Int. J. Nonlinear Anal. Appl. 15 (2024) 1, 313–320
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.23018.2456

LG-paracompactness of LG-fuzzy topological metric spaces

Marzieh Mostafavi

Faculty of Science, Department of Mathematics, University of Qom, Qom, Iran

(Communicated by Sirous Moradi)

Abstract

In this manuscript, we introduce LG^c-fuzzy Euclidean topological space in which L denotes a completely distributive lattice with a countable subset dense in it. We use the structure of LG-fuzzy topological space $(X,\ \mathfrak{T})$, which X is an L-fuzzy subset of the crisp set M and $\mathfrak{T} : L_X^M \to L$, is an L-gradation of openness on X to define the fundamental concepts of LG-fuzzy analysis such as LG-locally compactness and LG-paracompactness and prove several theorems. In consequence, we show that any second countable Hausdorff LG-fuzzy topological space that is LG-locally compact is LG-paracompact. Also from any given metric ρ on a crisp set M and L-fuzzy subset X of it, we construct an L-gradation of openness $\mathfrak{T}_\rho$ on X and obtain LG-fuzzy topological metric space $(X, \mathfrak{T}_\rho)$. Finally, we prove an interesting theorem: Every LG-fuzzy topological metric space, is LG-paracompact.

Keywords: LG^c-fuzzy Euclidean topological space, LG-locally compact, LG-fuzzy topological metric space, LG-paracompact
2020 MSC: 54A40, 06D72, 08A45

1 Introduction

The concept of fuzzy topological spaces was introduced by Chang [2] in 1968 and later was redefined in a somewhat different way by Shostak [31]. Chattopadhyay et. al. [3] introduced a concept of gradation of openness of fuzzy subsets of X in 1992 and Gregori and Vidal [10], defined fuzziness in Chang's fuzzy topological spaces. To develop this kind of fuzzy topology, we assumed in [28] that X is an L-fuzzy subset of the crisp set M, in Goguen's sense [9] where $L =< L, \leq, \bigwedge, \bigvee, ' >$ is a complete distributive lattice set with at least 2 elements and introduced an LG-fuzzy topological space $(X,\ \mathfrak{T})$, which $\mathfrak{T} : L_X^M \to L$, is an L-gradation of openness on X along with C^∞ L-fuzzy manifolds with L-gradation of openness which are defined and studied.

In the theory of fuzzy topological spaces, one of the main problems is to obtain an appropriate notion of a fuzzy metric space. Many authors have made significant contributions to the development of fuzzy metric space theory [5, 7, 8, 11, 12, 14, 15, 17, 24, 25, 33]. They have introduced different fuzzy metrics, which have applications in Economics, Geology, Artificial Intelligence and Computer Science. At present, the process of digital signals and images, and particularly colour image processing, is a problem widely studied. The techniques using fuzzy logic have been studied to solve the problem reducing impulse noise in colour and multichannel images and improve experimentally sharpness and the quality of the image, because fuzzy logic and fuzzy metrics can deal with the nonlinear nature of digital images and with the essential uncertainty in distinguishing between noise and image structures. (See [12, 6, 16, 30]). The concepts of compact Housdorff fuzzy topological spaces by Lowen [23] and L-fuzzy local compactness by Kudri

Email address: mmostafavi14279@gmail.com (Marzieh Mostafavi)

Received: March 2021 *Accepted:* August 2021

and Warner [18] were introduced. Since paracompactness describes the relation between a locally finite property and an entire property of spaces, this concept occupies an important position in general topology. As two references of paracompactness topological spaces, we refer the reader to [32] and [27]. In 1988 Luo [22] initiated the concept of paracompactness in fuzzy topological spaces and eleven years later Lupianez [20] studied the notion of fuzzy perfect map and fuzzy paracompactness. Tirado [26] in 2012, studied compactness and G-compactness in fuzzy metric spaces. In 2019 Lupianez defined and discussed three paracompactness-type properties of fuzzy topological spaces [21]. Recently Wali [34] investigated the compactness of Hausdorff fuzzy metric spaces. Our approach in this manuscripts is different from what they have constructed here, since we answer two questions: What will these structures look like if we assume that the fuzzy topological space X is itself an L-fuzzy subset of a crisp set and also if we consider L-gradation of openness of L-fuzzy subsets of X instead of the collection of fuzzy subsets of X as a topology on it?

In this article, we assume that the Lattice set L has a countable subset J, dence in L, and define the LG^c-fuzzy Euclidean topological space with countable basis. This in turn is used to construct LG-fuzzy topological space proved LG-compactness and LG-paracompactness.

LG-paracompactness of LG-fuzzy topological metric spaces appear naturally in many areas of mathematics which we need the existence of suitable LG-partitions of unity. To formulate the definition of LG-paracompactness, following Bourbaki, [1] and Engelking [4] we include the Hausdorff L-gfts assumption. Significant authors such as Munkres [29] do not include any separation assumptions. We prove in this paper that any second countable Hausdorff LG-fuzzy topological space that is LG-locally compact, is LG-paracompact. We recall the definition of concept of an LG-fuzzy topological space $(X, \mathfrak{T})$ of dimension n as we introduced in our previous article [28] and bring out the equivalence of LG-paracompactness of it with two properties of X: its connected LG-components are countable unions of LG-compact sets, its connected LG-components are second countable.

In the last section, we introduce the L-gradation of openness induced by the metric ρ on a crisp set and present the definition of an LG-fuzzy topological metric space $(X, \mathfrak{T}_{L\rho})$ and prove an important theorem: Every LG-fuzzy topological metric space, is LG-paracompact.

2 Preliminary

Let M be an nonempty set and X be an L-fuzzy subset of M. We denote by L_X^M the set of all L-fuzzy subsets of M, which are less than or equal to X (called L-fuzzy subsets of X).

Definition 2.1. If $\mathfrak{T} : L_X^M \to L$, be a mapping satisfying:

i) $\mathfrak{T}(X) = \mathfrak{T}(\tilde{0}) = 1$.

ii) $\mathfrak{T}(A \cap B) \geq \mathfrak{T}(A) \wedge \mathfrak{T}(B)$.

iii) $\mathfrak{T}(\bigcup_{j\in J} A_j) \geq \bigwedge_{j\in J} \mathfrak{T}(A_j)$

Then $\mathfrak{T}$ is called a L-gradation of openness on X and $(X,\ \mathfrak{T})$ is called an LG-fuzzy topological space (L-gfts).

Set supp $\mathfrak{T} = \{A \in L_X^M :\ \mathfrak{T}(A) > 0\}$, then A is called an LG-fuzzy open subset of X if $A \in$ supp $\mathfrak{T}$.

Definition 2.2. Let $(X, \mathfrak{T})$ be an LG-fuzzy topological space, $p \in X$ and A be an L-fuzzy subset of X,

i) An L-fuzzy subset V of X is called an LG-neighborhood of p, if there exists an LG-fuzzy open subset U of X such that $p \in U \leq N$.

ii) The union of all L-fuzzy subsets of X less or equal to A is called LG-interior set of A, denoted by LGA° and the intersection of all LG-closed subsets grater or equal to A is called an LG-closure of A, denoted by $LG\bar{A}$.

Definition 2.3. Let $B(a,\ r,\ b)$ be an L-fuzzy subset of $1_{\mathbb{R}^n}$, that is equal to zero outside or on the sphere $B_r(a)$ for $a \in \mathbb{R}^n$, $r \in \mathbb{R}^+$ and equal to the function b with values in L, inside $B_r(a)$. Let $\mathfrak{T}_{Ln}$ be any L-gradation of openness on $1_{\mathbb{R}^n}$, such that $supp\mathfrak{T} = \tau_{_{Ln}}$, where $\tau_{_{Ln}}$ is the L-fuzzy topology induced by

$$\beta_{Ln} = \{B(a, r, b),\ a \in \mathbb{R}^n,\ r \in \mathbb{R}^+,\ b : B_r(a) \to L\ is\ a\ function\}.$$

Then we call $(1_{\mathbb{R}^n},\ \mathfrak{T}_{Ln})$ the LG-fuzzy Euclidean topological space.

Example 2.4. As two usefull examples of L-gradations of openness on $1_{\mathbb{R}^n}$, we define

$$\mathfrak{T}_{Ln} \; : \; I_X^M \to L \qquad \mathfrak{T}_{Ln}(B) = \begin{cases} 1 & B \in \tau_{Ln}, \\ 0 & \text{elsewhere.} \end{cases} \tag{2.1}$$

and

$$\mathfrak{T}_{Linf} \; : \; L_X^M \to L, \qquad \mathfrak{T}_{Linf}(B) = \begin{cases} 1 & ZB = \tilde{0} \\ \inf\{B(x) : x \in M\} & \tilde{0} \neq B \in \tau_{Ln} \\ 0 & \text{elsewhere,} \end{cases} \tag{2.2}$$

Definition 2.5. Let $(X, \mathfrak{T})$ be an LG-fuzzy topological space and A be an L-fuzzy subset of X. Then

i) A is called an LG-compact if every LG-fuzzy open cover of A has a finite LG-fuzzy open subcover.

ii) A is called LG-locally compact if each $p \in A$ admits an LG-compact LG-neighborhood V such that $V \leq A$. It means that for each $p \in A$, there exists an LG-fuzzy open set U and an LG-compact set K with $p \in U \leq K$.

iii) A Hausdorff L-gfts X is said to be LG-paracompact if any LG-fuzzy open cover of it has a locally finite LG-fuzzy open refinement.

iv) X is LG-normal if for any two LG-fuzzy closed disjoint subsets $A, B \subseteq X$, there exist two disjoint LG-fuzzy open subsets of X containing A and B respectvely.

3 *LG*-paracompactness of second countable Hausdorff *LG*-fuzzy topological spaces

From now on we assume that there exists a countable subset J dence in the Lattice set L, hence $L = \bar{J}$.

Definition 3.1. We denote by β^c_{Ln} the set of all constant L-fuzzy subsets $B(a, r, b)$ defined in Example 2.2. Since for each real number, there exists an increasing sequences of rational numbers limited to it, hence the L-fuzzy topology τ^c_{Ln}, induced by β^c_{Ln} has a countable basis.

$$\{ \; B(a, r, b), \; a \in \mathbb{Q}^n, \; r \in \mathbb{Q}^+, \; b : B_r(a) \to \; J \;\; is \; a \; constant \; function \; \}$$

We call $(1_{\mathbb{R}^n}, \; \mathfrak{T}^c_{Ln})$, the LG^c-fuzzy Euclidean topological space.

Proposition 3.2. Each LG^c-fuzzy open covering $\{A_i\}$ of the LG^c-fuzzy Euclidean topological space can be refined to an LG^c-fuzzy open covering that is locally finite.

Proof . For each $x \in \mathbb{R}^n$ we can consider an LG^c-fuzzy open subset $B(x, r_x, b_x)$ contained in some $A_{i(x)}$ with $r_x \leq 1$ in this manner. Since $A_{i(x)} \in \tau^c_{In}$, then $A_{i(x)} = \bigcup_{j \in J} B(a_j, r_j, b_j)$. Hence there exists at least one $j_1 \in J$ such that $x \in B(a_{j_1}, r_{j_1}, b_{j_1})$. Setting $r_x = \min\{1, (r_{j_1} - \|x - a_{j_1}\|)\}$ and $b_x = b_{j_1}$, we have $r_x \leq 1$ and $B(x, r_x, b_x) \leq B(a_{j_1}, r_{j_1}, b_{j_1})$. If we have $x \in \bigcap_{k=1}^{s} B(a_{j_k}, r_{j_k}, b_{j_k})$, then $A_{i(x)}(x) = \sup\{ \; b_{j_k} \mid 1 \leq k \leq s\}$. Thus $B(x, r_x, b_x) \leq A_{i(x)}$.

For each integer $N > 0$ finitely many of LG^c-fuzzy open subsets $B(x, r_x, b_0)$ cover the LG-fuzzy compact set $\overline{B}(0, N, b_0) - B(0, N-1, b_0)$, say $B(x_1, r_{x_1}, b_0), \; \ldots, \; B(x_m, r_{x_m}, b_0)$. Hence we may write $\{V_{j,N}\}$ to denote these finitely many LG^c-fuzzy open subsets. As we rechange j and N, the $V_{j,N}$'s assuredly cover the whole $(1_{\mathbb{R}^n}, \; \mathfrak{T}^c_{In})$ (even the origin), and this covering refines $\{A_i\}$ in the sense that every $V_{j,N}$ lies in some A_i and the collection $V_{j,N}$ is locally finite in the sense that any point $x \in \mathbb{R}^n$ has an LG^c-neighborhood meeting only finitely many $V_{j,N}$'s. Indeed, since $V_{j,N}$ is an LG^c-fuzzy open subset of radius at most 1 and it intersects $\overline{B}(0, N, b_0) - B(0, N-1, b_0)$, by elementary investigation with the triangle inequality we see that a bounded region of $\mathbb{R}^n$ encounter only finitely many $V_{j,N}$'s. Thus, we have refined $\{A_i\}$ to an LG^c-fuzzy open covering that is locally finite. □

Corollary 3.3. The IG^c-fuzzy Euclidean topological space $(1_{\mathbb{R}^n}, \; \mathfrak{T}^c_{In})$ is LG^c-paracompact.

Example 3.4. Consider the IG^c-fuzzy Euclidean topological space $(1_{\mathbb{R}}, \; \mathfrak{T}^c_{I1})$, that $I = [0, 1]$. We define for each $q \in \mathbb{Q} \cap (0, 1)$ and any $n \in \mathbb{Z}$, the IG^c-fuzzy subset $A_{q,n}$ by

$$A_{q,n}(x) = \begin{cases} q & \text{if } \; |x - n| < q \\ \\ 0 & \text{elsewhere} \end{cases}$$

Since for each $x \in \mathbb{R}$, we have $n \leq x < n+1$ for some $n \in \mathbb{Z}$. Thus we have $(x-n) < \frac{3}{4}$ or $(n+1-x) < \frac{3}{4}$. These imply that $x \in A_{\frac{3}{4},n}$. Therefore $\{A_{q,n}\}$ is an IG^c-fuzzy open covering of $1_{\mathbb{R}}$. Therefore we have refined $\{A_{q,n}\}$ to an IG^c-fuzzy open covering $\{A_{\frac{3}{4},n}\}$ that is locally finite.

Proposition 3.5. If X is an LG-locally compact and Hausdorff L-gfts, then each LG-compact L-fuzzy subset K of X is LG-fuzzy closed.

Proof . We will show that $X-K$ is an LG-fuzzy open. Let $q \in X-K$. For each $p \in X$ there are LG-fuzzy open subsets U_p, V_p of X such that $p \in U_p$, $q \in V_p$ and $U_p \cap V_p = \phi$. Then we have $K = \bigcup_{p\in K} U_p$. Since K is LG-compact, there exist $p_1, \ldots, p_n$ elements of K such that $K \subseteq U_{p_1} \cup \ldots \cup U_{p_n}$. Set $W_q = V_{p_1} \cup \ldots V_{p_n}$. Then W_q is an LG-fuzzy open subset of X containing q. Suppose $x \in W_q \cap K$, then $x \in U_{p_i}$ for some i. Since $x \in W_q \subseteq V_{p_i}$, hence $x \in U_{p_i} \cap V_{p_i} = \phi$, a contradiction. Thus $W_q \cap K = \phi$. Therefore $W_q \subseteq (X-K)$. □

Proposition 3.6. An LG-fuzzy closed subset of an LG-paracompact L-gfts $(X, \mathfrak{T})$ is itself LG-paracompact.

Proof . Let $\mathcal{U}$ be an LG-fuzzy open cover of an LG-fuzzy closed subset C of X. Then $\mathcal{U}' = \mathcal{U} \cup \{X-C\}$ is an LG-fuzzy open covering of X. Hence $\mathcal{U}'$ has a locally finite LG-fuzzy open refinement, which also refines $\mathcal{U}$. □

Lemma 3.7. If $(X, \mathfrak{T})$ is an locally LG-compact Hausdorff space that is second countable, then it admits a countable base of LG-fuzzy open subsets $\{V_n\}$ with LG-compact LG-closures.

Proof . Since X is an LG-locally compact, each $p \in X$ admits an LG-compact LG-neighborhood N_p. Hence by Proposition 3.5, N_p is LG-fuzzy closed and so N_p contains the LG-closure of LGN_p° around p. Hence, in such cases every point $p \in X$ lies in an LG-fuzzy open subset U_p whose closure is LG-compact. Let $\{V_n\}$ be a countable base of LG-fuzzy open subsets of X. Then some $V_{n(p)}$ contains p and is contained in U_p. The LG-closure of $V_{n(p)}$ is an LG-closed subset of the LG-compact set $LG\overline{U}_p$, and so $V_{n(p)}$ is also LG-compact. Thus, the $\{V_n\}$'s with LG-compact closure are a countable base of LG-fuzzy open subsets. □

Theorem 3.8. Any second countable Hausdorff LG-fuzzy topological space $(X, \mathfrak{T})$ that is locally LG-compact is LG-paracompact.

Proof . Let V_n be a countable base of LG-fuzzy open subsets of X. Let $\{U_i\}$ be an LG-fuzzy open cover of X for which we search a locally finite refinement. Each $p \in X$ lies in some U_i and so there exists a $V_n(p)$ containing p with $V_n(p) \subseteq U_i$. The $V_n(p)$'s therefore organize a refinement of U_i that is countable. Since the exclusivity of one LG-fuzzy open covering refining another is transitive, we therefore lose no generality by finding locally finite refinements of countable LG-fuzzy covers. Assume that all $LG\overline{V}_n$ are LG-compact. Hence, we can curb our attention to countable covers by LG-fuzzy opens U_n for which $LG\overline{U}_n$ is LG-compact. Since closure commutes with finite unions, by replacing U_n with $\bigcup_{j<n} U_j$ we retain the LG-compactness condition (as a finite union of LG-compact subsets is LG-compact) and so we can suppose that U_n is an increasing collection of LG-opens with LG-compact closure (with $n \geq 0$). Since $LG\overline{U}_n$ is LG-compact yet is covered by the open U_i's, for sufficiently large N we have $LG\overline{U}_n \subseteq U_N$. If we recursively replace U_{n+1} with such a U_N for each n, then we can arrange that $LG\overline{U}_n \subseteq U_{n+1}$ for each n. Let $K_0 = LG\overline{U}_0$ and for $n \geq 1$ let $K_n = LG\overline{U}_n - U_{n-1} = LG\overline{U}_n \cap (X - U_{n-1})$, so K_n is LG-compact for every n (as it is LG-fuzzy closed subset in the LG-compact $\overline{U}_n$ but for any fixed N we see that U_N is disjoint from K_n for all $n > N$. Now we have a situation similar to the concentric shells in our earlier proof of paracompactness of $\mathbb{R}^n$, and so we can carry over the argument from LG^c-fuzzy Euclidean spaces as follows. We search a locally finite refinement of $\{U_n\}$. For $n \geq 2$ the LG-fuzzy open set $W_n = U_{n+1} - LG\overline{U}_n$ contains K_n, so for each $p \in K_n$ there exists some $V_m \subseteq W_n$ around p. There are finitely many such V_m's that cover the LG-compact K_n, and the collection of V_m's that arise in this way as we vary $n \geq 2$ is a locally finite collection of LG-fuzzy open subsets in X whose union contains $X - U_0$. Throwing in finitely many V_m's contained in U_1 that cover the LG-compact U_0 thereby gives an open cover of X that refines $\{U_i\}$ and is locally finite. □

Lemma 3.9. Let X be an LG-fuzzy topological space and $\mathcal{V} = \{V_k\}_{k\in K}$ be a locally finite covering of X. Then

$$LG\overline{\bigcup_{k\in K} V_k} = \bigcup_{k\in K} LG\overline{V_k}.$$

Proof . Since each $LG\overline{V_k} \subseteq LG\overline{\bigcup_{k\in K} V_k}$, hence $\bigcup_{k\in K} LG\overline{V_k} \subseteq LG\overline{\bigcup_{k\in K} V_k}$. To show the reverse inclusion let $p \in LG\overline{\bigcup_{k\in K} V_k}$ and choose a neighbourhood V of p meeting only finitely many of the V_k. Then given any LG-neighbourhood U of x, the set $U \cap V$ meets only finitely many of the V_k nontrivially, $V_1, \ldots, V_n$. This implies that U meets $\bigcup_{k\in K} V_k$ and since U was arbitrary we have $p \in LG\overline{\bigcup_{k=1}^{n} V_k}$. Therefore

$$p \in LG\overline{\bigcup_{k=1}^{n} V_k} = \bigcup_{k=1}^{n} LG\overline{V_k} \subseteq \bigcup_{k\in K} LG\overline{V_k}.$$

□

Proposition 3.10. An LG-paracompact L-gfts $(X, \mathfrak{T})$ is LG-normal.

Proof . We first show that X is LG-regular. Let F be an LG-fuzzy closed subset of X and $x \in X - F$. Using the Hausdorff assumption on X, for any point $y \in F$, there exist disjoint LG-fuzzy open neighbourhoods U_y of x and V_y of y. Since by Proposition 3.6 we have F is an LG-paracompact, hence LG-fuzzy open covering $\{V_y\}_{y\in F}$ of F can be refined to a locally finite family $\mathcal{V}$ covering F. Let W be the union of all the sets in $\mathcal{V}$. Then $F \subseteq W$ and $x \notin W$. Moreover, according to Lemma 3.9, the $LG\overline{W}$ is the union of the LG-closures of the sets in $\mathcal{V}$. This implies that $x \notin LG\overline{W}$, since for each element of $\mathcal{V}$ we can find a disjoint LG-neighbourhood of x. Thus W and $X - LG\overline{W}$ are the required separating LG-neighbourhoods of F and x respectively and X is LG-regular.

Let $B \subseteq X$ is a second LG-closed set disjoint from F. Then by LG-regularity of X, for each $y \in F$, we have LG-fuzzy open set W_y whose LG-closure is disjoint from B. Thus $W = \bigcup_{y\in F} W_y$ and $X - W$ are two disjoint LG-fuzzy open sets containing F and B respectively. Therefore X is LG-normal. □

In our discussion, we now define LG-fuzzy topological space of dimension n and indicate how Theorem 3.7 leads us to the a conclusion that every LG-fuzzy topological space of dimention n with two equivalent condition, can be LG-paracompact.

Definition 3.11. Let $X \in L^{M_1}$, $Y \in L^{M_2}$ such that $(X, \mathfrak{T})$, $(Y, \mathfrak{R})$ are LG-fuzzy topological spaces. suppose $f : M_1 \to M_2$ be a function. If we have $f[X] \leq Y$, then f is called an LG-related function from X to Y and the set of all these functions is denoted by $LGRf(X, Y)$. Further more if we have $\mathfrak{R}(H) \leq \mathfrak{T}(f^{-1}[H])$, for all LG-fuzzy open subset H of Y, then f is an L-gradation preserving LG-related function so it is called an LGP-related function from X to Y or briefly $f \in LGPRf(X, Y)$.

Definition 3.12. An LG-fuzzy topological space $(X, \mathfrak{T})$ is called an LG-fuzzy topological space of dimention n, if for any $x \in X$, there exists an LG-fuzzy open subset A of X such that $x \in A$ and $B \in \mathfrak{T}^c_{Ln}$ along with an LGP-homeomorphism $\psi \in LGPRf(A, B)$.

Corollary 3.13. Let $(X, \mathfrak{T})$ be an LG^c-fuzzy topological space of dimention n. The following properties of X are equivalent: its connected LG-components are countable unions of LG-compact sets, its connected LG-components are second countable, and it is LG-paracompact.

Proof . If $\{U, V\}$ is a separation of X and X is LG-paracompact then it is clear that both U and V are LG-paracompact. Hence, since the connected LG-components of X are LG-fuzzy open, X is LG-paracompact if and only if its connected LG-components are LG-paracompact. We may therefore restrict our attention to connected X. For such X, we claim that it is equivalent to require that X be a countable union of LG- compact sets, that X be second countable, and that X be LG-paracompact. By the preceding theorem, if X is second countable then it is LG-paracompact. Since X is connected, Hausdorff, and locally LG- compact, if it is LG-paracompact then it is a countable union of LG-compacts. Hence, to complete the cycle of implications it remains to check that if X is a countable union of LG-compacts then it is second countable. Let $\{K_n\}$ be a countable collection of LG-compacts that cover X, so if $\{U_i\}$ is a covering of X by LG-fuzzy open sets $LGPRf$-homeomorphic to an LG^c-fuzzy open set in the LG^c-fuzzy Euclidean space we may find finitely many U_i's that cover each K_n. As there are only countably many K_n's, in this way we find countably many U_i's that cover X. Since each U_i is certainly second countable, a countable base of LG-fuzzy opens for X is given by the union of countable bases of LG-fuzzy open subsets for each of the U_i's. Hence, X is second countable. □

4 *LG*-paracompactness of *LG*-fuzzy topological metric spaces

Definition 4.1. Let ρ be a metric on the nonempty set M and X be an L-fuzzy subset of M. Let $S(p,r)$ be the sphere with center p and radius r. Then the L-fuzzy topology τ_{L_ρ} induced by

$$\beta_{L\rho} = \{S(p,r,s),\, p \in X,\, r \in \mathbb{R}^+,\, s : S(p,r) \to L \text{ is a constant function less than or equal to } X\}.$$

is called L-fuzzy topology induced by the metric ρ. Also we call any L-gradation of openness on X, with support equal to τ_{L_ρ}, the L-gradation of openness induced by the metric ρ and denote it by $\mathfrak{T}_{L\rho}$. Also $(X, \mathfrak{T}_{L\rho})$ is called an LG-fuzzy topological metric space.

Example 4.2. Let $M = \mathbb{R}$ and $\rho(x,y) = |x-y|$ be the ordinary metring on it. Let X be an I-fuzzy subset of M defined by $X(x) = \dfrac{1}{|\lfloor x \rfloor| + 2}$ where $|\lfloor x \rfloor|$ denotes the absolute value of the greatest integer less than or equal to x. For each $x \in S(k,1)$, we have two cases: if $x \in (k-1,k)$, then $x \in \big(S(k-1,1) \cap S(k,1)\big)$, and so

$$X(x) = \frac{1}{k+1} = S\big(k-1, 1, \frac{1}{k+1}\big)(x) \bigvee S\big(k, 1, \frac{1}{k+2}\big)(x)$$

$$\text{if } x \in [k,\ k+1) \text{ then } \ x \in \big(S(k,1) \cap S(k+1,1)\big), \text{ and so}$$

$$X(x) = \frac{1}{k+2} = S(k, 1, \frac{1}{k+2})(x) \bigvee S\big(k+1, 1, \frac{1}{k+3}\big)(x).$$

Hence $X = \bigcup_{k \in \mathbb{Z}} S\big(k, 1, \frac{1}{k+2}\big)$. Therefore $(X, \tau_{L\rho})$ has an countable $L\rho$-fuzzy open covering.

Proposition 4.3. Let $(X, \mathfrak{T}_{L\rho})$ be an LG-fuzzy topological metric space and Z be an L-fuzzy subset of X. Define

$$\tau_{L\rho|_Z} = \{\ V \mid V = U \cap Z, \text{for some } U \in \tau_{L_\rho}\}$$

$$\mathfrak{T}_{L\rho_{|_Z}} : L_Z^M \to L, \quad \mathfrak{T}_{L\rho_{|_Z}}(W) = \mathfrak{T}_{L\rho}(W).$$

Then $\mathfrak{T}_{L\rho_{|_Z}}$ is an LG-topology on Z with support equal to $\tau_{L\rho|_Z}$.

Lemma 4.4. Let Z be an LG-topological subspace of LG-fuzzy topological metric space X. Then Z is LG-compact if and only if for every collection $\{U_i | i \in I\}$ of LG-fuzzy open sets of X such that $Z \subseteq \bigcup_{i \in I} U_i$ there is a finite subset J of I such that $Z \subseteq \bigcup_{i \in J} U_i$

Proposition 4.5. Let Z be an L-fuzzy subset of an LG-fuzzy topological metric space X. If Z is LG-compact (in the subspace LG-topology $\mathfrak{T}_{L\rho_{|_Z}}$) then Z is LG-bounded.

Proof . Let $x_0 \in Z$. We show that $X = \bigcup_{n=1}^{\infty} S\big(x_0, n, X_{|_{S(x_0,n)}}\big)$. Clearly each $S\big(x_0, n, X_{|_{S(x_0,n)}}\big) \leq X$. Let $x \in X$ be any point, pick a positive integer $n > \rho(x, x_0)$. Then we have $x \in S(x_0, n)$. Hence $X \leq S\big(x_0, n, X_{|_{S(x_0,n)}}\big)$. Now suppose that Z is LG-compact. Then $Z \subseteq \bigcup_{n=1}^{\infty} S\big(x_0, n, X_{|_{S(x_0,n)}}\big)$ and by Lemma 4.4, there exist finitely many LG-fuzzy open subsets

$$Z \subseteq S\big(x_0, n_1, X_{|_{S(x_0,n_1)}}\big) \bigcup \ldots \bigcup S\big(x_0, n_k, X_{|_{S(x_0,n_k)}}\big).$$

Let $m = \max\{n_1, n_2, ..., n_k\}$. Then we have $Z \subseteq S\big(x_0, m, X_{|_{S(x_0,m)}}\big)$. Now for $z_1, z_2 \in Z$ we have

$$\rho(z_1, z_2) \leq \rho(z_1, x_0) + \rho(z_0, z_2) \leq m + m = 2m.$$

Hence Z is LG-bounded. □

Theorem 4.6. Let $(X, \mathfrak{T}_{L\rho})$ be an LG-fuzzy topological metric space. Then X is LG-paracompact.

Proof . Assume that $\{U_\alpha\}$ is an LG-fuzzy open cover of X indexed by ordinals. For each positive integer n, define $A_{\alpha n}$ to be the union of all LG-fuzzy subsets $S(p, 2^{-n}, s_n)$ such that:

(i) α is the smallest ordinal with $p \in U_\alpha$,

(ii) $p \notin A_{\beta j}$ if $j < n$,

(iii) $S\big(p,\ 3.2^{-n},\ s_n\big) \subseteq U_\alpha$.

We show that $\{A_{\alpha n}\}$ is a locally finite refinement of $\{U_\alpha\}$ which covers X and therefore X is LG-paracompact. Let $p \in X$. There is a smallest ordinal such that $p \in U_\alpha$ and an n so large that (iii) holds, hence , by (ii), $p \in A_{\beta j}$ for some $j \geq n$. Hence $\{A_{\alpha n}\}$ cover X. For each $p \in X$ assume that α be smallest ordinal such that $p \in A_{\alpha n}$ and choose j so that $S(p, 2^{-j}, s_j) \subseteq A_{\alpha n}$. We prove that $\{A_{\alpha n}\}$ is locally finite by showing that:

(1) if $i \geq n + j$, $S(p, 2^{-n-j}, s_{n+j})$ intersects no $A_{\beta i}$,

(2) if $i < n + j$, $S(p, 2^{-n-j}, s_{n+j})$ intersects $A_{\beta i}$, for at most one β.

Proof of (1). Since $i > n$, by (2), every one of the LG-fuzzy subsets $S(q, 2^{-i}, s_i)$ used in the definition of A_{β_i} has its center q outside of A_{α_n}. And since $S(p, 2^{-j}, s_j) \subseteq A_{\alpha_n}$, so $\rho(p, q) \geq 2^{-j}$. But $i \geq n + j \geq j + 1$, so $S(p, 2^{-n-j}, s_{n+j}) \cap S(q, 2^{-i}, s_i) = \phi$.

Proof of (2). Suppose that $x \in A_{\beta_i}$, $y \in A_{\gamma_i}$ and $\beta \leq \gamma$; we want to show that $\rho(x, y) > 2^{-n-i+1}$. There are points u and v such that $x \in S(u, 2^{-i}, s_i) \subseteq A_{\beta_i}$, $y \in S(v, 2^{-i}, t_i) \subseteq A_{\gamma_i}$; and by (3), $S(u, 3.2^{-i}, s_i) \subseteq U_\beta$ but, by (2), $v \notin U_\beta$. So $\rho(u, v) \geq 3.2^{-i}$ and $\rho(x, y) > 2^{-i} \geq 2^{-n-j+1}$. □

5 Conclusion

To gain a new understanding of the notion of fuzzy topological spaces, we have introduced in [28] an LG-fuzzy topological space $(X,\ \mathfrak{T})$, which $\mathfrak{T} : L_X^M \to L$, is an L-gradation of openness on X. The main motivation of this paper is to provide an intrinsic study about LG-paracompactness of LG-fuzzy topological spaces that is an extraordinarily most useful than LG-compactness. A key feature of LG-paracompactness is the existence of suitable LG-partitions of unity which plays a very important role in LG-fuzzification of Riemannian Geometry and Finsler Geometry. However we find the conditions under which such LG-fuzzy topological spaces are LG-paracompact. Further, we introduce the L-gradation of openness induced by any metric on a set and construct an LG-fuzzy topological metric space and we show that it is LG-paracompact. We also give some examples to clarify the notions and results. We recently wrote an article entitled LG-fuzzy partitions of unity and are submitting it.

Now for a development of knowledge frontiers, an interesting question is that under what conditions we can construct LG-fuzzy Minkowski or Finsler manifolds?

Acknowledgements

The authors would like to thank the anonymous referees for carefully reading the manuscript and making their valuable comments.

References

[1] N. Bourbaki, *General Topology I*, Addison-Wesley, 1966.

[2] C.L. Chang, *Fuzzy topological spaces*, J. Math.l Anal. Appl. **24** (1968), 182–190.

[3] K.C. Chattopadhyay, R.N. Hazra, and S.K. Samanta, *Gradation of openness: fuzzy topology*, Fuzzy Sets Syst. **49** (1992), no. 2, 234–242.

[4] R. Engelking, *General Topology*, Heldermann Verlag, 1989.

[5] A. George and P. Veeramani, *On some results of analysis for fuzzy metric spaces*, Fuzzy Sets Syst. **90** (1997), no. 3, 365–368.

[6] J.G. Camarena and S. Morillas, *On the importance of the choice of the (fuzzy) metric for a practical application*, Proc. Workshop Appl. Topology WiAT'10, 2011, pp. 67–73.

[7] V. George and S. Romaguera, *Some properties of fuzzy metric space*, Fuzzy Sets Syst. **115** (2000), no. 3, 485–489.

[8] V. George and S. Romaguera, *On completion of fuzzy metric Spaces*, Fuzzy Sets Syst. **130** (2002), no. 3, 399–404.

[9] J.A. Goguen, *L-Fuzzy sets*, J. Math. Anal. Appl. **18** (1967), 145–174.

[10] V. Gregori and A. Vidal, *Fuzziness in Chang's fuzzy topological spaces*, Rend. Istit. Mat. Univ. Trieste **30** (1999), 111–121.

[11] V. Gregori, *Some results in fuzzy metric spaces*, XVI Encuentro de Topología, 2009.

[12] V. Gregori, S. Morillas, and A. Sapena, *On a class of completable fuzzy metric spaces*, Fuzzy Set Syst. **161** (2010), no. 5, 2193–2205.

[13] V. Gregori, S. Morillas, and A. Sapena, *Examples of fuzzy metrics and applications*, Fuzzy Sets Syst. **170** (2011), no. 1, 95–111.

[14] V. Gregori, S. Romgueraa and M. Sanchis, *An identification theorem for the completion of the Hausdorff fuzzy metric spaces*, Ital. J. Pure Appl. Math. **37** (2017), 49–58.

[15] O. Kaleva and S. Seikkala, *On fuzzy metric spaces*, Fuzzy Sets Syst. **12** (1984), no. 3, 215–229.

[16] Ch.Ch. Kang and W.J. Wang, *Fuzzy reasoning-based directional median filter design*, Signal Process. **9** (2009), 344-–351.

[17] O. Kramosil and J. Michale, *Fuzzy metric and statistical metric spaces*, Kybernetica **11** (1975), no. 5, 336–334.

[18] S.R.T. Kudri and M.W. Warner, *L-fuzzy local compactness*, Fuzzy Sets Syst. **67** (1994), no. 3, 337—345.

[19] R. Lowen, *Compact Housdorff fuzzy topological spaces*, Fuzzy Sets Syst. **67** (1981), 337–345.

[20] F.G. Lupianez, *Fuzzy perfect maps and fuzzy paracompactness*, Fuzzy Sets Syset. **98** (1998), 137–140.

[21] F.G. Lupianez, *On some paracompactness-type properties of fuzzy topological spaces*, Soft Comput. **23** (2019), no. 20, 9881–9883.

[22] M.K. Luo, *Paracompactness in fuzzy topological spaces*, J. Math. Anal. Appl. **130** (1988), no. 1, 55–77.

[23] R. Lowen, *Compact Housdorff fuzzy topological spaces are topological*, Topology Appl. **12** (1981), 65–74.

[24] T. Onera, *Some topological properties of fuzzy strong b-metric spaces*, J. Linear Topological Alg. **8** (2019), no. 2, 127–131.

[25] D. Qiu, R. Dong, and H. Li, *On metric spaces induced by fuzzy metric spaces*, Iran. J. Fuzzy Syst. **13** (2016), no. 2, 145–160.

[26] P. Tirado, *On compactness and G-compactness in fuzzy metric spaces*, Iran. J. Fuzzy Sets Syst. **9** (2012), no. 4, 151–158.

[27] E. Michael, *A note on paracompact spaces*, Proc. Amer. Math. Soc. **4** (1953), 831–838.

[28] M. Mostafavi, *C^∞ L-Fuzzy manifolds with gradation of openness and C^∞ LG-fuzzy mappings of them*, Iran. J. Fuzzy Syst. **17** (2020), no. 6, 157–174.

[29] J. Munkres, Topology, *Second Edition*, Prentice Hall., 2000.

[30] S. Schulte, S. Morillas, V. Gregori, and E.E. Kerre, *A new fuzzy color correlated impulsive noise reduction method*, IEEE Trans. Image Process. **16** (2007), no. 10, 2565—2575.

[31] A.P. Shostak, *On a fuzzy topological structure*, Rendi. Ciec. Mat. Palermo Ser. II **11** (1985), 89–103.

[32] A.H. Stone, *Paracompactness and product spaces*, Bull. Amer. Math. Soc. **54** (1948), 977–982.

[33] G. Valentin and R. Salvador, *Some properties of fuzzy metric spaces*, Fuzzy Sets Syst. **115** (2000), no. 3, 485–489.

[34] H.M. Wali, *Compactness of Hausdorff fuzzy metric spaces*, Int. J. Appl. Math. **51** (2021), no. 1, 17–24.