Int. J. Nonlinear Anal. Appl. 15 (2024) 7, 289–298
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2023.30970.4529

On the structure of the equitably nondominated set of multi-objective optimization problems

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(Communicated by Haydar Akca)

Abstract

This paper is mainly concerned with some of the theoretical aspects of equitable multi-objective optimization. By using the equitability preference structure, we discuss some properties of the equitably nondominated set, such as nonemptiness, external stability and connectedness. Also, we introduce the concept of proper equitable nondominance, and show that these solutions can be obtained by minimizing a weighted sum of the sort of objective functions where all weights are positive and decreasing. Moreover, we present a hybrid scalarization problem to generate equitably nondominated solutions. This method also provides a necessary condition for the existence of properly equitable nondominated solutions.

Keywords: Nondominancy, Proper nondominance, Equitability, External stability, Connectedness, Multi-objective programming

2020 MSC: Primary 90B50; Secondary 91B08, 90C29

1 Introduction

The equitable efficiency tends to strengthen the notion of Pareto efficiency by imposing additional conditions on the preference structure defining the Pareto preference. It is especially designed to solve multi-objective optimization problems in which the criteria are uniform in the sense of the scale used and their values are directly comparable. Equitability is based on the assumption that the criteria are not only comparable (measured on a common scale) but also anonymous (impartial). The latter makes the distribution of outcomes among the criteria more important than the assignment of outcomes to specific criteria, and therefore models are equitable allocation of resources.

The equitable preference was first known as the generalized Lorenz dominance [5, 7]. Kostreva and Ogryczak [3] are the first ones who introduced the concept of equitability into multi-objective programming. They have shown equitable efficiency to be a refinement of Pareto efficiency by adding, to the reflexivity, strict monotonicity and transitivity of the Pareto preference order, the requirements of impartiality and satisfaction of the principle of transfers. Moreover, they have analyzed the structure of the equitably efficient set of a linear multi-objective optimization problem and obtained some properties of the equitably efficient set. These include sufficient conditions for existence, connectivity of the equitably efficient set, and characterizations related to weighting problems. Then, Kostreva et al. [4] presented the theory of equitable efficiency in greater generality. They have developed scalarization approaches to generate equitably efficient solutions of linear and nonlinear multi-objective programs. Singh [14] has developed some scalarization based

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Received: May 2023 *Accepted:* July 2023

methods of generating equitably efficient solutions and showed how equitably efficient solutions arise in the context of a particular type of linear complementarity problem and matrix games.

More recently, Foroutannia and Mahmodinejad [2] have introduced the concept of equitably B-efficient solution, where B is a partition of the index of the objective functions. The equitable optimization method applied to problems such as portfolio, location, telecommunications and resource allocation [8, 9, 10, 11, 12]. It should be noted that some authors have used the term “fair” rather than “equitable”.

The relationships between the properties of nonemptiness, $R^p_{\geqq}$-(semi)compactness, external stability and connectedness of the set of nondominated solutions are established by Pourkarimi and Soleimani-damaneh in [13]. In this paper, we want to examine these properties for the equitably nondominated set of multi-objective optimization. The paper is organized as follows. In the second section, terminology is introduced and basic concepts are defined. Some properties of the equitably nondominated set such as existence, external stability and connectedness, will be discussed in third section. In the fourth section, we introduce the concept of proper equitable nondominance and characterize these solutions by the weighted sum problem. Finally, the final section concludes the paper.

2 Terminology

Let R^p be the Euclidean vector space and $y', y'' \in R^p$. We use the following componentwise orders, $y' \leqq y''$ denotes $y'_i \leq y''_i$ for $i = 1, 2, \ldots, p$ and $y' < y''$ denotes $y'_i < y''_i$ for $i = 1, 2, \ldots, p$, and also $y' \leq y''$ denotes $y' \leqq y''$ but $y' \neq y''$. With the relations $\geqq$, $\geq$ and $>$ defined accordingly, then the nonnegative, the nonnegative nonzero and the positive orthants are denoted by $R^p_{\geqq} = \{y \in R^p : y \geqq 0\}$, $R^p_{\geq} = \{y \in R^p : y \geq 0\}$ and $R^p_{>} = \{y \in R^p : y > 0\}$.

We consider a multi-objective optimization problem which minimizes a set of objective functions with p members. The image of the feasible set under the objective functions is called the image space, and it is usually denoted by Y.

Definition 2.1. Let $Y \subset R^p$. The point $\hat{y} \in Y$ is called a nondominated solution of Y if there does not exist $y \in Y$ such that $y \leq \hat{y}$. The set of all nondominated solutions of Y is denoted by Y_N and called the nondominated set.

The concept of proper nondominance plays an important role in multi-objective optimization, from both theoretical and practical points of view. There are different definitions for proper nondominance in the literature, see [1]. Among them, we use the following ones.

Definition 2.2. Let $Y \subset R^p$. The vector $\hat{y} \in Y$ is called a properly nondominated solution of Y in the Geoffrion's sense, if $\hat{y} \in Y_N$ and there is a real number $M > 0$ such that for all $y \in Y$ and $i \in \{1, 2, \ldots, p\}$ satisfying $y_i < \hat{y}_i$ there exists an index $j \in \{1, 2, \ldots, p\}$ such that $\hat{y}_j < y_j$ and

$$\frac{\hat{y}_i - y_i}{y_j - \hat{y}_j} \leq M.$$

The set of all properly nondominated solutions of Y is denoted by Y_{PN}.

For a set $S \subset R^p$, we use the notations $cl(S)$ and $cone(S)$ for the closure and the convex conic hull of S, respectively.

Definition 2.3. Let $Y \subset R^p$. The vector $\hat{y} \in Y$ is called a properly nondominated in Benson's sense, if

$$cl(cone(Y + R^p_{\geqq} - \hat{y})) \bigcap -R^p_{\geqq} = \{0\}\,.$$

According to Theorem 2.48 from [1] proper dominancy in Geoffrion's sense is equivalent to proper dominancy in Benson's sense. The following definition is a necessary notion for the concepts of solution of interest in this paper.

Definition 2.4. Let $y \in \mathbb{R}^p$.

1. The function $\theta : \mathbb{R}^p \to \mathbb{R}^p$ is called an ordering map iff $\theta(y) = (\theta_1(y), \theta_2(y), \ldots, \theta_p(y))$, where $\theta_1(y) \geq \theta_2(y) \geq \ldots \geq \theta_p(y)$ in which $\theta_i(y) = y_{\tau(i)}$ for $i = 1, 2, \ldots, p$, and τ is a permutation of the set $\{1, 2, \ldots, p\}$.
2. The function $\bar{\theta} : \mathbb{R}^p \to \mathbb{R}^p$ is called a cumulative ordering map iff $\bar{\theta}(y) = (\bar{\theta}_1(y), \bar{\theta}_2(y), \ldots, \bar{\theta}_p(y))$, where $\bar{\theta}_i(y) = \sum_{j=1}^{i} \theta_j(y)$ for $i = 1, 2, \ldots, p$.

Definition 2.5. For any two vectors $y', y'' \in Y$, we say that y' equitably dominates y'', and denote by $y' \prec_e y''$ iff $\bar{\theta}(y') \leq \bar{\theta}(y'')$. A vector $\hat{y} \in Y$ is called equitably nondominated solution of Y if there is no $y \in Y$ such that $y \prec_e \hat{y}$. The set of all equitably nondominated solutions is denoted by Y_{EN} and called the equitably nondominated set.

It should be noted the weak equitable preference and indifference equitable preference relations are respectively defined by

$$\begin{aligned} y' \preceq_e y'' \quad &\Longleftrightarrow \quad \bar{\theta}(y') \leqq \bar{\theta}(y''), \\ y' \simeq_e y'' \quad &\Longleftrightarrow \quad \bar{\theta}(y') = \bar{\theta}(y''). \end{aligned}$$

The weak equitable dominance relation is illustrated by the following sets.

Definition 2.6. Let $\hat{y} \in Y$. The set of points

1. weakly equitable dominated by $\hat{y}$ is defined as $D(\hat{y}) = \{y \in R^p : \bar{\theta}(\hat{y}) \leqq \bar{\theta}(y)\}$;
2. weakly equitable preferred to $\hat{y}$ is defined as $P(\hat{y}) = \{y \in R^p : \bar{\theta}(y) \leqq \bar{\theta}(\hat{y})\}$.

The domination structure of the equitable dominance depends on the location of a vector relative to the absolute equity line $(y_1 = y_2 = \ldots y_p)$. In the general case, the set $D(y)$ is not a cone and it is not convex. In the following figure, we show these sets for the point $\hat{y} = [3, 1]^T$, where $Y = R^2$. The red and cyan shaded regions represents the set $D(\hat{y})$ and $P(\hat{y})$, respectively.

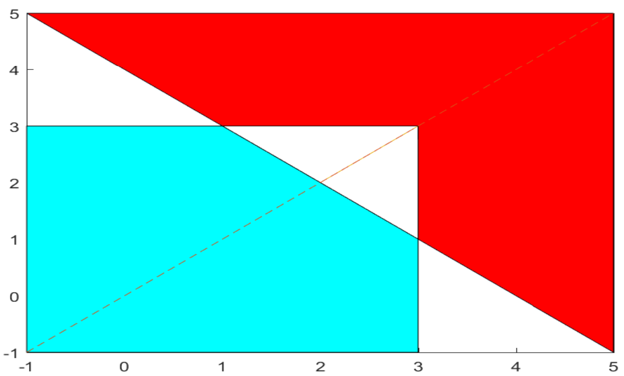

Figure 1: The $P(\hat{y})$, $D(\hat{y})$ with $\hat{y} = [3, 1]^T$ for $Y = R^2$

3 Structure of the equitably nondominated set

In this section, we first provide some sufficient conditions which guarantee that the equitably nondominated set is nonempty. Then we investigate the connections between nonemptiness, $\bar{\theta}$-semicompactness, $\bar{\theta}$-compactness, $\bar{\theta}$-external stability and connectedness of the equitably nondominated set.

Theorem 3.1. Suppose there is some $y^{\circ} \in Y$ such that the set $P(y^{\circ}) \bigcap Y$ is compact. Then $Y_{EN} \neq \emptyset$.

Proof . Consider the following optimization problem:

$$\begin{aligned} \min \quad & \sum_{i=1}^{p} \bar{\theta}_i(y) \\ s.t. \quad & y \in P(y^{\circ}) \bigcap Y. \end{aligned} \tag{3.1}$$

The set of feasible solutions to Problem (3.1) is $P(y^{\circ}) \bigcap Y$ which is compact. Because the objective function of this problem is continuous, there exists a feasible solution $\hat{y} \in P(y^{\circ}) \bigcap Y$ such that $\hat{y}$ is an optimal solution of Problem

(3.1). We claim that $\hat{y} \in Y_{EN}$. If $\hat{y} \notin Y_{EN}$, then there is $\bar{y} \in Y$ such that $\bar{y} \prec_e \hat{y} \preceq_e y^\circ$. Hence $\bar{y}$ is a feasible solution to Problem (3.1) and

$$\sum_{i=1}^{p} \bar{\theta}_i(\bar{y}) < \sum_{i=1}^{p} \bar{\theta}_i(\hat{y}).$$

This contradicts the optimality of $\hat{y}$ for (3.1) and the proof is completed. □ Note that the condition given in the above theorem is essential. For example, if

$$Y = \{(x_1, x_2) \colon x_1^2 + x_2^2 \leqslant 1, -1 \leqslant x_1 \leqslant 0, -1 \leqslant x_2 \leqslant 0\} - \{(-1/\sqrt{2}, -1/\sqrt{2})\},$$

then $P(y) \bigcap Y$ is not compact for any $y \in Y$, and $Y_{EN} = \emptyset$.

In the following, we considers open covers with special sets and define $\bar{\theta}$-semicompactness, similar to the concept of $R^p_{\geq}$-semicompactness from [1].

Definition 3.2. $Y \subseteq R^p$ is called $\bar{\theta}$-semicompact if every open cover of Y of the form $\{P(y^i)^C : y^i \in Y, i \in I\}$ has a finite subcover, where I is an index set. This means that whenever $Y \subset \cup_{i \in I} P(y^i)^C$ there exist a positive integer t and $\{i_1, i_2, \dots, i_t\} \subset I$ such that $Y \subset \cup_{k=1}^{t} P(y^{i_k})^C$.

Here $P(y^i)^C$ denotes the complement $R^p - P(y^i)$ of $P(y^i)$. Note that these sets are always open.

Let $(S, \preceq)$ be a preordered set, i.e, $\preceq$ is reflexive and transitive. $(S, \preceq)$ is inductively ordered, if every totally ordered subset of $(S, \preceq)$ has a lower bounded. Now, by applying Zorn's Lemma, we can prove that $\bar{\theta}$-semicompactness guarantees existence of an equitable nondominated solution, similar to Theorem 2.12 of [1].

Theorem 3.3. If $Y \neq \emptyset$ is $\bar{\theta}$-semicompact, then $Y_{EN} \neq \emptyset$.

Proof . We show that Y is inductively ordered with respect to $\preceq_e$ and apply Zorn's lemma. If Y is not inductively ordered, then there is a totally ordered subset Y' of Y which has no lower bound. Let $Y' = \{y^i : i \in I\}$, we have

$$\bigcap_{i \in I} (P(y^i) \bigcap Y) = \emptyset, \tag{3.2}$$

because any element in this intersection would be lower bound of Y'. Since $P(y^i)$ is closed, the relation (3.2) deduces that the collection $\{P(y^i)^C : i \in I\}$ is an open cover of Y. The assumption of $\bar{\theta}$-semicompactness implies that there is a finite subcover of $\{P(y^i)^C : i \in I\}$. On the other hand, we have $P(y^i) \subseteq P(y^j)$ if and only if $y^i \preceq_e y^j$ and the sets of the cover are totally ordered by inclusion because Y' is a totally ordered subset. So there exists a single $y^\star \in Y'$ such that $Y \subseteq P(y^\star)^C$. This concludes $y^\star \preceq_e y^i$ for all i and $y^\star \notin Y$, which is not possible. Therefore Y is inductively ordered. Now according Zorn's lemma Y contains minimal element i.e. there is $y^\circ \in Y$ such that $y \preceq_e y^\circ$ implies that $y^\circ \preceq_e y$. It remains to be shown that $y^\circ \in Y_{EN}$. If $y^\circ \notin Y_{EN}$, then there would be some $\bar{y} \in Y$ with $\bar{y} \prec_e y^\circ \preceq_e \bar{y}$, which is a contradiction. □

It is usually not easy to check the condition of $\bar{\theta}$-semicompactness in Theorem 3.3. Because this we present the stronger condition of $\bar{\theta}$-compactness.

Definition 3.4. $Y \subseteq R^p$ is called $\bar{\theta}$-compact if $P(\hat{y}) \bigcap Y$ is compact for all $\hat{y} \in Y$.

Theorem 3.5. If Y is $\bar{\theta}$-compact, then Y is $\bar{\theta}$-semicompact, hence $Y_{EN} \neq \emptyset$.

Proof . The proof is similar to Proposition 2.14 of [1]. If $\{P(y^i)^C \colon y^i \in Y, i \in I\}$ is an open cover of Y. For arbitrary $y^{i^\star} \in Y$, the collection

$$\{P(y^i)^C \colon y^i \in Y,\ i \in I,\ i \neq i^\star\},$$

defines an open cover of $P(y^{i^\star}) \bigcap Y$. Since Y is $\bar{\theta}$-compact, this cover contains a finite subcover of $P(y^{i^\star}) \bigcap Y$. This finite subcover together with $P(y^{i^\star})^C$ yields a finite cover of

$$Y = (P(y^{i^\star}) \bigcap Y) \bigcup (P(y^{i^\star})^C \bigcap Y),$$

of the structure required for $\bar{\theta}$-semicompact. Thus Y is $\bar{\theta}$-semicompact, and Theorem 3.3 follows that $Y_{EN} \neq \emptyset$. □

We continue this section by introducing a new concept, external stability of the equitably nondominated set.

Definition 3.6. Let $Y \subseteq R^p$. The set Y_{EN} is called $\bar{\theta}$-externally stable if $Y \subseteq \bigcup_{y \in Y_{EN}} D(y)$, i.e. for any $y \in Y \backslash Y_{EN}$ there is $\hat{y} \in Y_{EN}$ such that $\bar{\theta}(\hat{y}) \leqq \bar{\theta}(y)$.

Similar to Theorem 4.1 of [13], we give the following theorem.

Theorem 3.7. If $Y \subseteq R^P$ is nonempty and $\bar{\theta}$-semicompact, then Y_{EN} is $\bar{\theta}$-externally stable.

Proof . For any $y^\circ \in Y$, we show that the set $Y^\circ = P(y^\circ) \bigcap Y$ is $\bar{\theta}$-semicompact. To do this, assume that

$$Y^\circ \subseteq \bigcup_{i \in I} P(y^i)^C.$$

We have

$$Y \subseteq (\bigcup_{i \in I} P(y^i)^C) \bigcup P(y^\circ)^C.$$

The $\bar{\theta}$-semicompactness assumption of Y implies that there exists $m \in N$ such that

$$Y \subseteq (\bigcup_{i=1}^m P(y^i)^C) \bigcup P(y^\circ)^C.$$

Hence

$$Y^\circ \subseteq \bigcup_{i=1}^m P(y^i)^C,$$

and Y° is $\bar{\theta}$-semicompact. Now by using Theorem 3.3, $Y^\circ_{EN} \neq \emptyset$. Thus, there exists $\bar{y} \in Y^\circ_{EN}$. We claim that $\bar{y} \in Y_{EN}$, because otherwise there is $y^\star \in Y$ such that $y^\star \prec_e \bar{y} \preceq_e y^\circ$. Hence there exists $y^\star \in Y^\circ$ such that $y^\star \prec_e \bar{y}$, which contradict $\bar{y} \in Y^\circ_{EN}$. Therefore $\bar{y} \in Y^\circ_{EN} \bigcap Y_{EN}$ and

$$y^\circ \in D(\bar{y}) \subseteq \bigcup_{y \in Y_{EN}} D(y).$$

Since $y^\circ \in Y$ is arbitrary, we have

$$Y \subseteq \bigcup_{y \in Y_{EN}} D(y).$$

□

By applying Theorem 3.5 and Theorem 3.7, the following result is obtained.

Corollary 3.8. If $Y \subset R^p$ be nonempty and $\bar{\theta}$-compact. Then Y_{EN} is $\bar{\theta}$-externally stable.

In the next, we investigate the connection between the notions introduced so far. To do this the following lemmas are required.

Lemma 3.9. We have the following statements:

(i) If Y is closed, then $\bar{\theta}(Y)$ and $\theta(Y)$ are closed.
(ii) If Y is an unbounded subset of R^p, then $\bar{\theta}(Y)$ is unbounded.

Proof . The proof is obvious. □

Lemma 3.10 ([6]). Assume that $Y \subseteq R^p$ is a closed convex set. We have the following statements:

(i) Y is unbounded if and only if $d \in Y^\infty \neq \emptyset$, where

$$Y^\infty = \{d \in R^p \colon d \neq 0, y + \alpha d \in Y, \forall y \in Y, \forall \alpha > 0\},$$

is the set of recession directions of Y.
(ii) If there exists $\hat{y} \in Y$ and $d \in R^p$ such that $\hat{y} + \alpha d \in Y$ for any $\alpha > 0$, then $d \in Y^\infty$.

Theorem 3.11. Let $Y \subseteq R^p$ be nonempty, convex and closed. If $P(y) \bigcap Y$ is closed for every $y \in Y$, then the following statement are equivalent:

(i) $Y_{EN} \neq \emptyset$.
(ii) $P(y) \bigcap Y$ is bounded for every $y \in Y$.

(iii) Y is $\bar{\theta}$-compact.
(iv) Y is $\bar{\theta}$-semicompact.
(v) Y_{EN} is $\bar{\theta}$-externally stable.

Proof . We only prove that $(i) \Rightarrow (ii)$. The remaining part resulted from Theorems 3.5 and 3.7. By contradiction assume that there exists $\hat{y} \in Y$ such that $P(\hat{y}) \bigcap Y$ is unbounded. Using Lemma 3.9, conclude that the set

$$\bar{\theta}(P(\hat{y}) \bigcap Y) = (\bar{\theta}(\hat{y}) - R^p_{\geqq}) \bigcap \bar{\theta}(Y),$$

is unbounded, so $(\bar{\theta}(\hat{y}) - R^p_{\geqq}) \bigcap (\bar{\theta}(Y) + R^p_{\geqq})$ is an unbounded closed convex set. Hence, the part (i) of Lemma 3.10 implies that there exists nonzero vector $d \in R^p$ such that

$$\bar{\theta}(\hat{y}) + \alpha d \in (\bar{\theta}(\hat{y}) - R^p_{\geqq}) \bigcap (\bar{\theta}(Y) + R^p_{\geqq}), \qquad (\forall \alpha > 0).$$

Therefore $\bar{\theta}(\hat{y}) + \alpha d \in \bar{\theta}(Y) + R^p_{\geqq}$ for all $\alpha > 0$. The part (ii) of Lemma 3.10 concludes that

$$z + d \in \bar{\theta}(Y) + R^p_{\geqq}, \qquad (\forall z \in \bar{\theta}(Y) + R^p_{\geqq}).$$

Since $\bar{\theta}(\hat{y}) + \alpha d \in \bar{\theta}(\hat{y}) - R^p_{\geqq}$, we have $d \leq 0$ which results $z + d \leq z$ for each $z \in \bar{\theta}(Y) + R^p_{\geqq}$. These imply $(\bar{\theta}(Y) + R^p_{\geqq})_N = (\bar{\theta}(Y))_N = \emptyset$, which contradict the assumption. □

Consider a multi-objective optimization problem as follows:

$$\min\left(f_1(x), f_2(x), \ldots, f_p(x)\right), \quad \textit{subject to } x \in X \tag{3.3}$$

where $X \subset R^n$ is a nonempty set and f is a vector function that maps the feasible set X into the objective space R^p. The image of X under f is denoted by $Y = f(X)$ and is referred to image space. We recall that if $\hat{y} = f(\hat{x})$ is a nondominated solution of Y, then $\hat{x}$ is an efficient solution of problem (3.3). Hence

$$X_E = \{x \in X : f(x) \in Y_N\},$$

is called the set of all efficient solutions of problem (3.3).

Kostreva and Ogryczak in [3] express equitable nondominance in terms of the standard efficiency for the multi-objective optimization problem with objectives $\overline{\theta}(y)$

$$\min\left(\bar{\theta}_1(y), \bar{\theta}_2(y), \ldots, \bar{\theta}_p(y)\right) \qquad \text{subject to} \quad y \in Y. \tag{3.4}$$

Corollary 3.12. ([3], Corollary 2.2.) A vector $\hat{y} \in Y$ is an equitably nondominated solution if and only if $\hat{y}$ is an efficient solution of problem (3.4).

Now, we examine the connectedness property of Y_{EN}. The connectedness is topological property that can make the task of selecting a final compromise solution among the set of equitably nondominated solutions easier, as there are no gaps in the equitably nondominated set. To derive a connectedness result for Y_{EN}, we need the following statement.

Theorem 3.13 ([1], Theorem 3.40). Let $X \subseteq R^p$ be a convex and compact set. Assume that all objective functions f_k are convex. Then X_E is connected.

Corollary 3.14. If $Y \subseteq R^p$ is a convex and compact set, then Y_{EN} is connected.

Proof . The function $\bar{\theta}_i$ is a convex function for each i, because

$$\begin{aligned} \bar{\theta}_i(\lambda y' + (1-\lambda)y'') &= \max_{|I|=i, I \subset \{1,2,\ldots,p\}} \sum_{j \in I} (\lambda y'_j + (1-\lambda) y''_j) \\ &\leq \lambda \max_{|I|=i, I \subset \{1,2,\ldots,p\}} \sum_{j \in I} y'_j + (1-\lambda) \max_{|I|=i, I \subset \{1,2,\ldots,p\}} \sum_{j \in I} y''_j \\ &= \lambda \bar{\theta}_i f(x) + (1-\lambda) \bar{\theta}_i f(y). \end{aligned} \tag{3.5}$$

Now, the desired result follows by using Theorem 3.13 and Corollary 3.12. □

Lemma 3.15 ([1], Lemma 3.32). If $\{S_i : i \in I\}$ is a family of connected sets with $\bigcap_{i\in I} S_i \neq \emptyset$ then $\bigcup_{i\in I} S_i$ is connected.

Finally, we prove the connectedness of Y_{EN} under the weaker condition of $\bar{\theta}$-compactness.

Theorem 3.16. If $Y \subseteq R^p$ is closed, convex, and $\bar{\theta}$-compact, then Y_{EN} is connected.

Proof . Let $d \in R^p_>$. For any $\alpha \in R$, we define $y(\alpha) = \alpha d$. We claim that for all $y \in R^p$ there is a real number $\alpha > 0$ such that $y \in P(y(\alpha))$. Because otherwise the nonempty convex sets $\{\bar{\theta}(y) - \alpha\bar{\theta}(d) : \alpha > 0\}$ and $-R^p_{\geqq}$ can be separated. Thus there exists some $y^\star \in R^p - \{0\}$ with

$$\langle y^\star, \bar{\theta}(y) - \alpha\bar{\theta}(d)\rangle \geqslant 0, \qquad (\forall \alpha > 0), \tag{3.6}$$

$$\langle y^\star, -d'\rangle \leqslant 0, \qquad (\forall d' \in R^p_{\geqq}), \tag{3.7}$$

by using separation theorem. So $\langle y^\star, d'\rangle \geqslant 0$, for all $d' \in R^p_{\geqq}$, in particular $\langle y^\star, \bar{\theta}(d)\rangle > 0$ because $d \in R^p_>$. Hence $\langle\langle y^\star, \bar{\theta}(y) - \alpha\bar{\theta}(d)\rangle < 0$ for α sufficiently large, but that is contradiction to (3.6). By using Theorem 3.11, we have $Y_{EN} \neq \emptyset$. So according to the claim proved, we can choose $\hat{y} \in Y_{EN}$ and appropriate $\hat{\alpha} > 0$ such that $\hat{y} \in P(y(\hat{\alpha}))$, which means that $P(y(\hat{\alpha})) \bigcap Y_{EN} \neq \emptyset$. We define

$$Y(\alpha) := P(y(\alpha)) \bigcap Y.$$

With this notation, the claim above implies in particular that

$$Y_{EN} = \bigcup_{\alpha \geqslant \hat{\alpha}} Y(\alpha)_{EN}.$$

Also, it is obvious that $Y(\hat{\alpha})_{EN} \subseteq Y(\alpha)_{EN}$ for $\alpha \geqslant \hat{\alpha}$, therefore

$$\bigcap_{\alpha \geqslant \hat{\alpha}} Y(\alpha)_{EN} = Y(\hat{\alpha})_{EN} \neq \emptyset.$$

Since Y is $\bar{\theta}$-compact, $Y(\alpha)$ is compact. We now apply Corollary 3.14 to get $Y(\alpha)_{EN}$ is connected. We have expressed Y_{EN} as union of a family of connected sets with nonempty intersection. Hence Lemma 3.15 concludes that Y_{EN} is a connected set. □

4 Proper equitable nondominance

The purpose of this section is to introduce of the concept of proper equitable nondominance. We prove that the weighted sum scalarization method is able to find properly equitable nondominated solutions. Moreover, we present a hybrid method for generating the equitable nondominated solution and we obtain a necessary condition for existence of properly equitable nondominated solutions via this method.

To introduce the concept of proper equitable nondominance, we use the definition of properly nondominated in Geoffrion's sense which in trade-offs of problem with objectives $\bar{\theta}(y)$ are bounded.

Definition 4.1. A vector $\hat{y} \in Y_{EN}$ is called properly equitable nondominated (in Geoffrion sense), if there is a real number $M > 0$ such that for all $i \in \{1, 2, \ldots, p\}$ and $y \in Y$ satisfying $\bar{\theta}_i(y) < \bar{\theta}_i(\hat{y})$ there exists an index $j \in \{1, 2, \ldots, p\}$ such that $\bar{\theta}_j(\hat{y}) < \bar{\theta}_j(y)$ and

$$\frac{\bar{\theta}_i(\hat{y}) - \bar{\theta}_i(y)}{\bar{\theta}_j(y) - \bar{\theta}_j(\hat{y})} \leq M.$$

The set of all properly equitable nondominated solutions is denoted by Y_{PEN}. This definition allows us to express the proper equitable nondominance in terms of the standard proper efficiency.

Corollary 4.2. The vector $\hat{y} \in Y$ is a properly equitable nondominated solution if and only if $\hat{y}$ is a properly efficient solution of problem (3.4).

Kostreva and Ogryczak in [3] showed that the equitably nondominated set is contained within the nondominated set. On the other hand, Definition 4.1 implies that $Y_{PEN} \subset Y_{EN}$, so $Y_{PEN} \subset Y_N$. Hence, to reduce nondominated solutions, we can use properly equitable nondominated solutions. The following example shows this fact.

Example 4.3 ([3], Example 2.1). Let's consider the problem

$$\min \{(y_1, y_2) \colon 3y_1 + y_2 \geqslant 21, 4y_1 + 5y_2 \geqslant 72, y_1 \geqslant 0, y_2 \geqslant 0\}.$$

It is obvious that

$$\begin{aligned} Y_N &= \{(y_1, y_2) \colon 3y_1 + y_2 = 21, 4y_1 + 5y_2 = 72, y_1 \geqslant 0, y_2 \geqslant 0\}, \\ Y_{EN} &= Y_{PEN} = \{(y_1, y_2) \colon 4y_1 + 5y_2 = 72, 3 \leqslant y_1 \leqslant 8\}, \end{aligned}$$

hence $Y_{PEN} \subset Y_{EN} \subset Y_N$.

The weighted sum method is one of the most common ways of finding nondominated solutions of multi-objective problem. Kostreva et al. [4] have proven every optimal solution of the weighted sum problem with strictly decreasing positive weights and ordering map $\theta(y)$, is an equitably nondominated solution.

Theorem 4.4. [3, Proposition 3.2] Let $\hat{y}$ be an optimal solution of weighted sum optimization problem

$$\min_{y \in Y} \sum_{k=1}^{p} \lambda_k \theta_k(y). \tag{4.1}$$

If $\lambda_1 > \lambda_2 > \ldots > \lambda_p > 0$, then $\hat{y} \in Y_{EN}$.

In the following, we show that this theorem also holds for proper equitable nondominance. Furthermore, we prove that the converse of the theorem is true when the set Y is convex. For this purpose, the below statements are useful.

Lemma 4.5. Let x and λ be two vectors in R^p. If $\lambda_1 > \lambda_2 \ldots > \lambda_p > \lambda_{p+1} = 0$, then

$$\sum_{i=1}^{p} \lambda_i x_i = \sum_{i=1}^{p} (\lambda_i - \lambda_{i+1}) \sum_{k=1}^{i} x_k.$$

Proof . The proof is obvious. □

By using Lemma and setting $\alpha_k = \lambda_k - \lambda_{k+1}$ for $k = 1, 2, \ldots p-1$ and $\alpha_p = \lambda_p$, one can see the weighted sum optimization problem (4.1) is equivalent to

$$\min_{y \in Y} \sum_{k=1}^{p} \alpha_k \bar{\theta}_k(y). \tag{4.2}$$

It is interesting to note here that $\alpha_k > 0$ and $\lambda_k = \sum_{i=k}^{p} \alpha_i$ for $k = 1, 2, \ldots p$.

We recall that if $\hat{y} = f(\hat{x})$ is a properly nondominated solution of Y, then $\hat{x}$ is a properly efficient solution of problem (3.3).

Theorem 4.6 ([1], Theorem 3.15). If $\lambda \in R^p_>$ and $\hat{x} \in X$ is an optimal solution of problem

$$\min_{x \in X} \sum_{k=1}^{p} \lambda_k f_k(x). \tag{4.3}$$

then $\hat{x}$ is a properly efficient solution of problem (3.3). Conversely, let X be a convex set, and let f_j be convex functions for $j = 1, 2, \ldots, p$. If $\hat{x}$ is a properly efficient solution of problem (3.3), then there exists some $\lambda \in R^p_>$ such that $\hat{x}$ is an optimal solution of problem (4.3).

Theorem 4.7. Let $\lambda \in R^p_>$ be a vector with $\lambda_1 > \lambda_2 > \ldots > \lambda_p > 0$. If $\hat{y} \in Y$ is an optimal solution of problem (4.1), then $\hat{y} \in Y_{PEN}$. Conversely, if Y is a convex set and $\hat{y} \in Y_{PEN}$, then there exists some $\lambda \in R^p_>$ with $\lambda_1 > \lambda_2 > \ldots > \lambda_p > 0$, such that $\hat{y}$ is an optimal solution of Problem (4.1).

Proof . If $\hat{y} \in Y$ is an optimal solution of problem (4.1), then it is an optimal solution of problem (4.2). By applying Theorem 4.6 to problem (4.2), we deduce that $\hat{y}$ is a properly efficient solution of problem (3.4). Now Corollary 4.2 implies that $\hat{y} \in Y_{PEN}$. Conversely, let $\hat{y} \in Y_{PEN}$. Using the Corollary 4.2 concludes that $\hat{y}$ is properly efficient solution of problem (3.4). The function $\bar{\theta}$ is convex, according to relation (3.5), hence by Theorem 4.6 there exists some $\alpha \in R^p_>$ such that $\hat{y}$ is an optimal solution of problem (4.2). By setting $\lambda_k = \sum_{i=k}^{p} \alpha_i$ for all $k = 1, 2, \ldots p$, we have the desired result. □

Now, we find equitable nondominated solutions by a hybrid method which is defined by combining the weighted sum method with the ϵ-constraint method. In this method, the scalarized problem to be solved has a weighted sum objective and constraints on all objectives $\bar{\theta}(y)$. For an arbitrary point $y^0 \in Y$, consider the following problem:

$$\min \sum_{k=1}^{p} \lambda_k \left(\sum_{j=1}^{k} \theta_j(y) \right), \qquad \text{subject to} \quad \bar{\theta}(y) \leqq \bar{\theta}(y^0), y \in Y, \tag{4.4}$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p) \in R^p_>$. Then we have the following theorem.

Theorem 4.8. Suppose that $y \in Y$ is an optimal solution of (4.4), then y is an equitably nondominated solution. The converse is true when $y = y^0$.

Proof . Let $y \in Y$ be an optimal solution of (4.4) and $y \notin Y_{EN}$. There is a feasible solution $y' \in Y$ such that $\bar{\theta}(y') \leq \bar{\theta}(y)$. We have

$$\sum_{i=1}^{p} \lambda_i \left(\sum_{j=1}^{i} \theta_j(y') \right) < \sum_{i=1}^{p} \lambda_i \left(\sum_{j=1}^{i} \theta_j(y) \right),$$

Therefore y can not be an optimal solution of (4.4). Conversely, suppose y^0 is not an optimal solution of (4.4). So there is a feasible solution $y' \in Y$ of (4.4) such that

$$\sum_{i=1}^{p} \lambda_i \left(\sum_{j=1}^{i} \theta_j(y') \right) < \sum_{i=1}^{p} \lambda_i \left(\sum_{j=1}^{i} \theta_j(y^0) \right).$$

Hence we deduce that $\bar{\theta}(y') \leq \bar{\theta}(y^0))$, which is contradiction with $y^0 \in Y_{EN}$. □

Finally, we obtain a necessary condition for existence of properly equitable nondominated solutions by scalarization problem (4.4).

Theorem 4.9. If $Y_{PEN} \neq \emptyset$, then for any feasible solution $y^0 \in Y$, Problem (4.4) has a finite value.

Proof . If the theorem is not true, there exist $y^0 \in Y$ such that Problem (4.4) is unbounded. Because of $Y_{PEN} \neq \emptyset$, there is $\hat{y} \in Y_{PEN}$. Hence $\hat{y}$ is a properly efficient solution of problem (3.4), by Corollary 4.2. Since proper efficiency in Geoffrion's sense is equivalent to proper efficiency in Benson's sense, we have

$$cl(cone(\bar{\theta}(Y) + R^p_{\geqq} - \bar{\theta}(\hat{y}))) \cap -R^p_{\geqq} = \{0\}. \tag{4.5}$$

As problem (4.4) is unbounded, there is some sequence $\{y_n\} \subseteq Y$ such that $\bar{\theta}(y_n) < \bar{\theta}(y^0)$ and $\sum_{i=1}^{p} \lambda_i \bar{\theta}_i(y_n) \leq -n$, for $n = 1, 2, \ldots$. Hence we have $\|\bar{\theta}(y_n) - \bar{\theta}(y^0)\| \to \infty$ when $n \to \infty$.

Since the sequence $\frac{\bar{\theta}(y_n)-\bar{\theta}(y^0)}{\|\bar{\theta}(y_n)-\bar{\theta}(y^0)\|}$ is bounded in R^p, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that

$$\lim_{k\to\infty} \frac{\bar{\theta}(y_{n_k}) - \bar{\theta}(\hat{y})}{\|\bar{\theta}(y_{n_k}) - \bar{\theta}(y^0)\|} = d$$

where d is a nonzero vector in $R^p_{\geqq}$. Therefore for k_0 sufficiently large we have

$$\frac{\bar{\theta}(y_{n_k}) - \bar{\theta}(\hat{y})}{\|\bar{\theta}(y_{n_k}) - \bar{\theta}(y^0)\|} \in -R^p_{\geqq} - \{0\}\,, \qquad (\forall k \geq k_0),$$

which contradicts the relation (4.5). □

Therefore the unboundedness of the problem (4.4) shows that no properly equitable nondominated solution exists.

5 Conclusion

In this paper, we investigated the properties non-emptiness, external stability and connectedness for the equitable nondominated set, Y_{EN}, by imposing the conditions $\bar{\theta}$-semicompactness and $\bar{\theta}$-compactness on Y, and we established the equivalence between these concepts. Also, we introduced the concept of properly equitable nondominated solution and characterized them by minimizing a weighted sum of the sort of objective functions where the vector of weight is positive and decreasing. Moreover, a hybrid scalarization problem is presented to generate equitably nondominated solutions and the existence of properly efficient solutions via this method is studied.

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