z, w) is called the Cartesian product of L and L it self; notation $L \times L$, where $z, w \in L$.

Remark 2.4. [8] Let L be a non-empty set. For $z \neq w$ the points (z, w) and (w, z) of $L \times L$ are different.

Definition 2.5. [16] The relation R is called equivalence in L if,

(1) $(z, z) \in R$ for all $z \in L$ (R is reflexive),

(2) $(z, w) \in R$ and $(w, z) \in R(R$ is symmetric),

(3) $(z, w) \in R$ and $(w, e) \in R$ implies $(z, e) \in R(R$ is transitive).

Definition 2.6. [16] The relation R is called a partial ordering if,

(1) $(z, z) \in R$ for all $z \in L(R$ is reflexive),

(2) $(z, w) \in R$ and $(w, e) \in R$ implies $(z, e) \in R(R$ is transitive).

(3) $(z, w) \in R$ and $(w, z) \in R$ implies $z = w(R$ is anti-symmetric).

If R is a relation of partial ordering, we write $z \leq w$ or $z \leq w$ instead of $(z, w) \in R$.

Remark 2.7. [9]

(1) If R is a relation of partial ordering. If $z \leq w$ or $z \geq w$ then we say that z and w are comparable and if neither $z \leq w$ nor $z \geq w$ then z and w are incomparable, For each paire of points z, w satisfying $z \neq w$ is an incomparable paire

(2) If each points in L are comparable then the partial ordering is a linear ordering. For each paire of points z, w satisfying $z \neq w$ is an incomparable paire.

Example 2.8. [16] $L = R^2 = R \times R$, if $z, w \in S$ then $z = (z_1, z_2)$ and $w = (w_1, w_2)$, then $z \leq w$ if and only if $z_1 \leq w_1$ and $z_2 \leq w_2$, and the points $(0, 1)$ and $(1, 0)$ are incomparable point.

Definition 2.9. [8] Let L be a vector space, and assume that there is a reflexive, anti-symmetric and transitive relation $\leq$ on L such that,

(1) If $z, w \in L$ then $z \leq w$ and $z + k \leq w + k$ for all $k \in L$.

(2) If $z \in L, z \geq 0$, then for all $\delta \in R, \delta \geq 0$ we have $\delta z \geq 0$.

Then L be an ordered vector space.

Definition 2.10. [16] If L be an ordered vector space has the property that any set $\{y, z\}$ consisting of two elements $y, z \in L$ has both its maximum and minimum, then L is called a Riesz space or a Vector lattice, where $\{y, z\} = y \vee z$ and $\{y, z\} = y \wedge z$.

Any element z in a Riesz space L has the representation $z = z^+ - z^-$, where z^+be a positive part and $z^-$the negative part of z. For the element $z^+$and $z^-$the usual notation are $z \vee 0$ and $(-z) \vee 0$ and the absolute value of z is $|z| = z^+ \vee z^-$. For any subset A of L and the maximum of A is exist, then the elements $(-A), (A)$ are exist and $(A) = -(-A)$.

Definition 2.11. [8] A vector lattice L is said to be Archimedean, if $z, w \in L$ and $kz \leq w$ for all $k \in N$ then $z \leq 0$.

Remark 2.12. [8] If $z \in L$ and $z > 0$ then $\delta_n z \downarrow 0$, where $\delta_n \in R$ for each n and $\delta_n \downarrow 0$.

Definition 2.13. [12] Let Ω be an arbitrary set and h, k are real valued functions on Ω then we define,
$h \vee k = \{h, k\} = \max\{h - k, 0\} + k$ and $h \wedge k = \{h, k\} = (h + k) - \{h, k\}$, where 0 is the zero function.

Remark 2.14. [13, 16]

(1) If L be a vector space of real valued function on Ω. Then L be an ordered vector space if L is partially ordered by defining that $h \leq j$ in L whenever $h(x) \leq j(x)$ for all $x \in \Omega$.

(2) Suppose that L is a set of all real valued function on a set Ω. Then L is a real linear space under the following addition and scalar multiplication

  (a) $(h + j)(x) = h(x) + j(x)$ for all $h, j \in L$,

  (b) $(\lambda h)(x) = \lambda h(x)$ for all $h \in L$ and for all $\lambda \in R$.

Definition 2.15. [9] A real linear space L of real valued functions on Ω is called an ordered vector space if L is partially ordered in such a manner that the partial ordering is compatible with the algebraic structure of L. That is,

(1) $h \leq j$ implies $h + f \leq j + f$ for every $f \in L$,

(2) $f \geq 0$ implies $af \geq 0$ for every real number $a \geq 0$.

The ordered vector space S is called a Riesz space if for every pair h and j in S, the maximum $\max\{h, j\}$ minimum $\min\{h, j\}$ with respect to the partial ordering exists in S.

Remark 2.16. [7]

(1) Let L be a linear space of functions $(h : F \to R)$. Then L is a vector lattice (Riesz space) if $\max\{h, 0\} \in S$ for all $h \in$ L.

(2) If h is a real valued function in Riesz space then $|h|$ is also in a Riesz space.

Definition 2.17. [2] Let Ω be any set and $h : FR$ a function, we define the positive and negative parts $h^+$and $h^-$by $h^+ \max\{h, 0\}$ and $h^- \min\{h, 0\} \max\{h, 0\}$, where, $h^+(x) = \{h(x), \quad h(x) \geq 0\}$ and $h^-(x) = \{-h(x), \quad h(x) \leq 0\}$ the following relations for h^+ and h^- are hold

(1) $h = h^+ - h^-$and $|h| = h^+ + h^- = h^+ + (-h)^-$

(2) $h^+ = \frac{1}{2}(|h| + h)$ and $h^- = \frac{1}{2}(|h| - h)$

(3) $(-h)^+ = h^-$and $(-h)^- = h^+$

(4) If $\lambda > 0$, then $(\lambda h)^+ = \lambda h^+$and $(\lambda h)^- = \lambda h^-$

Definition 2.18. [1] Let R be the set of real numbers. The extended real numbers system consists of the real numbers system to be the real number with two symbols, $+\infty$ and ∞. and it is denoted by $\underline{R}$, that is,

$$\underline{R} = R \cup \{\infty\} \cup \{-\infty\} = \{-\infty, \infty\}$$

The following algebraic relation among them and real numbers

$$x : -\infty < x < \infty$$

(1) $z + \infty = \infty + z = \infty, x + (-\infty) = -\infty + z = -\infty$,

(2) If $z = 0$, then $z(\infty) = 0$ and $z(-\infty) = 0$,

(3) If $z > 0$, then $z(\infty) = \infty$ and $z(-\infty) = \infty$,

(4) If $z < 0$, then $z(\infty) = -\infty$ and $z(-\infty) = -\infty$,

(5) $\infty + \infty = \infty, -\infty + (-\infty) = -\infty, \infty - (-\infty) = \infty, -\infty - \infty = -\infty$,

(6) An infinite sum with one or more terms ∞ and no terms of $-\infty$ is equal to ∞.

We may notice here that in $\underline{R}$, every increasing sequence of real numbers has a limit, where we define $x_n = \infty$ if the sequence is not bounded

Example 2.19. $[16, 9]$

(1) If V is a Riesz space then V^n is a Riesz space,

(2) $\underline{R}$ is a Riesz space and R is a Riesz space.

Definition 2.20. [5] Let L be a Riesz space of functions defined on Ω. A linear functional $D : \Omega \to R$ is called,

(1) Positive if $D(h) \geq 0$ whenever $h \in L$ and $h \geq 0$,

(2) Continuous under monotone limits if for every increasing sequence $\{h_n\}$ of functions in L and $h \in L$ such that $h(x) \leq h_n(x)$ for all $x \in \Omega$, then $D(h)D(h_n)$,

Then D is Daniell functional (Daniell integral) whenever D is positive and continuous under monotone limit.

Remark 2.21. [14] If D is positive, then $D(h) \leq D(j)$ for each $h \in L$ and $h \leq j$. Remark (2.22):

(1) A triple (Ω, L, D) is a Daniell space if Ω is a nonempty set, L is a Riesz space of real valued functions on Ω, and $D : L \to R$ is a Daniell functional.

(2) D is continuous under monotone limit if and only if $D(h_n) \downarrow 0$ whenever $h_n \downarrow 0$ for each $h_n \in L$.

3 Main Results

Definition 3.1. [11] Let L be a vector lattice. A norm on L is a function $\|\| : .L \to R$, having the following properties,

(1) $\|f\| \geq 0$ for all $f \in L$,

(2) $\|f\| = 0$ if and only if $f = 0$,

(3) $\|\lambda f\| = |\lambda|\|f\|$ for all $f \in L$ and $\lambda \in R$,

(4) $\|f + g\| \leq \|f\| + \|g\|$ for all $f, g \in L$.

A Riesz space S together with $\|\cdot\|$ is called a normed space and it is denoted by $(L, \|\cdot\|)$.

Remark 3.2. [4, 5]

(1) If (Ω, L, D) be a Daniell space. A norm on L is a function $\|\| : .L \to R$ which is defined by $\|f\| = D(|f|)$.

(2) A norm on a vector lattice S is called a lattice norm if $|f| \leq |g|$ implies $\|f\| \leq \|g\|$.

Theorem 3.3. If L is a normed lattice then $\||f|\| = \|f\|$ for every $f \in L$.

Proof . Let $f, g \in L$ such that $g = |f|$, then we have $|f| \leq |(|f|)|$ and hence $\|f\| \leq \||f|\|$, by taking $|f|$ and $g = f$, we also have $|(|f|)| \leq |f|$ and hence $\||f|\| = \|f\|$. □

Definition 3.4. A vector lattice L is called a Banach lattice if it is complete under a lattice norm and denoted by L_B.

Theorem 3.5. Let (Ω, L_B, D) be a space, where Ω be a nonempty set, L_B Banach lattice and D is a function such that $D : L_B \to R$, then (Ω, L_B, D) be a Daniell space

Proof . We will show that D is positive and continuous under monotone limit. Let $f \geq 0$, then $|f| \geq 0$ implies $\|f\| \geq 0$, so $D(|f|) \geq 0$. To prove that D is continuous under monotone limit. Let $\{f_n\}$ be a sequences in L_B with $f_n \downarrow 0$, then $f_n \to 0$ in norm implies that, there is $\epsilon > 0$, since $0 = f_n$, there exist $k \in Z^+$such that $D(|f|) < \epsilon$ for all $n \geq \epsilon$ implies $D(f_n) \to 0$ in norm, since L_B is complete, then $D(f_n) \downarrow 0$. Therefore, the space (Ω, L_B, D) is a Daniell space. □

Definition 3.6. Given a Daniell space (Ω, L_B, D), let ${L_B}^*$ be the class of all extended real valued functions on Ω for which there exists a sequence of functions $f_1, f_2, \ldots \in L_B$ such that $f = \sum_{n=1}^{\infty} f_n$.

Now, we want to show that $(\Omega, {L_B}^*, D)$ is a complete Daniell space, where the integral of $f = \sum_{n=1}^{\infty} f_n$ is defined as $D(f) = \sum_{n=1}^{\infty} D(f_n)$. That is, (if L_B is a Banach lattice, then $h \in {L_B}^*$ if and only if $h : \Omega \to \underline{R}$ a function and there exists a sequence $\{h_n\}$ of monotone increasing sequences of functions in L_B such that $h_n \to h$ in norm.

Definition 3.7. [5] Let f be a real function on Ω. if there exist a function $f_n \in L, n \in N$, such that

(1) $\sum_{n=1}^{\infty} D(|f_n|) < \infty$,

(2) $f(x) = \sum_{n=1}^{\infty} f_n(x)$ for every $x \in \Omega$ and $\sum_{n=1}^{\infty} |f_n(x)| < \infty$, then we write $f = \sum_{n=1}^{\infty} f_n$.

Definition 3.8. [5] A Daniell space (Ω, L, D) is called complete if $f = \sum_{n=1}^{\infty} f_n$ for some $f_1, f_2, \ldots \in L$, implies that $f \in L$.

Theorem 3.9. Let (Ω, L_B, D) be a Daniell space, f_n and g_n are non-decreasing sequences in L_B with $f_n(x) \leq g_n(x)$ for every $x \in \Omega$, then $\lim_{n\to\infty} \|f_n\| \leq \lim_{n\to\infty} \|g_n\|$.

Proof . Let $k \in N$, the function $f_k - (f_k \wedge g_n), n \in N$, for a non-increasing sequence which converges to zero at ever point in Ω, we have, $D(|f_k|) - D(|f_k \wedge g_n|) = 0$ and hence $\|f_k\| = \lim_{n\to\infty} \|f_k \wedge g_n\| \leq \|g_n\|$, if we let $k \to \infty$, then $\lim_{n\to\infty} \|f_n\| \leq \lim_{n\to\infty} \|g_n\|$. □

Theorem 3.10. Let (Ω, L_B, D) be a Daniell space. If $f = \sum_{n=1}^{\infty} f_n$ and $f \geq 0$, then $\sum_{n=1}^{\infty} \|f_n\| \geq 0$.

Proof . Define $g_n = f_1 + f_2 + \cdots, +f_p + |f_{p+1}| + |f_{p+2}| + \cdots + |f_{p+n}|$ and $h_n = g_n \wedge 0$ suppose that g_n and h_n are non-decreasing sequences in L_B and $f_n = g_n$ then we have $\lim_{n\to\infty} \|f_n\| \geq 0$, so $\|f_1\| + \|f_2\| + \cdots, + \|f_p\| + \|f_{p+1}\| + \|f_{p+2}\| + \cdots + \|f_{p+n}\| + \cdots \geq 0$, if we let $p \to \infty$, we obtain $\sum_{n=1}^{\infty} \|f_n\| \geq 0$. □

Theorem 3.11. Let (Ω, L_B, D) be a Daniell space. If $f = \sum_{n=1}^{\infty} f_n$ and $f = \sum_{n=1}^{\infty} f_n$, then $\sum_{n=1}^{\infty} f_n \| = \sum_{n=1}^{\infty} \| g_n\|$.

Proof . Since $0 = f_1 - g_1 + f_2 - g_{21} \ldots$, we have $\sum_{n=1}^{\infty} \|f_n\| - \sum_{n=1}^{\infty} \|g_n\| \geq 0$, similarly, we have $\sum_{n=1}^{\infty} \|g_n\| - \sum_{n=1}^{\infty} \|f_n\| \geq 0$. Therefore $\sum_{n=1}^{\infty} \|f_n\| = \sum_{n=1}^{\infty} \|g_n\|$. □

Theorem 3.12. Let (Ω, L_B, D) be a Daniell space. If $f, g \in L_B{}^*$ and $f \leq g$, then $\|f\| \leq \|g\|$.

Proof . Since $f, g \in L_B{}^*$, then $f = \sum_{n=1}^{\infty} f_n$ and $g = \sum_{n=1}^{\infty} g_n$, then $f + g = f_1 + g_1 + f_2 + g_2 + \cdots$, since $D(f+g) = D(f) + D(g)$. If $f, g \in L_B{}^*$ and $f \leq g$, then $g - f \geq 0$. Hence $D(g-f) \geq 0$, therefore $\|f\| \leq \|g\|$. □

Theorem 3.13. Let (Ω, L_B, D) be a Daniell space. If $f \in L_B{}^*$, then $\|f\| \in L_B{}^*$ and $|D(f)| \leq \|f\|$. Moreover, if $\|f\| = \|f_1 + f_2 + \cdots + f_n\|$.

Proof . Let $f = \sum_{n=1}^{\infty} f_n$. Define $A = \{x \in \Omega : \sum_{n=1}^{\infty} f_n\| < \infty\}$ and $S_n = f_1 + f_2 + \cdots + f_n$. Then, $f(x) = S_{n n \to \infty}$ for all $x \in A$. In other word, $\|f\| = \|S_1(x)\| + (\|S_2(x)\| - \|S_1(x)\|) + (\|S_3(x)\| - \|S_2(x)\|) + \cdots$ for $x \in A$. Let $g_1 = \|S_1(x)\|$ and $g_n = \|S_n(x)\| - \|S_{n-1}(x)\|$ for $n \geq 2$, we claim that $\|f\| = g_1 + f_1 - f_1 + g_2 + f_2 - f_2 + \cdots$. We will show that $\sum_{n=1}^{\infty} D(|g_n|) < \infty$ and that $\|f\| = \sum_{n=1}^{\infty} |: g_n$ for all $x \in A$.

First, for $n \geq 2$, we have $\|g_n\| \leq \|S_n(x) - S_{n-1}(x)\| = \|f\|$, thus $\sum_{n=1}^{\infty}(|g_n|) \leq \sum_{n=1}^{\infty} D(|f_n|) < \infty$, since $f = \sum_{n=1}^{\infty} f_n$ and $\|f\| = \sum_{n=1}^{\infty} g_n$ for all $x \in A$. If $x \notin A$ then the sum is not absolutely convergent. There fore $f \in L_B{}^*$.

Since $f \leq |f|$ and $-f \leq |f|$, we have $D(f) \leq \|f\|$ and $-D(f) \leq \|f\|$ by theorem 3.12. Thus $|D(f)| \leq \|f\|$. Lastly, we have $\|f\| = \sum_{n=1}^{\infty} D(g_n) = \lim_{n\to\infty} \|S_n\| = \|f_1 + f_2 + \cdots + f_n\|$. □

Theorem 3.14. $L_B{}^*$ is closed under Banach lattice operations.

Proof . For,$g \in L_B^*$.

$$f \vee g = \frac{1}{2}(f + g + |f - g|), f \wedge g = \frac{1}{2}(f + g - |f - g|).$$

These two identities, the fact that $L_B{}^*$ is a vector space (theorem 3.12) and (theorem 3.13) gives our proof. □

Theorem 3.15. If $f \in L_B{}^*$ then for every $\varepsilon > 0$ there exists a sequence of functions $f_1, f_2, \ldots \in L_B$ such that $f = \sum_{n=1}^{\infty} f_n$ and $\sum_{n=1}^{\infty} \|f_n\| \leq D(f) + \varepsilon$.

Proof . Let $\varepsilon > 0$ be given and let $f = \sum_{n=1}^{\infty} g_n$. Choose $n_1 \in N$ such that $\sum_{n_1+1}^{\infty} \|g_n\| < \frac{\varepsilon}{2}$. By theorem 2.13 we have $\|f\| = \|g_1 + g_2 + \cdots + g_n\|$, so there exists an $n_2 \in N$ such that $\|g_1 + g_2 + \cdots + g_n\| < \|f\| + \frac{\varepsilon}{2}$, for every $n \geq n_2$. Let $n_\circ = \max(n_1, n_2)$ and define $f_1 = g_1 + g_2 + \cdots + g_{n_0}, f_n = g_{n_\circ + n - 1}$, for $n \geq 2$. Then $f = \sum_{n=1}^{\infty} f_n$ and $\sum_{n=1}^{\infty} \|f_n\| = \|g_1 + g_2 + \cdots + g_{n_\circ}\| + \sum_{n=1}^{\infty} \|g_n\| \leq \|f\| + \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$, which complete the proof. □

Theorem 3.16. If $f = \sum_{n=1}^{\infty} f_n$ where $f_n \in L_B{}^*$, then $f \in L_B{}^*$ and $D(f) = \sum_{n=1}^{\infty} D(f_n)$.

Proof . Let $f = \sum_{n=1}^{\infty} f_n$ with $f_n \in L_B{}^*$. Choose $g_{in} \in L_B$, where $i, n \in N$, such that $f_i = \sum_{n=1}^{\infty} h_n$ and $\sum_{n=1}^{\infty} \|g_{in}\| \leq \|f_i\| + 2^{-i}$ for $i = 1, 2, \ldots$.

Let $\{h_n\}$ be a sequence arranged from all the functions g_{in}. Then clearly $f = \sum_{n=1}^{\infty} h_n$ which implies $f \in L_B{}^*$ and $D(f) = \sum_{n=1}^{\infty} D(f_n)$. □

Theorem 3.17. For every non-increasing sequence of functions $f_n \in L_B{}^*$ such that $f_n \to 0$ in norm then $D(f_n) \to 0$.

Proof . The observation of $0 = f_1 + (f_2 - f_1) + (f_3 + f_2) + \cdots$ combine with Theorem 3.16 gives our proof. □

Theorem 3.18. Let $f_1, f_2, \ldots \in L_B{}^*$, if $\sum_{n=1}^{\infty} \|f_n\| < \infty$, then there exists $f \in L_B{}^*$ such that $f = f_1 + f_2 + \cdots$

Proof . The function f can be defined as $f(x) = \{\sum_{n=1}^{\infty} f_n(x), \sum_{n=1}^{\infty} \|f_n\| < \infty \quad 0$ otherwise □

Theorem 3.19. Every Daniell space (Ω, L_B, D) can be extended to a complete Daniell space $(\Omega, L_B{}^*, D)$.

Proof . To show $(\Omega, L_B{}^*, D)$ is a Daniell space, we need to satisfy conditions (1) and (2) from Definition 2.20. Both of these conditions are satisfied as a result of Theorem 3.10 and theorems 3.12, 3.13 and 3.17. A direct result of Theorem 3.16 shows that $(\Omega, L_B{}^*, D)$ is complete. □

Definition 3.20. (1) Let $f \in L_B{}^*$ is lower Daniell integral of f if satisfy $\underline{D}(f) = \sup\{\|g\| : g \in L_B{}^{**}, |g| \leq |f|\}$

(2) Let $f \in L_B^*$ is upper Daniell integral of f if satisfy $\underline{D}(f) = \inf\inf\{\|h\| : h \in L_B{}^*, |h| \geq |f|\}$.

References

[1] H. Amann and J. Escher, *Analysis III*, Birkhauser Basel, 2009.

[2] R.B. Ash, *Real analysis and probability*, University of Illinois, Academics Press, 1972.

[3] R.B. Ash, *Real analysis and probability*, Academics Press, New York, 1992.

[4] J. Banasiak, *Banach Lattices in Applications*, University of Pretoria, Pretoria, South Africa, Available at https://www.up.ac.za/media/shared/259/Documents/Teaching%20material/ablbook.zp158048.pdf

[5] E. Blackstone and P. Mikusinski, *The Daniell integral*, arXiv:1401.0310v1[math.CA].

[6] G.A.M. Jeurnink, *Integration of functions with Values in a Banach Lattice*, Catholic university, Nijmegen, 1982.

[7] J.L. Kelley, *General Topology*, Van Nostrand Company Inc., 1955.

[8] S.S. Kutateladze, *Nonstandard Analysis and Vector Lattices*, Springer Science+Business Media Dordrecht, 2000.

[9] W.A.J. Luxemburg and A.C. Zaanen, *Riesz Spaces*, North-Holland Publishing Company, 1971.

[10] F.A. Noori, *Measure Theory and Applications*, Deposit number in the National Library and Archives in Baghdad, 1780. First edition, Iraq, 2018.

[11] A.A. Pedgaonkar, *Daniell Integration for Banach Space Valued Vector Maps*, Int. J. Latest Res. Sci. Technol. **4** (2015), no. 6, 21–23.

[12] R.E. Shermoen, *An Introduction to General Integrals*, M.S.C. thesis, the Faculty of the Oklahoma State University, 1965.

[13] D.H.Tucker and H.B. Maynard, *Vector and Operator Valued Measures and Applications*, University of Utah, 1973.

[14] E.M. Wadsworth, *Daniell integral*, Msc. thesis, University of Montana, 1965.

[15] M.R. Weber, *Finite Elements in Vector Lattices*, Walter de Gruyter GmbH, Berlin/Boston, 2014.

[16] A.C. Zaanem, *Introduction to Operator Theory in Riesz Spaces*, springer-Verlag Berlin Heidelberg, 1997.